On natural derivatives and the curvature formula in fibre bundles

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Abstract

In a fibre bundle, natural derivatives of a section are defined as tangent vector fields on the image of a section of the fibre bundle. A local extension to vector fields in the tangent bundle leads to a direct proof of the formula expressing the curvature of a connection in terms of covariant derivatives. The result is based on a tensoriality argument and extends to nonlinear connections on fibre bundles a well-known formula for linear connections on vector bundles.

Key words: fibre bundles,, natural derivatives, connections, integrability, covariant derivatives.

1. Introduction

The notion of connection on a fibre bundle was introduced by Charles Ehresmann [2] in 1950 and investigated by Paulette Libermann in [7], [8], [9]. Standard references on this topic are the article by Kobayashi [3] and the text [4]. The analysis developed in this paper makes also reference to the treatment of the matter presented in [5]. Let us recall some well-known facts. In the tangent bundle to a fibre bundle the vertical distribution is naturally defined by considering, at each point of the manifold, the vectors tangent to the fibre through that point. The vertical distribution is always integrable and the leaves of the induced foliation are the fibres themselves. The general definition of a connection as a (regular) field of projections on the vertical subspaces of the tangent spaces to a fibre bundle, splits each tangent space into two complementary subspaces, the vertical and the horizontal ones. This leads naturally to the question about integrability of the horizontal distribution. The involutivity condition provided by Frobenius theorem leads to the definition of the curvature as obstruction against integrability of the horizontal distribution [5]. In this context I provide a new result, stated hereafter in Theorem 4.1. This result builds a direct bridge between the expression of the curvature in terms of horizontal lifts, which is the one naturally steaming out of Frobenius involutivity condition, and the expression of the curvature in terms of covariant derivatives, more suitable for applications. The expression, which is well-known for linear connections on vector or principal bundles, is extended by the new result to general connections on fibre bundles. The proof is based on the novel definition of natural derivative vector fields, on an extension to a vector field in the tangent bundle and on a direct, powerful tensoriality argument. The analysis moves along the same line of thought as for instance the one declared in [10], by trying to avoid unnecessary recourse to additional geometric structures. In this respect the assumptions and the result of our Theorem 4.1 should be compared with the ones in [4] Chapter III Theorem 5.1, in [1] Chapter V-bis Section A.5, in [11] Chapter 2 Section 2.4, and in [13] Corollary 19.16, dealing with the curvature of linear connections on vector bundles.

2. Connection on a fibre bundle

Let us recall some definitions and notations [15], [6], [14]. Given two differentiable manifolds $\mathbb{M}\,,\mathbb{N}$, the related tangent bundles with projections $\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{M})$, $\boldsymbol{\tau}_{\mathbb{N}} \in \mathrm{C}^1(\mathbb{TN}\,;\mathbb{N})$ and a morphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, a vector field $\mathbf{X} \in \mathrm{C}^1(\boldsymbol{\varphi}(\mathbb{M})\,;\mathbb{TN})$ is $\boldsymbol{\varphi}$-related to a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ if $\mathbf{X} \circ \boldsymbol{\varphi} = T\boldsymbol{\varphi} \circ \mathbf{v}$ where T is the tangent functor. For a diffeomorphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ the push and pull operations are then defined by $\mathbf{X} = \boldsymbol{\varphi}\!\uparrow\!\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\varphi}\!\downarrow\!\mathbf{X}$. The usual notation is $\boldsymbol{\varphi}\!\uparrow = \boldsymbol{\varphi}_*$ and $\boldsymbol{\varphi}\!\downarrow = \boldsymbol{\varphi}^*$ but then too many stars do appear in the geometrical sky (push, duality, Hodge star). A fibre bundle is a surjective submersion $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ with $\mathbb{E}$ the total manifold and $\mathbb{M}$ the base manifold, i.e. $\mathbf{im}(\mathbf{p}) = \mathbb{M}$ and $\mathbf{im}(T\mathbf{p}(\mathbf{e})) = \mathbb{T}_{\mathbf{p}(\mathbf{e})}\mathbb{M}$ for all $\mathbf{e} \in \mathbb{E}$. The vertical distribution is $\mathbb{VE} := \mathbf{ker}(T\mathbf{p})$. A section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ is such that $\mathbf{p} \circ \mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ is the identity. The fibre at $\mathbf{x} \in \mathbb{M}$ is the set $\mathbb{E}_{\mathbf{x}} := \mathbf{p}^{-1}(\mathbf{x})$ which is assumed to be isomorphic to a standard fibre manifold. The pull-back bundle of the tangent bundle $\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E})$ by a section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ is the fibre bundle $\mathbf{s}\!\downarrow\!\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbf{s}\!\downarrow\!\mathbb{TE}\,;\mathbb{M})$ whose fibre at $\mathbf{x} \in \mathbb{M}$ is the tangent space $\mathbb{T}_{\mathbf{s}(\mathbf{x})}\mathbb{E}$ of $\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E})$.

Definition 1 (Connection). *A connection* $\mathrm{P_V} \in \Lambda^1(\mathbb{E}\,;\mathbb{TE})$ *in a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ *is an idempotent vector-valued one-form, which is pointwise a projector on vertical subspaces:* $\mathrm{P_V} \circ \mathrm{P_V} = \mathrm{P_V}$ *with* $\mathbf{im}(\mathrm{P_V}(\mathbf{e})) = \mathbf{ker}(T\mathbf{p}(\mathbf{e}))$. *Horizontal vectors are the ones in the kernel* $\mathbf{ker}(\mathrm{P_V}(\mathbf{e}))$ *of the connection. The projector on the horizontal distribution* $\mathbb{HE}$ *is denoted by* $\mathrm{P_H} = \mathbf{id}_{\mathbb{TE}} - \mathrm{P_V}$*, so that* $\mathrm{P_H} \circ \mathrm{P_H} = \mathrm{P_H}$ *and* $\mathrm{P_H} \circ \mathrm{P_V} = \mathrm{P_V} \circ \mathrm{P_H} = 0$.

The tangent to a section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ of a fibre bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ along a vector field $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$ is a section $T\mathbf{s} \cdot \mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{s}\!\downarrow\!\mathbb{TE})$ of the pull-back bundle $\mathbf{s}\!\downarrow\!\mathbf{p} = \mathrm{C}^1(\mathbf{s}\!\downarrow\!\mathbb{TE}\,;\mathbb{M})$.

Definition 2 (Natural derivative). *In a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$*, the natural derivative of a section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ *according to a vector field* $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$ *is the tangent vector field* $T_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{TE})$ *in the tangent bundle* $\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E})$ *defined by*

$$T_{\mathbf{v}} \circ \mathbf{s} := T\mathbf{s} \cdot \mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TE})\,.$$

For any $\mathbf{x} \in \mathbb{M}$ we have that $T_{\mathbf{v}}(\mathbf{s}_{\mathbf{x}}) = T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} \in \mathbb{T}_{\mathbf{s}_{\mathbf{x}}}\mathbb{E}$. The natural derivative $T_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{TE})$ is $\mathbf{p}$-related to the vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$, because:

$$T\mathbf{p} \circ T_{\mathbf{v}} = \mathbf{v} \circ \mathbf{p} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{TM})\,,$$

$$\mathbf{p} \circ \mathbf{Fl}^{T_{\mathbf{v}}}_{\lambda} = \mathbf{Fl}^{\mathbf{v}}_{\lambda} \circ \mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})\,.$$

It is also apparent that the natural derivative is tensorial in $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$ since the differential $T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} \in \mathbb{T}_{\mathbf{s}(\mathbf{x})}\mathbb{E}$ is linearly dependent on the vector $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$. The next statement enunciates a well known property of naturality of the Lie bracket with respect to relatedness, (see e.g. [5] Lemma 3.10 or [14] Lemma 1.3.4).

Lemma 2.1 (Morphism-related vector fields and Lie brackets). *Let the vector fields* $\mathbf{X}, \mathbf{Y} \in \mathrm{C}^1(\mathbb{N}\,;\mathbb{TN})$ *be related to vector fields* $\mathbf{u}, \mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ *by a morphism* $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, *viz:*

$$\mathbf{X} \circ \boldsymbol{\varphi} = T\boldsymbol{\varphi} \circ \mathbf{u}\,, \qquad \mathbf{Y} \circ \boldsymbol{\varphi} = T\boldsymbol{\varphi} \circ \mathbf{v}\,.$$

Then also their Lie brackets are $\boldsymbol{\varphi}$*-related:*

$$[\mathbf{X}, \mathbf{Y}] \circ \boldsymbol{\varphi} = T\boldsymbol{\varphi} \circ [\mathbf{u}, \mathbf{v}]\,.$$

Setting $T_{\mathbf{v}} \circ \boldsymbol{\varphi} := T\boldsymbol{\varphi} \circ \mathbf{v}$ *for any morphism* $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, *we have that* $T\boldsymbol{\varphi} \circ [\mathbf{u}, \mathbf{v}] = T_{[\mathbf{u},\mathbf{v}]} \circ \boldsymbol{\varphi}$ *and the result may be stated as* $[T_{\mathbf{u}}, T_{\mathbf{v}}] = T_{[\mathbf{u},\mathbf{v}]}$.

Tensoriality is a crucial property of a multilinear scalar or vector valued map, meaning that it *lives at points* [16], i.e. that its point-values depend only on the values of the argument fields at that point. A standard tensoriality criterion for multilinear forms on $\mathbb{M}$ is provided by $\mathrm{C}^\infty(\mathbb{M}\,;\Re)$-linearity (see [5] Lemma 7.3 or [6] Lemma 2.3 of Ch. VIII).

Although not needed in evaluating the Lie bracket $[T_{\mathbf{u}}, T_{\mathbf{v}}]$ on $\mathbf{s}(\mathbb{M})$, for the developments illustrated in Theorem 4.1 it is essential to extend the domain of the natural derivatives $T_{\mathbf{u}}, T_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{TE})$ outside the range $\mathbf{s}(\mathbb{M}) \subset \mathbb{E}$ of the section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$, so that they can be considered as (local) tangent vector fields $T_{\mathbf{u}}, T_{\mathbf{v}} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{TE})$ with the further property of being projectable. This task can be accomplished by the following construction.

Lemma 2.2 (Extension by foliation). *The natural derivative of a section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ *of a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$, *according to a vector field* $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$, *can be extended, in the bundle* $\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E})$, *to a (local) tangent vector field* $T_{\mathbf{v}} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{TE})$ *which projects on the vector field* $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$, *i.e. we have that, locally in* $\mathbb{E}$:

$$\boldsymbol{\tau}_{\mathbb{E}} \circ T_{\mathbf{v}} = \mathbf{id}_{\mathbb{E}}\,,$$

$$T\mathbf{p} \circ T_{\mathbf{v}} = \mathbf{v} \circ \mathbf{p}\,.$$

Proof. The extension may be performed by considering a (local) foliation of the total manifold $\mathbb{E}$, whose leaves are transversal to the fibres and include the folium $\mathbf{s}(\mathbb{M})$. The existence of at least a local foliation with these characteristics can be inferred by acting with a local bundle chart, which maps (locally) the image of the section into the trivial bundle image of the chart, and, subsequently, with a local chart which maps (locally) the fibres in their linear model space. The foliation is then performed by translation in the linear image of the fibres and the resulting leaves are mapped back to get the leaves in the total manifold. It is thus possible to define the map $\boldsymbol{\sigma} \in \mathrm{C}^1(\mathbb{E}\,;\mathrm{C}^1(\mathbb{M}\,;\mathbb{E}))$ which to each $\mathbf{e} \in \mathbb{E}$ associates the (local) section $\boldsymbol{\sigma}_\mathbf{e} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ by

$$\boldsymbol{\sigma}_\mathbf{e}(\mathbf{x}) := \boldsymbol{\Sigma}_\mathbf{e} \cap \mathbb{E}_\mathbf{x}\,, \quad \forall\, \mathbf{e} \in \mathbb{E}\,,$$

whose range is the leaf $\boldsymbol{\Sigma}_\mathbf{e}$ through $\mathbf{e} \in \mathbb{E}$. The extension of $T_\mathbf{v}$ is then (locally) defined by $T_\mathbf{v}(\mathbf{e}) := T_{\mathbf{p}(\mathbf{e})}\boldsymbol{\sigma}_\mathbf{e} \cdot \mathbf{v}_{\mathbf{p}(\mathbf{e})}$ and gives a vector field since $\boldsymbol{\tau}_\mathbb{E}(T_{\mathbf{p}(\mathbf{e})}\boldsymbol{\sigma}_\mathbf{e} \cdot \mathbf{v}_{\mathbf{p}(\mathbf{e})}) = \mathbf{e}$ for all $\mathbf{e} \in \mathbb{E}$. Moreover this extension projects on $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{TM})$ since

$$T_{\mathbf{p}(\mathbf{e})}\mathbf{p} \cdot T_\mathbf{v}(\mathbf{e}) = T_{\mathbf{p}(\mathbf{e})}\mathbf{p} \cdot T_{\mathbf{p}(\mathbf{e})}\boldsymbol{\sigma}_\mathbf{e} \cdot \mathbf{v}_{\mathbf{p}(\mathbf{e})} = T_{\mathbf{p}(\mathbf{e})}(\mathbf{p} \circ \boldsymbol{\sigma}_\mathbf{e}) \cdot \mathbf{v}_{\mathbf{p}(\mathbf{e})} = \mathbf{v}_{\mathbf{p}(\mathbf{e})}\,.$$

Being $\boldsymbol{\sigma}_\mathbf{e}(\mathbf{p}(\mathbf{e})) = \mathbf{e}$ the extension $T_\mathbf{v}(\mathbf{e}) := T_{\mathbf{p}(\mathbf{e})}\boldsymbol{\sigma}_\mathbf{e} \cdot \mathbf{v}_{\mathbf{p}(\mathbf{e})}$ may be written as $(T_\mathbf{v} \circ \boldsymbol{\sigma}_\mathbf{e})(\mathbf{p}(\mathbf{e})) = (T\boldsymbol{\sigma}_\mathbf{e} \circ \mathbf{v})(\mathbf{p}(\mathbf{e}))$ which, by surjectivity of $\mathbf{p}$, means that (locally)

$$T_\mathbf{v} \circ \boldsymbol{\sigma}_\mathbf{e} = T\boldsymbol{\sigma}_\mathbf{e} \circ \mathbf{v}\,, \quad \forall\, \mathbf{x} \in \mathbb{M}\,.$$

If $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{E}$ are such that $\boldsymbol{\Sigma}_{\mathbf{e}_1} = \boldsymbol{\Sigma}_{\mathbf{e}_2}$, then $\boldsymbol{\sigma}_{\mathbf{e}_1} = \boldsymbol{\sigma}_{\mathbf{e}_2}$. If $\mathbf{e} \in \mathbf{s}(\mathbb{M})$, the section $\boldsymbol{\sigma}_\mathbf{e} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ is in fact coincident with $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$. ■

Definition 3 (Horizontal lift). *In a bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ *the horizontal lift* $\mathbf{H} \in \mathrm{C}^1(\mathbb{E} \times_\mathbb{M} \mathbb{TM}\,;\mathbb{TE})$ *is a right inverse of* $(\boldsymbol{\tau}_\mathbb{E}\,, T\mathbf{p}) \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E} \times_\mathbb{M} \mathbb{TM})$ *such that the map* $\mathbf{H}_{\mathbf{s}_\mathbf{x}} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{TE})$, *defined by* $\mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{v}_\mathbf{x}) = \mathbf{H}(\mathbf{s}_\mathbf{x}\,, \mathbf{v}_\mathbf{x})$ *for all* $\mathbf{v}_\mathbf{x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$, *is a linear homomorphism from the tangent bundle* $\boldsymbol{\tau}_\mathbb{M} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{M})$ *to the tangent bundle* $\boldsymbol{\tau}_\mathbb{E} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{E})$, *i.e:*

$$(\boldsymbol{\tau}_\mathbb{E}\,, T\mathbf{p}) \circ \mathbf{H} = \mathbf{id}_{\mathbb{E} \times_\mathbb{M} \mathbb{TM}}\,,$$

$$\mathbf{H}_{\mathbf{s}_\mathbf{x}}(\alpha\,\mathbf{u}_\mathbf{x} + \beta\,\mathbf{v}_\mathbf{x}) = \alpha\,\mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{u}_\mathbf{x}) + \beta\,\mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{v}_\mathbf{x}) \in \mathbb{T}_{\mathbf{s}_\mathbf{x}}\mathbb{E}\,,$$

with $\mathbf{s}_\mathbf{x} \in \mathbb{E}_\mathbf{x}$ *and* $\mathbf{u}_\mathbf{x}, \mathbf{v}_\mathbf{x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$ *and* $\alpha, \beta \in \Re$.

Lemma 2.3 (Horizontal lifts and horizontal projectors). *Given a horizontal projector* $\mathrm{P_H} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{TE})$, *the induced horizontal lift is defined by*

$$\mathbf{H}(\mathbf{s}_\mathbf{x}\,, \mathbf{v}_\mathbf{x}) := \mathrm{P_H} \cdot T_\mathbf{x}\mathbf{s} \cdot \mathbf{v}_\mathbf{x} \in \mathbb{H}_{\mathbf{s}_\mathbf{x}}\mathbb{B}\,, \quad \forall\, \mathbf{s}_\mathbf{x} \in \mathbb{E}_\mathbf{x}, \quad \mathbf{v}_\mathbf{x} \in \mathbb{T}_\mathbf{x}\mathbb{M}\,,$$

where $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ *is an arbitrary section extension of* $\mathbf{s}_\mathbf{x} \in \mathbb{E}_\mathbf{x}$. *Vice versa, a horizontal lift* $\mathbf{H} \in \mathrm{C}^1(\mathbb{E} \times_\mathbb{M} \mathbb{TM}\,;\mathbb{TE})$ *induces a horizontal projector given by* $\mathrm{P_H} := \mathbf{H} \circ (\boldsymbol{\tau}_\mathbb{E}\,, T\mathbf{p})$.

Proof. The former formula yields a horizontal lift since:

$$((\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) \circ \mathbf{H})(\mathbf{s_x}\,, \mathbf{v_x}) = (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) \cdot \mathrm{P_H} \cdot T_{\mathbf{x}}\mathbf{s} \cdot \mathbf{v_x} = (\mathbf{s_x}\,, \mathbf{v_x})\,,$$

and the latter formula yields a horizontal projector because the homomorphism $\mathrm{P_H} := \mathbf{H} \circ (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p})$ is idempotent by $\mathrm{P_H} \circ \mathrm{P_H} = \mathbf{H} \circ (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) \circ \mathbf{H} \circ (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) = \mathbf{H} \circ \mathbf{id}_{\mathbb{E}\times_{\mathbb{M}}\mathbb{TM}} \circ (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) = \mathrm{P_H}$ and horizontal by the identity $((\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) \circ \mathrm{P_H})(\mathbf{X}) = ((\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}) \circ \mathbf{H} \circ (\boldsymbol{\tau}_{\mathbb{E}}\,, T\mathbf{p}))(\mathbf{X}) = (\boldsymbol{\tau}_{\mathbb{E}}(\mathbf{X})\,, T\mathbf{p}(\mathbf{X}))\,.$ ■

Definition 4 (Covariant derivative). *The covariant derivative is the vertical component of the natural derivative:*

$$\overline{\nabla}_{\mathbf{v}}\mathbf{s} := \mathrm{P_V} \circ T_{\mathbf{v}} \circ \mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{VE})\,.$$

Setting $\mathbf{Hs} = \mathrm{P_H} \circ T\mathbf{s}$ and $\overline{\nabla}\mathbf{s} = \mathrm{P_V} \circ T\mathbf{s}$, it is $T\mathbf{s} = \overline{\nabla}\mathbf{s} + \mathbf{Hs} \in \mathrm{C}^1(\mathbb{TM}\,; \mathbb{TE})$ and $T_{\mathbf{v}} = \overline{\nabla}_{\mathbf{v}} + \mathbf{H_v} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,; \mathbb{TE})$ with $\overline{\nabla}_{\mathbf{v}} = \mathrm{P_V} \circ T_{\mathbf{v}}$ and $\mathbf{H_v} = \mathrm{P_H} \circ T_{\mathbf{v}}\,.$

Lemma 2.4 (Projectability). *The horizontal lift* $\mathbf{H_v} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,; \mathbb{HE})$ *is* $\mathbf{p}$-*related to the vector field* $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$ *:* $T\mathbf{p} \circ \mathbf{H_v} = \mathbf{v} \circ \mathbf{p} \in \mathrm{C}^0(\mathbf{s}(\mathbb{M})\,; \mathbb{TM})\,.$

Proof. From the decomposition $T_{\mathbf{v}} = \overline{\nabla}\mathbf{v} + \mathbf{H_v} \in \mathrm{C}^1(\mathbb{E}\,; \mathbb{TE})$ it follows that: $T\mathbf{p} \circ T_{\mathbf{v}} = T\mathbf{p} \circ \overline{\nabla}_{\mathbf{v}} + T\mathbf{p} \circ \mathbf{H_v} = T\mathbf{p} \circ \mathbf{H_v}$ being, by definition $T\mathbf{p} \circ \overline{\nabla}_{\mathbf{v}} = 0\,.$ The $\mathbf{p}$-relatedness of $\mathbf{H_v}$ to $\mathbf{v}$ is then inferred from that of $T_{\mathbf{v}}\,.$ ■

Naturality of Lie brackets with respect to relatedness and Lemma 2.4 give:

$$T\mathbf{p} \circ [\mathbf{H_u}, \mathbf{H_v}] = [T\mathbf{p} \circ \mathbf{H_u}, T\mathbf{p} \circ \mathbf{H_v}] = [\mathbf{u} \circ \mathbf{p}, \mathbf{v} \circ \mathbf{p}] = [\mathbf{u}, \mathbf{v}] \circ \mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,; \mathbb{TM})\,.$$

Lemma 2.5 (Injectivity). *The horizontal lift* $\mathbf{Hs} \in \mathrm{C}^1(\mathbb{TM}\,; \mathbb{HE})$*, along a cross section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{E})$ *of a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,; \mathbb{M})$*, is a fibrewise injective homomorphism, i.e.* $\mathbf{H_x s} \in BL(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,; \mathbb{H}_{\mathbf{s(x)}}\mathbb{E})$ *is an injective linear map at each* $\mathbf{x} \in \mathbb{M}\,.$

Proof. We must prove that $\mathbf{ker}(\mathbf{H_x s}) = \{0\}\,.$ We first investigate the linear differential $T_{\mathbf{x}}\mathbf{s} \in BL(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,; \mathbb{T}_{\mathbf{s(x)}}\mathbb{E})\,.$ By the characteristic property of a section, $\mathbf{p}\circ\mathbf{s} = \mathbf{id}_{\mathbb{M}}$ it is: $T_{\mathbf{s(x)}}\mathbf{p}\cdot T_{\mathbf{x}}\mathbf{s}\cdot \mathbf{v_x} = T_{\mathbf{x}}(\mathbf{p}\circ\mathbf{s})\cdot\mathbf{v_x} = \mathbf{v_x}$ for all $\mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}\,.$ It follows that $\mathbf{ker}(T_{\mathbf{x}}\mathbf{s}) = \{0\}$ and $\mathbf{im}(T_{\mathbf{x}}\mathbf{s}) \cap \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p}) = \{0\}\,.$ The injectivity of $T_{\mathbf{x}}\mathbf{s}$ implies that: $\dim \mathbf{im}(T_{\mathbf{x}}\mathbf{s}) = \dim \mathbb{T}_{\mathbf{x}}\mathbb{M}\,.$ Being $T_{\mathbf{x}}\mathbf{s} = \overline{\nabla}_{\mathbf{x}}\mathbf{s} + \mathbf{H_x s}$ with $\mathbf{im}(\overline{\nabla}_{\mathbf{x}}\mathbf{s}) \subseteq \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p})$, we have that $T_{\mathbf{s(x)}}\mathbf{p}\cdot\mathbf{H_x s}\cdot\mathbf{v_x} = T_{\mathbf{s(x)}}\mathbf{p}\cdot T_{\mathbf{x}}\mathbf{s}\cdot\mathbf{v_x} = \mathbf{v_x}$ for all $\mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}\,.$ It follows that $\mathbf{ker}(\mathbf{H_x s}) = \{0\}$ and $\mathbf{im}(\mathbf{H_x s}) \cap \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p}) = \{0\}$ with $\dim \mathbf{im}(\mathbf{H_x s}) = \dim \mathbb{T}_{\mathbf{x}}\mathbb{M}\,.$ ■

Theorem 2.1 (Homomorphism). *The horizontal lift* $\mathbf{Hs} \in \mathrm{C}^1(\mathbb{TM}\,; \mathbb{HE})$ *along a section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{E})$ *of a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,; \mathbb{M})$ *is a vector bundle homomorphism between the bundle* $\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{TM}\,; \mathbb{M})$ *and the pull-back bundle* $\mathbf{s}{\downarrow}\boldsymbol{\tau}_{\mathbb{E}} \in \mathrm{C}^1(\mathbf{s}{\downarrow}\mathbb{HE}\,; \mathbb{M})$ *which is fibrewise invertible and tensorial in* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{E})\,.$

Proof. Let $\dim \mathbb{M} = \dim \mathbb{T}_{\mathbf{x}}\mathbb{M} = m$ and $\dim \mathbb{F} = f$ where $\mathbb{F}$ is the typical fibre. Then $\dim \mathbb{E} = \dim \mathbb{T}_{\mathbf{s}(\mathbf{x})}\mathbb{E} = m + f$. So that $\dim \mathbb{V}_{\mathbf{s}(\mathbf{x})}\mathbb{E} = f$ and $\dim \mathbb{H}_{\mathbf{s}(\mathbf{x})}\mathbb{E} = m$. By reasons of dimensions the injectivity of $\mathbf{H}_{\mathbf{x}}\mathbf{s} \in BL(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{H}_{\mathbf{s}(\mathbf{x})}\mathbb{E})$ implies then its surjectivity. Moreover let $\overline{\mathbf{s}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ be another section such that $\overline{\mathbf{s}}(\mathbf{x}) = \mathbf{s}(\mathbf{x})$. Then, for any $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$, being $T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}, T_{\mathbf{v}_{\mathbf{x}}}\overline{\mathbf{s}} \in \mathbb{T}_{\mathbf{s}(\mathbf{x})}\mathbb{E}$, we have that $T\mathbf{p} \circ (T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} - T_{\mathbf{v}_{\mathbf{x}}}\overline{\mathbf{s}}) = 0$ and hence that $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} = \mathrm{P}_{\mathrm{H}} \circ T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} = \mathrm{P}_{\mathrm{H}} \circ T_{\mathbf{v}_{\mathbf{x}}}\overline{\mathbf{s}} = \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\overline{\mathbf{s}} \in BL(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{H}_{\mathbf{s}(\mathbf{x})}\mathbb{E})$. To a tangent vector $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ there corresponds a horizontal vector $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} \in \mathbb{H}_{\mathbf{s}(\mathbf{x})}\mathbb{E}$ which depends only on the value of $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ at $\mathbf{x} \in \mathbb{M}$. ■

3. Curvature of a connection

The vertical distribution of a fibre bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ is integrable and the leaves of the induced foliation are the fibres of the bundle. By FROBENIUS theorem [5], [6], integrability of vertical distribution is inferred form the vanishing of the vector-valued *cocurvature* form: $\mathbf{R}^{\mathbf{c}}(\mathbf{X},\mathbf{Y}) := -\mathrm{P}_{\mathrm{H}} \circ [\widehat{\mathrm{P}_{\mathrm{V}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{V}}\mathbf{Y}}] = 0$ for any $\mathbf{X},\mathbf{Y} \in \mathbb{TE}$. Here $(\widehat{\mathrm{P}_{\mathrm{V}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{V}}\mathbf{Y}}) \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{TE})$ is any pair of vector fields extension of the vectors $\mathrm{P}_{\mathrm{V}}\mathbf{X}, \mathrm{P}_{\mathrm{V}}\mathbf{Y} \in \mathbb{TE}$, since tensoriality follows from the $\mathrm{C}^\infty(\mathbb{E}\,;\Re)$-linearity of the cocurvature form. The involutivity condition: $[\widehat{\mathrm{P}_{\mathrm{H}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{H}}\mathbf{Y}}] \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{HE})$, to be imposed for the integrability of the horizontal distribution, is equivalently expressed by the vanishing of the *curvature* defined by [5]:

$$\mathbf{R}(\mathbf{X},\mathbf{Y}) := -\mathrm{P}_{\mathrm{V}} \circ [\widehat{\mathrm{P}_{\mathrm{H}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{H}}\mathbf{Y}}]\,, \quad \forall\, \mathbf{X},\mathbf{Y} \in \mathbb{TE}\,.$$

Again tensoriality follows from the $\mathrm{C}^\infty(\mathbb{E}\,;\Re)$-linearity of the curvature form, as shown below. Let us denote by $\Lambda^k(\mathbb{M}\,;\mathbb{TM})$ the space of tangent-valued k-forms on a manifold $\mathbb{M}$.

Proposition 3.1 (Tensoriality of the curvature). *The curvature of a connection $\mathrm{P}_{\mathrm{V}} \in \Lambda^1(\mathbb{E}\,;\mathbb{TE})$ in a fibre bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ is a vertical-vector valued, horizontal 2-form $\mathbf{R} \in \Lambda^2(\mathbb{E}\,;\mathbb{VE})$, that is a 2-form vanishing on vertical vectors and taking values in the vertical distribution.*

Proof. A direct verification of the tensoriality, based on $\mathrm{C}^\infty(\mathbb{E}\,;\Re)$-linearity, yields the result:

$$\begin{aligned} -\mathbf{R}(\mathbf{X}, f\mathbf{Y}) &:= \mathrm{P}_{\mathrm{V}} \circ [\widehat{\mathrm{P}_{\mathrm{H}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{H}}\mathbf{Y}}] \\ &= f\,\mathrm{P}_{\mathrm{V}} \circ [\widehat{\mathrm{P}_{\mathrm{H}}\mathbf{X}}, \widehat{\mathrm{P}_{\mathrm{H}}\mathbf{Y}}] + (\mathcal{L}_{\mathrm{P}_{\mathrm{H}}\mathbf{X}} f)\,(\mathrm{P}_{\mathrm{V}} \circ \mathrm{P}_{\mathrm{H}})(\mathbf{Y}) \\ &= -f\,\mathbf{R}(\mathbf{X},\mathbf{Y})\,, \quad \forall\, f \in \mathrm{C}^1(\mathbb{E}\,;\Re)\,, \end{aligned}$$

since $\mathrm{P}_{\mathrm{V}} \circ \mathrm{P}_{\mathrm{H}} = 0$. Similarly $\mathbf{R}(f\mathbf{X},\mathbf{Y}) = f\,\mathbf{R}(\mathbf{X},\mathbf{Y})$. ■

Theorem 3.1. *For any given section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$, *the curvature of a connection* $\mathrm{P_V} \in \Lambda^1(\mathbb{E};\mathbb{VE})$ *is expressed by a 2-form* $\textsc{curv}_\mathbf{s} \in \Lambda^2(\mathbb{M};\mathbf{s}{\downarrow}\mathbb{VE})$ *with values in the pull-back of the vertical distribution by the section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$, *defined in terms of horizontal lifts by:*

$$\overline{\textsc{curv}}_\mathbf{s}(\mathbf{u},\mathbf{v}) := \mathbf{R}(\mathbf{H_u},\mathbf{H_v}) \circ \mathbf{s} = (\mathbf{H}_{[\mathbf{u},\mathbf{v}]} - [\mathbf{H_u},\mathbf{H_v}]) \circ \mathbf{s}\,, \quad \forall\, \mathbf{u},\mathbf{v} \in \Lambda^0(\mathbb{M};\mathbb{TM})\,,$$

The 2*-form* $\textsc{curv}_\mathbf{s} \in \Lambda^2(\mathbb{M};\mathbf{s}{\downarrow}\mathbb{VE})$ *is tensorial in* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$.

Proof. We rely on the properties of tensoriality and horizontality of the curvature two-form $\mathbf{R} \in \Lambda^2(\mathbb{E};\mathbb{VE})$ stated in Proposition 3.1 and on the tensorial isomorphism of the horizontal lifts stated in Theorem 2.1. Accordingly, the point value of the curvature $\mathbf{R}(\mathbf{X},\mathbf{Y}) = -\mathrm{P_V} \circ [\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}]$ at $\mathbf{b} \in \mathbb{E}_\mathbf{x}$ depends only on the vectors $\mathrm{P_H}\mathbf{X_b}, \mathrm{P_H}\mathbf{Y_b} \in \mathbb{T}_\mathbf{b}\mathbb{E}$. Moreover, by Theorem 2.1, given any section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ such that $\mathbf{s_x} = \mathbf{b}$, there exists a uniquely determined pair of vectors $\mathbf{u_x},\mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$, such that $\mathbf{H_{u_x}s} = (\mathrm{P_H}\mathbf{X})(\mathbf{s_x})$, $\mathbf{H_{v_x}s} = (\mathrm{P_H}\mathbf{Y})(\mathbf{s_x})$ and the pair $\mathbf{u_x},\mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$ does not depend on the choice of the section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ such that $\mathbf{s_x} = \mathbf{b}$. Then the curvature two-form $\mathbf{R} \in \Lambda^2(\mathbb{E};\mathbb{VE})$, evaluated on pairs of horizontal lifts, defines the field $\textsc{curv}_\mathbf{s}(\mathbf{u},\mathbf{v}) := -\mathrm{P_V} \circ [\mathbf{H_u},\mathbf{H_v}] \circ \mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{VE})$ for any pair of vector fields $\mathbf{u},\mathbf{v} \in \mathrm{C}^0(\mathbb{M};\mathbb{TM})$ on the tangent bundle and any section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ of the fibre bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E};\mathbb{M})$. By tensoriality, for any section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ the field $\textsc{curv}_\mathbf{s} \in \Lambda^2(\mathbb{M};\mathbb{VE})$ is a vector-valued two-form on $\mathbb{M}$ with values in $\mathbf{s}{\downarrow}\mathbb{VE}$ and for any pair $\mathbf{u},\mathbf{v} \in \mathrm{C}^0(\mathbb{M};\mathbb{TM})$ the field $\textsc{curv}(\mathbf{u},\mathbf{v}) \in \Lambda^1(\mathbb{M};\mathbf{s}{\downarrow}\mathbb{VE})$ is a vertical-valued vector field along $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$. Moreover, by Lemma 2.4, the horizontal lifts are projectable and we have the relations:

$$\left.\begin{aligned} T\mathbf{p} \circ [\mathbf{H_u},\mathbf{H_v}] &= [\mathbf{u},\mathbf{v}] \circ \mathbf{p} \\ T\mathbf{p} \circ \mathbf{H}_{[\mathbf{u},\mathbf{v}]} &= [\mathbf{u},\mathbf{v}] \circ \mathbf{p} \end{aligned}\right\} \implies T\mathbf{p} \circ ([\mathbf{H_u},\mathbf{H_v}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]}) = 0\,.$$

Then $\mathbf{H}_{[\mathbf{u},\mathbf{v}]}$ is the horizontal component of $[\mathbf{H_u},\mathbf{H_v}]$ and we get the equality: $[\mathbf{H_u},\mathbf{H_v}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_V} \circ [\mathbf{H_u},\mathbf{H_v}] \iff \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_H} \circ [\mathbf{H_u},\mathbf{H_v}]$. ∎

4. Covariant derivative

Lemma 4.1 (Covariant derivative as Lie derivative). *In a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E};\mathbb{M})$ *with a connection, the covariant derivative may be defined as the generalized* Lie *derivative:*

$$\overline{\nabla}_\mathbf{v}\mathbf{s} = \mathcal{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}_\lambda^{(\mathbf{H_v},\mathbf{v})}{\downarrow}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}_{-\lambda}^{\mathbf{H_v}} \circ \mathbf{s} \circ \mathbf{Fl}_\lambda^{\mathbf{v}}\,.$$

Proof. By Leibniz rule $\mathcal{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} = T\mathbf{s} \circ \mathbf{v} - \mathbf{H_v}\mathbf{s} = T_\mathbf{v}\mathbf{s} - \mathbf{H_v}\mathbf{s}$. Then, being $\mathcal{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{VE})$ and $\mathbf{H_v}\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{HE})$, by uniqueness of the vertical-horizontal split, we get that $\overline{\nabla}_\mathbf{v}\mathbf{s} := \mathrm{P_V} \circ T_\mathbf{v}\mathbf{s} = \mathcal{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s}$. ∎

Definition 5 (Parallel transport). *Let* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ *be a fibre bundle with a connection. The parallel transport* $\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ *of a section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ *along the flow* $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ *is defined by:*

$$\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s} := \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda} \circ \mathbf{s} = (\mathbf{Fl}^{(\mathbf{H}_{\mathbf{v}},\mathbf{v})}_{\lambda}\uparrow\mathbf{s}) \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda}\,,$$

so that $\mathbf{p} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s} = \mathbf{p} \circ \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda} \circ \mathbf{s} = \mathbf{Fl}^{\mathbf{v}}_{\lambda} \circ \mathbf{p} \circ \mathbf{s} = \mathbf{Fl}^{\mathbf{v}}_{\lambda}$ *.*

From the definition of parallel transport and Lemma 4.1 we infer that the covariant derivative and the horizontal lift are given by:

$$\overline{\nabla}_{\mathbf{v}}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{-\lambda} \circ \mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{-\lambda}\Uparrow\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda}\,,$$

$$\mathbf{H}_{\mathbf{v}}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda} \circ \mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s}\,.$$

Since the horizontal lift $\mathbf{H}_{\mathbf{v}}$ is defined pointwise in $\mathbb{M}$, the parallel transport along a curve in $\mathbb{M}$ of a section defined only on that curve is meaningful and so is for the covariant derivative.

Definition 6 (Geodesic). *A curve* $c \in \mathrm{C}^1(I\,;\mathbb{M})$ *in a manifold with a connection is a geodesic if the velocity field of the curve* $\mathbf{v} \in \mathrm{C}^1(I\,;\mathbb{TM})$ *fulfils the condition*

$$\overline{\nabla}_t\mathbf{v} := \partial_{\tau=t}\,c_{t,\tau}\Uparrow\mathbf{v}_{\tau} = 0\,,$$

where $\overline{\nabla}$ *is the covariant derivative, the velocity is given by* $\mathbf{v}_t := \partial_{\tau=t}\,c_{\tau}$ *and* $c_{t,\tau}\Uparrow$ *is the parallel transport from* c_{τ} *to* c_t *along the curve.*

Definition 7 (Spray). *A section* $\mathbf{X} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{TTM})$ *of the tangent bundle* $\boldsymbol{\tau}_{\mathbb{TM}} \in \mathrm{C}^1(\mathbb{TTM}\,;\mathbb{TM})$ *is called a spray if it is also a section of the bundle* $T\boldsymbol{\tau} \in \mathrm{C}^1(\mathbb{TTM}\,;\mathbb{TM})$*, that is if* $T\boldsymbol{\tau} \circ \mathbf{X} = \boldsymbol{\tau}_{\mathbb{TM}} \circ \mathbf{X} = \mathbf{id}_{\mathbb{TM}}$ *.*

Lemma 4.2 (Geodesics and sprays). *Let* $\mathbf{S} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{TTM})$ *be a spray and* $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ *a tangent vector. Then the base curve through* $\mathbf{x} \in \mathbb{M}$ *below the flow line of the spray through a vector* $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ *is a geodesic curve for any connection compatible with the spray, i.e. such that* $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{v}_{\mathbf{x}}) = \mathbf{S}(\mathbf{v}_{\mathbf{x}})$ *.*

Proof. Let $\mathbf{v}_{\mu} = \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\mu,\lambda}(\mathbf{v}_{\mathbf{x}})$ be the flow line of the spray through the vector $\mathbf{v}_{\lambda} = \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$. The projected curve on the base manifold is then $c_{\mu} = (\boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda})(\mathbf{v}_{\mathbf{x}})$, with $c_{\lambda} = \mathbf{x} \in \mathbb{M}$. Its velocity field $\mathbf{v} \in \mathrm{C}^1(I\,;\mathbb{TM})$ is given by

$$\begin{aligned}\mathbf{v}_{\mu} &= \partial_{\xi=\mu}\,c_{\xi}(\mathbf{x}) = \partial_{\xi=\mu}\,(\boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{Fl}^{\mathbf{S}}_{\xi,\lambda})(\mathbf{v}_{\mathbf{x}})\\ &= T\boldsymbol{\tau}_{\mathbb{M}} \cdot \mathbf{S}(\mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda}(\mathbf{v}_{\mathbf{x}})) = \boldsymbol{\pi}_{\mathbb{TM}}(\mathbf{S}(\mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda}(\mathbf{v}_{\mathbf{x}}))) = \mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda}(\mathbf{v}_{\mathbf{x}})\,,\end{aligned}$$

and $\mathbf{S}(\mathbf{v}_{\lambda}) = \partial_{\mu=\lambda}\,\partial_{\xi=\mu}\,c_{\xi}$. Being $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{v}_{\mathbf{x}}) = \mathbf{S}(\mathbf{v}_{\mathbf{x}})$, the formula for the time-covariant derivative yields:

$$\begin{aligned}\overline{\nabla}_{\lambda}\mathbf{v} &= \partial_{\mu=\lambda}\,\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda,\mu}(\mathbf{v}_{\mu}) = \partial_{\mu=\lambda}\,\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda,\mu}(\mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda}\mathbf{v}_{\lambda})\\ &= \partial_{\mu=\lambda}\,(\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda,\mu} \circ \mathbf{Fl}^{\mathbf{S}}_{\mu,\lambda})(\mathbf{v}_{\lambda}) = \mathbf{S}(\mathbf{v}_{\lambda}) - \mathbf{H}_{\mathbf{v}_{\lambda}}(\mathbf{v}_{\lambda}) = 0\,.\end{aligned}$$

Hence the curve $c \in \mathrm{C}^1(I\,;\mathbb{M})$ is a geodesic. ■

A similar proof shows that the base curve through $\mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$ below the tangent-flow line of a spray is the velocity field of a geodesic, in any connection compatible with the spray, and that the velocity field of the base points of the line is a JACOBI field [12].

The next original result is the main contribution of this paper. It provides, in the general context of fibre bundles, the expression of the curvature in terms of covariant derivatives.

Theorem 4.1 (Curvature and covariant derivatives). *For a given section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ *of a fibre bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E};\mathbb{M})$ *and any pair of vector fields* $\mathbf{u},\mathbf{v} \in \mathrm{C}^1(\mathbb{M};\mathbb{TM})$ *, the following identity holds on* $\mathbf{s}(\mathbb{M}) \subset \mathbb{E}$ *:*

$$[\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v}] - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + [\mathbf{H_u}, \mathbf{H_v}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = [\mathbf{H_v}, \overline{\nabla}_\mathbf{u}] + [\overline{\nabla}_\mathbf{v}, \mathbf{H_u}] = 0\,.$$

Accordingly , the vertical-valued curvature two-form $\overline{\mathrm{CURV}}_\mathbf{x}(\mathbf{s})(\mathbf{u},\mathbf{v}) \in \mathbb{V}_{\mathbf{s(x)}}\mathbb{E}$ *is given by*

$$\overline{\mathrm{CURV}}(\mathbf{s})(\mathbf{u},\mathbf{v}) = [\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v}](\mathbf{s}) - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]}(\mathbf{s})\,.$$

Proof. By Lemma 2.1 we know that on $\mathbf{s}(\mathbb{M}) \subset \mathbb{E}$ it is $[T_\mathbf{u}, T_\mathbf{v}] = T_{[\mathbf{u},\mathbf{v}]}$. By performing an extension of the natural derivatives, e.g. by the foliation method envisaged in Lemma 2.2, the covariant derivatives of a section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ are consequently extended to (local) vector fields $\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v} \in \mathrm{C}^1(\mathbb{E};\mathbb{VE})$. Then, being

$$T_\mathbf{u} = \overline{\nabla}_\mathbf{u} + \mathbf{H_u}\,, \qquad T_\mathbf{v} = \overline{\nabla}_\mathbf{v} + \mathbf{H_v}\,, \qquad T_{[\mathbf{u},\mathbf{v}]} = \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathbf{H}_{[\mathbf{u},\mathbf{v}]}\,,$$

by bilinearity of the LIE bracket we get

$$\begin{aligned}[\overline{\nabla}_\mathbf{u} + \mathbf{H_u}, \overline{\nabla}_\mathbf{v} + \mathbf{H_v}] &= [\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v}] + [\mathbf{H_u}, \mathbf{H_v}] + [\overline{\nabla}_\mathbf{u}, \mathbf{H_v}] + [\mathbf{H_u}, \overline{\nabla}_\mathbf{v}] \\ &= \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathbf{H}_{[\mathbf{u},\mathbf{v}]}\,,\end{aligned}$$

which, being $[\mathbf{H_u}, \mathbf{H_v}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_V} \cdot [\mathbf{H_u}, \mathbf{H_v}]$, can be written as:

$$[\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v}] - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathrm{P_V} \cdot [\mathbf{H_u}, \mathbf{H_v}] = [\mathbf{H_v}, \overline{\nabla}_\mathbf{u}] + [\overline{\nabla}_\mathbf{v}, \mathbf{H_u}]\,.$$

The tensoriality of the curvature $\mathrm{P_V} \cdot [\mathbf{H_u}, \mathbf{H_v}]$, as a function of the horizontal lifts $\mathbf{H_u}$ and $\mathbf{H_v}$, has the following implication. Let the local vector fields $\mathcal{F}^\mathbf{x}_\mathbf{u}, \mathcal{F}^\mathbf{x}_\mathbf{v} \in \mathrm{C}^1(\mathbb{E};\mathbb{TE})$ be generated by dragging the vectors $\mathbf{H}_{\mathbf{u_x}}, \mathbf{H}_{\mathbf{v_x}} \in \mathbb{T}_{\mathbf{s_x}}\mathbb{E}$ along the flows of the extended covariant derivatives $\overline{\nabla}_\mathbf{u}, \overline{\nabla}_\mathbf{v} \in \mathrm{C}^1(\mathbb{E};\mathbb{TE})$:

$$\begin{aligned}\mathcal{F}^\mathbf{x}_\mathbf{u} \circ \mathbf{Fl}^{\overline{\nabla}_\mathbf{v}}_\lambda &:= T\mathbf{Fl}^{\overline{\nabla}_\mathbf{v}}_\lambda \circ \mathbf{H}_{\mathbf{u_x}}\,, \\ \mathcal{F}^\mathbf{x}_\mathbf{v} \circ \mathbf{Fl}^{\overline{\nabla}_\mathbf{u}}_\lambda &:= T\mathbf{Fl}^{\overline{\nabla}_\mathbf{u}}_\lambda \circ \mathbf{H}_{\mathbf{v_x}}\,.\end{aligned}$$

By tensoriality, in evaluating the r.h.s. of the previous equality at a point $\mathbf{s(x)} \in \mathbb{E}$, the horizontal lifts $\mathbf{H_u}, \mathbf{H_v} \in \mathrm{C}^1(\mathbb{E};\mathbb{TE})$ can be substituted by the vector fields $\mathcal{F}^\mathbf{x}_\mathbf{u}, \mathcal{F}^\mathbf{x}_\mathbf{v} \in \mathrm{C}^1(\mathbb{E};\mathbb{TE})$. Then, by definition:

$$[\mathcal{F}^\mathbf{x}_\mathbf{v}, \overline{\nabla}_\mathbf{u}]_\mathbf{x} = 0\,, \qquad [\overline{\nabla}_\mathbf{v}, \mathcal{F}^\mathbf{x}_\mathbf{u}]_\mathbf{x} = 0\,,$$

so that

$$[\mathbf{H}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}]_{\mathbf{x}} + [\overline{\nabla}_{\mathbf{v}}, \mathbf{H}_{\mathbf{u}}]_{\mathbf{x}} = [\mathcal{F}^{\mathbf{x}}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}]_{\mathbf{x}} + [\overline{\nabla}_{\mathbf{v}}, \mathcal{F}^{\mathbf{x}}_{\mathbf{u}}]_{\mathbf{x}} = 0\,.$$

The result holds for any extension of the natural derivatives and the formula for the curvature is independent of the extension, since, by tensoriality, it depends only on the values of the covariant derivatives at $\mathbf{s}(\mathbf{x})$. ■

5. Connection on a vector bundle

Let us resume the peculiar properties of linear connections on a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ to infer the relevant special expression of the curvature form.

Definition 8 (Linear connection). *In a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ a connection is linear if the pair made of the horizontal lift $\mathbf{H}_{\mathbf{v}} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{HE})$ and of the vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is a linear vector bundle homomorphism from the vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ to the vector bundle $T\mathbf{p} \in \mathrm{C}^1(\mathbb{TE}\,;\mathbb{TM})$. This means that, given two sections $\mathbf{s}_1, \mathbf{s}_2 \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$, the following property of $\mathbf{p}$-$T\mathbf{p}$-linearity holds:*

$$\begin{cases} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,. \end{cases}$$

Being

$$\begin{cases} T_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ T_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,, \end{cases}$$

the $\mathbf{p}$-$T\mathbf{p}$-linearity of the horizontal lift $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}$ is equivalent to $\mathbf{p}$-$T\mathbf{p}$-linearity of the covariant derivative $\overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}$:

$$\begin{cases} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,. \end{cases}$$

The distinguishing feature with respect to a connection on a general fibre bundle is that, by the identification $\mathbb{VE} \simeq \mathbb{E}$, the covariant derivative $\overline{\nabla}_{\mathbf{v}}\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{VE})$ of a section $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ along a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ may be considered as a section $\nabla_{\mathbf{v}}\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})$ of the vector bundle and the covariant derivative $\nabla_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{E})$ as an operator. The result stated below in proposition 5.1 makes appeal to this identification and is a basic property of the covariant derivative in a linear connection (see e.g. [3], [4]).

Proposition 5.1 (Leibniz rule for the covariant derivative). *In a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$ endowed with a linear connection, the covariant derivative $\nabla_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbb{M})\,;\mathbb{E})$ fulfils* LEIBNIZ *rule:*

$$\nabla_{\mathbf{v}}(f\,\mathbf{s}) = (\nabla_{\mathbf{v}} f)\,\mathbf{s} + f\,(\nabla_{\mathbf{v}}\mathbf{s})\,.$$

Proof. Let us recall from Lemma 4.1 the expression $\overline{\nabla}_{\mathbf{v}}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{H_v}}_{-\lambda}\circ\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda}$. The linearity of the connection implies that the flow $\mathbf{Fl}^{\mathbf{H_v}}_{\lambda}$ is a one parameter family of automorphisms. Then

$$\overline{\nabla}_{\mathbf{v_x}}(f\mathbf{s}) = \partial_{\lambda=0}\, f(\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\mathbf{x}))\cdot\mathbf{Fl}^{\mathbf{H_v}}_{-\lambda}(\mathbf{s}(\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\mathbf{x})))\,.$$

To shorten the expressions we set $\overline{f}_{\lambda} := f(\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\mathbf{x}))$ and $\overline{\mathbf{s}}_{\lambda} := \mathbf{Fl}^{\mathbf{H_v}}_{-\lambda}(\mathbf{s}(\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\mathbf{x})))$ so that

$$\begin{aligned}\overline{\nabla}_{\mathbf{v_x}}(f\mathbf{s}) = \partial_{\lambda=0}\,\overline{f}_{\lambda}\cdot\overline{\mathbf{s}}_{\lambda} &= \lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda}\overline{\mathbf{s}}_{\lambda} - \overline{f}_{\lambda}\overline{\mathbf{s}}_{0} + \overline{f}_{\lambda}\overline{\mathbf{s}}_{0} - \overline{f}_{0}\overline{\mathbf{s}}_{0})\\ &= \lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda}(\overline{\mathbf{s}}_{\lambda} - \overline{\mathbf{s}}_{0})) + \lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda}\overline{\mathbf{s}}_{0} - \overline{f}_{0}\overline{\mathbf{s}}_{0})\,.\end{aligned}$$

Hence, observing that

$$\begin{aligned}&\lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda}(\overline{\mathbf{s}}_{\lambda} - \overline{\mathbf{s}}_{0})) = \overline{f}_{0}\,\partial_{\lambda=0}\,\overline{\mathbf{s}}_{\lambda}\,, = f(\mathbf{x})\,\overline{\nabla}_{\mathbf{v_x}}\mathbf{s}(\mathbf{x})\,,\\ &\lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda}\overline{\mathbf{s}}_{0} - \overline{f}_{0}\overline{\mathbf{s}}_{0}) = \lim_{\lambda\to 0}\lambda^{-1}(\overline{f}_{\lambda} - \overline{f}_{0})\,\overline{\mathbf{s}}_{0} = (\nabla_{\mathbf{v_x}}f)\,\mathbf{s}(\mathbf{x})\,,\end{aligned}$$

the result follows. ■

In a vector bundle $\mathbf{p}\in C^1(\mathbb{E};\mathbb{M})$ the iterated and the second covariant derivatives according to a given connection are meaningful. Hence, for any section $\mathbf{s}\in C^1(\mathbb{M};\mathbb{E})$, the curvature form may be written as

$$\begin{aligned}\text{CURV}_{\mathbf{s}}(\mathbf{u},\mathbf{v}) &= (\nabla_{\mathbf{u}}\circ\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\circ\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]})\,\mathbf{s}\\ &= (\nabla^2_{\mathbf{uv}} - \nabla^2_{\mathbf{vu}} + \nabla_{\text{TORS}(\mathbf{u},\mathbf{v})})\,\mathbf{s}\,,\end{aligned}$$

in terms of the second covariant derivative $\nabla^2_{\mathbf{uv}} := \nabla_{\mathbf{u}}\circ\nabla_{\mathbf{v}} - \nabla_{\nabla_{\mathbf{u}}\mathbf{v}}$ and of the torsion $\text{TORS}(\mathbf{u},\mathbf{v}) := \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{[\mathbf{u},\mathbf{v}]}$ which are both tensor fields. Tensoriality may be proved by relying on Leibniz rule to verify $C^{\infty}(\mathbb{E};\Re)$-linearity.

Remark 5.1. *We underline that on a fibre bundle, in writing the formula:* $\text{CURV}_{\mathbf{s}}(\mathbf{u},\mathbf{v}) = [\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}](\mathbf{s}) - \nabla_{[\mathbf{u},\mathbf{v}]}(\mathbf{s})$*, provided by Theorem 4.1, the term* $[\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}](\mathbf{s})$ *cannot be written as* $(\nabla_{\mathbf{u}}\circ\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\circ\nabla_{\mathbf{u}})\,\mathbf{s}$ *since, being* $\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}\in C^1(\mathbb{E};\mathbb{VE})$*, the compositions* $\nabla_{\mathbf{u}}\circ\nabla_{\mathbf{v}}$ *and* $\nabla_{\mathbf{v}}\circ\nabla_{\mathbf{u}}$*, are not defined, unless the bundle is a vector bundle and the identification* $\mathbb{VE}\simeq\mathbb{E}$ *can be made.*

6. Conclusions

Connections on fibre bundles and their torsion and curvature forms are of primary importance in many basic issues of mathematical physics, as witnessed by a vast number of contributions in literature (see e.g. [11]). The topic has been revisited here with the aim of providing a direct proof to the relation between the integrability condition provided by Frobenius theorem and the expression of the curvature field in terms of covariant derivatives. This result, in the general form provided here, appears to be new, since classical treatments consider the special case of linear connections on vector bundles. Our proof is based on the notion of natural derivative of a section, on a suitable extension, by foliation, to a vector field in the tangent bundle, and on a simple but powerful tensoriality argument.

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