Some Representation Theorem for nonreflexive Banach space ultrapowers under the Continuum Hypothesis.

Piotr WILCZEK

Abstract

In this paper it will be shown that assuming the *Continuum Hypothesis* (CH) every nonreflexive Banach space ultrapower $E^I/\mathcal{U}$ is isometrically isomorphic to the space of continuous, bounded and real-valued functions on the Stone-Cech remainder ω^*. This Representation Theorem will be helpful in proving some facts from geometry and topology of nonreflexive Banach space ultrapowers.

Keywords and phrases. Banach space ultrapowers, Stone-Cech remainder, dual space, extreme points, smooth points, complemented subspaces.

2010 *Mathematical Subject Classification*. $46B08, 46B20, 46B25$.

1. Introduction

Although the ultrapower construction was initially developed within *Model Theory* it exerted the great impact on almost all other branches of mathematics (especially such as *Algebra* and *Set Theory*). In the field rendered as *Banach Space Theory* this construction was introduced in the mid 60s by Bretagnolle, Dacunha-Castelle and Krivine ([4, 5]). Recall that a Banach space E is said to be finite dimensional if and only if its unit ball is compact, i.e., if and only if for every bounded family $(x_i)_{i\in I}$ and for every ultrafilter $\mathcal{U}$ on the set I the so-called $\mathcal{U}-$limit

$$\lim_{i,\mathcal{U}} x_i$$

exists. But if a Banach space E is infinite dimensional, then it is possible to enlarge E to a Banach space $\widehat{E}$ by adjoining to every bounded family $(x_i)_{i\in I}$ in E an element $\widehat{x}\in\widehat{E}$ such that $\|\widehat{x}\| = \lim_{i,\mathcal{U}}\|x_i\|$. This construction is termed *Banach space ultrapower*. Suppose that $(E_i)_{i\in I}$ is an index family of Banach spaces. Then define

$$\ell_\infty(E_i) = \{(x_i) : x_i \in E_i \text{ and } \|(x_i)\|_\infty < \infty\}.$$

It is easily observed that $\ell_\infty(E_i)$ is the Banach space of all bounded families $(x_i)\in\prod_{i\in I} E_i$ endowed with the norm given by $\|(x_i)\|_\infty = \sup_{i\in I}\|x_i\|_{E_i}$. If $\mathcal{U}$

is an ultrafilter on the index set I, then it is always possible to determine $\lim_{i,\mathcal{U}} \|x_i\|_{E_i}$. Then it is seen that $\mathcal{N}((x_i)) = \lim_{i,\mathcal{U}} \|x_i\|_{E_i}$ is a seminorm on $\ell_\infty(E_i)$. Consequently, the kernel of $\mathcal{N}$ is given by

$$\mathcal{N}_\mathcal{U} = \left\{ x = (x_i) \in \ell_\infty(E_i) : \lim_{i,\mathcal{U}} \|x_i\| = 0 \right\}.$$

It follows that $\mathcal{N}_\mathcal{U}$ is a closed ideal in the Banach space $\ell_\infty(E_i)$. Thus it is possible to define the quotient space of the form:

$$\ell_\infty(E_i)/\mathcal{N}_\mathcal{U}.$$

This quotient is said to be the *ultraproduct* of the family of Banach spaces $(E_i)_{i\in I}$. If $E_i = E$ for every $i \in I$, then the space $\ell_\infty(E_i)/\mathcal{U}$ is called the *ultrapower* of E and is symbolized by $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (or by $E^I/\mathcal{U}$). Therefore, it can be observed that - from the model-theoretical point of view - the above construction can be considered as the consequence of eliminating the elements of infinite norm from an ordinary (i.e., algebraic) ultrapower and dividing it by infinitesimal ([1, 3, 4, 5, 10, 11]).

Recall that the famous *Łoś Theorem* - also known as the *Fundamental Theorem on Ultraproducts* - asserting that any first-order formula is true in the ultraproduct $\mathcal{M} = \prod_{i\in I} M_i/\mathcal{U}$ (where $\mathcal{M}$ is any first order structure) if and only if the set of indices i such that the formula is true in M_i is a member of $\mathcal{U}$ can be easily adapted to the case of the so-called *positive bounded formulas* which are more adequate for considering metric structures (cf. Proposition 9.2 in [11]).

If (x_i) is a family in $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$, then let us denote by $(x_i)_\mathcal{U}$ the *equivalence class* of (x_i) in the ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. If E is any Banach space and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is its ultrapower, then the mapping $x \to (x_i)_\mathcal{U}$, where $x_i = x$ for every $i \in I$, constitutes an *isometry* of E into $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. Consequently, it becomes obvious that E is a subspace of $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. The above mentioned isometric embedding generally is not *onto*. On the other hand, the above isometry is surjective if the ultrafilter $\mathcal{U}$ is principal or the space E is finite dimensional ([11]).

Recall that an index family $(x_i)_{i\in I}$ in any topological space converges to the point x with respect to an ultrafilter $\mathcal{U}$, i.e., $\lim_{i,\mathcal{U}} = x$ if for every open set V containing the point x it follows that the set $\{i \in I : x_i \in V\}$ belongs to $\mathcal{U}$.

It is known that structural questions about Banach space ultrapowers are really interesting only when it is assumed that the considered ultrafilter $\mathcal{U}$ is *countably incomplete*, i.e., it is possible to single out a sequence (U_n) of members of $\mathcal{U}$ such that $\bigcap_n U_n = \varnothing$. This requirement holds especially for free ultrafilters on ω ([1, 2, 10]).

A Banach space E is called superreflexive if and only if each ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is reflexive ([1, 10]).

In this study we denote by ω the set of all natural numbers, i.e., $\omega = \{1, 2, ...\}$ and its *Stone-Cech compactification* by $\beta\omega$. Then the so-called *Stone-Cech remainder* of $\beta\omega$, i.e., the space $\beta\omega \backslash \omega$ is denoted by ω^* ([14]). The space $\beta\omega$ can be identified with the set of ultrafilters on ω under the topology generated by the sets of the form $\{F : U \in F\}$ for all sets $U \subseteq \omega$. On the other hand, it should be mentioned that the set ω corresponds to the set of principal ultrafilters and the Stone-Cech remainder ω^* corresponds to the set of free ultrafilters on ω ([14]). From *Set-Theoretical Topology* it is known that the so-called *Parovicenko space* X is identified with a topological space satisfying the following conditions: 1) X is compact and Hausdorff, 2) X has no isolated points, 3) X has the weight $\mathfrak{c}$, 4) every nonempty G_δ subset of X has nonempty interior, 5) every two disjoint open F_σ subsets of X have disjoint closures. In 1963 I. I. Parovičenko proved that assuming the continuum hypothesis (CH) every Parovicenko space X is isomorphic to ω^* ([14]).

Since Banach space ultrapowers were introduced into the field of *Functional Analysis* a considerable numbers of paper employing this methodology can be observed ([1, 3, 4, 5, 10, 11] and papers cited there). But on the other hand, it should be noted that there exists relatively little papers concerning the topological and geometrical structure of these model-theoretical objects. Consequently, it is hoped that the following article will be useful in solving the problems concerning these spaces.

In our studies it will be shown that under CH every Banach space ultrapower can be alternatively represented in the form of the space of continuous, bounded and real-valued functions defined on the Parovicenko space ω^*. Namely, assuming CH the *Representation Theorem* for nonreflexive Banach space ultrapowers will be obtained. This new result ascertains that if CH holds and E is any infinite dimensional nonsuperreflexive Banach space, then the Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is isometrically isomorphic to the space of continuous, bounded and real-valued functions on the Stone-Cech remainder ω^*. The congruence $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \cong C(\omega^*)$ which holds under CH is of central importance in our further studies. This isometric isomorphism will enable to prove that nonreflexive Banach space ultrapowers are never dual spaces, the unit ball of any nonreflexive Banach space ultrapower has an abundance of extreme points and no smooth points. Also the structure of complemented subspaces of $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ will be elucidated.

In this place it should be mentioned that our results are obtained under the assumption that all considered infinite dimensional Banach spaces are nonsuperreflexive and - consequently - their ultrapowers are nonreflexive. It is unknown if the condition of nonsuperreflexivity can be weakened (or modified) in order to formulate our Representation Theorem.

2. Notation

Dual of any Banach space E is denoted by E^*. The *unit ball* of E is symbolized by B_E and the *unit sphere* of E by S_E. Denote by A any convex subset in a Banach space E. Then the *convex hull* of A, denoted by $co(A)$, is identified with the smallest convex set containing A. The *closed convex hull* of A, denoted by $\overline{co}(A)$, is the smallest closed convex set which contains the subset A. If A is convex, then any point $x \in A$ is said to be an *extreme point* of A if whenever $x = \lambda x_1 + (1 - \lambda)x_2$ for $0 < \lambda < 1$, then $x = x_1 = x_2$. Alternatively, the point x is an extreme point of the subset A if $A \backslash \{x\}$ is still convex. Denote by $\partial_e(A)$ the set of all extreme points of the subset A. On the other hand, the *smooth points* of the unit ball of the space $C(T)$, where T is compact and Hausdorff, are identified with the functions $f \in C(T)$ such that $\|f\| = \sup\{|f(t)| : t \in T\} = 1$. This means that these functions peak at some $t_0 \in T$, i. e., $|f(t_0)| = 1 > |f(t)|$ for all $t \in T$ such that $t \neq t_0$. If some function f peaks at isolated points of the space T, then f is said to be the *point of Fréchet differentiability* of the supremum norm on the space $C(T)$ ([2]). The symbol $\cong$ is used in order to denote the relation of isometric isomorphism between Banach spaces.

If E and F are two Banach spaces, then an operator $Q : X \to Y$ is said to be *compact* (*weakly compact*, respectively) if the closure of $Q(B_E)$ is compact (weakly compact, respectively) ([2]).

3. The Representation Theorem for nonreflexive Banach space ultrapowers

Identifying any Tychonoff space T with completely regular and Hausdorff space it can be shown that its Stone-Cech compactification βT can be represented in the following form. Suppose that $C(T)$ is the Banach space of all continuous, bounded and real-valued functions on T with the norm defined by $\|f\| = \sup\{|f(t)| : t \in T\}$ and assume that $B_{C(T)^*}$ denotes the closed unit ball of the dual space $C(T)^*$. If we identify every element $t \in T$ with the evaluation functional $\phi_t \in B_{C(T)^*}$, where $\phi_t(f) = f(t)$ for $f \in C(T)$, then it is possible to represent βT as the weak*-closure of the set $\{\phi_t : t \in T\}$ in $B_{C(T)^*}$ and T can be understood as a dense subset of βT. Consequently, every function $f \in C(T)$ has a unique norm-preserving extension $\widehat{f} \in C(\beta T)$ (cf. [12, 13]). If the set of all natural numbers ω has the discrete topology and E is any Banach space, then $\ell_\infty(E) = C(\omega)$. If we define the restriction mapping $R : C(\beta\omega) \to C(\omega)$ by $R(\widehat{f}) = \widehat{f} \upharpoonright \omega$ for each $\widehat{f} \in C(\beta\omega)$, then it can be concluded that R is a linear isometry of $C(\beta\omega)$ onto $C(\omega)$. Hence, the following proposition can be asserted:

Proposition 1 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and let $\ell_\infty(E)$ be the $\ell_\infty-$sum of countably many copies of E. Then $\ell_\infty(E)$ is isometrically isomorphic to the space $C(\beta\omega)$. Symbolically*

$$\ell_\infty(E) \cong C(\beta\omega).$$

Now suppose that I is the closed ideal in the space $C(\beta\omega)$ consisting of functions which vanish on ω^*, i.e., $I = \left\{\widehat{f} \in C(\beta\omega) : \widehat{f}(t) = 0 \text{ for all } t \in \omega^*\right\}$. Next, assume that the space $c_0(\omega)$ consists of functions in $C(\omega)$ which vanish at infinity, i.e., $c_0(\omega) = \{f \in C(\omega) : \text{ for each } \varepsilon > 0, \ \{t \in \omega : |f(t)| > \varepsilon\} \text{ is finite}\}$. On the other hand, it is known that if $\mathcal{U}$ is a nontrivial ultrafilter on ω and the sequence (x_n) converges to the point x in the topology of the space E, then (x_n) converges to x with respect to $\mathcal{U}$, i.e., $\lim\limits_{\mathcal{U}} x_n = x$. This follows from the simple observation that if V denotes any neighborhood of x, then the set $\{i : x_i \notin V\}$ is finite and the nontriviality of $\mathcal{U}$ implies that the set $\{i : x_i \in V\}$ belongs to $\mathcal{U}$ (cf. Proposition 2.2 in [1]). Basing on this facts it can be claimed that $c_0(\omega) = \mathcal{N}_{\mathcal{U}}$. Also the restriction mapping $R : I \to \mathcal{N}_{\mathcal{U}}$ defines a linear isometry from I onto $\mathcal{N}_{\mathcal{U}}$. Consequently, we arrive at the following proposition:

Proposition 2 (CH). *Let E be any infinite dimensional nonsuperreflexivee Banach space, $N_{\mathcal{U}} = \left\{(x_i) \in \ell_\infty(E) : \lim\limits_{\mathcal{U}} \|x_i\| = 0\right\}$ be the closed ideal in the $\ell_\infty-$sum of countably many copies of E and*
$I = \left\{\widehat{f} \in C(\beta\omega) : \widehat{f}(t) = 0 \text{ for all } t \in \omega^*\right\}$ be the closed ideal in the space $C(\beta\omega)$. Then both ideals are isometrically isomorphic. Symbolically

$$\mathcal{N}_{\mathcal{U}} \cong I.$$

Further, assume that the mapping $\sigma : C(\beta\omega)/I \to C(\omega^*)$ defined by $\sigma\left(\widehat{f} + I\right) = \widehat{f} \restriction \omega^*$ for each function $\widehat{f} \in C(\beta\omega)$ constitutes a linear isometry from $C(\beta\omega)/I$ onto $C(\omega^*)$. Then the following corollary can be easily obtained:

Corollary 3 (CH). *Let E be any infinite dimensional nonsuperreflexivee Banach space and $\ell_\infty(E)/N_{\mathcal{U}}$ be its ultrapower. Then the Banach space ultrapower is isometrically isomorphic to the space of continuous, bounded and real-valued functions defined on the Stone-Cech remainder $C(\omega^*)$, i.e.,*

$$\ell_\infty(E)/\mathcal{N}_{\mathcal{U}} \cong C(\omega^*).$$

Corollary 3 can be regarded as the *Representation Theorem* since it asserts that - under CH- each Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}$ (where E is any infinite dimensional nonsuperreflexive Banach space) can be isometrically isomorphic represented in the form of the space of continuous, bounded and real-valued functions on the Parovicenko space $C(\omega^*)$.

4. Nonreflexive Banach space ultrapowers are never dual spaces

Suppose that T is a compact Hausdorff space. Then $\mathcal{B}$ denotes the $\sigma-$algebra of Borel subsets of T and $rca(T, \mathcal{B})$ is the Banach space of regular and countably additive Borel measures μ on T endowed with bounded variation. This norm is given by the variation μ on T, i.e., $\|\mu\| = |\mu|(T) = \sup \sum_{i=1}^{n} |\mu(A_i)|$ where the supremum ranges over all finite partitions $\{A_1, A_2, ..., A_n\}$ of the space T.

Now, define in $rca(T,\mathcal{B})$ the norm closed proper cone containing positive normal measures. Denote this cone by $N^{+}(T,\mathcal{B})$. The measure μ is said to be normal if $\mu(B)=0$ for each Borel set B which is meager in T. Also assume that $N^{+}(T,\mathcal{B})$ generates the closed ideal in the space $rca\,(T,\mathcal{B})$ which is denoted by $N\,(T,\mathcal{B})$. Every measure μ in the space $rca\,(T,\mathcal{B})$ is supported on the set of the form $S(\mu)=\bigcap\{F\subseteq T : F \text{ is closed and } |\mu|\,(F)=|\mu|\,(T)\}$. Recall that the compact Hausdorff space is said to be *hyperstonian* if T is extremally disconnected and the sum $\bigcup\{S(\mu):\mu\in N^{+}(T,\mathcal{B})\}$ constitutes the dense set in T. A. Grothendieck in ([9]) proved that any compact Hausdorff space T is congruent to a dual space if T is hyperstonian. Namely, the following theorem can be formulated:

Theorem 4 (Grothendieck). *If T is a compact Hausdorff space and X is any Banach space, then $L: C(T)\to X^{*}$ is an isometric isomorphism of $C(T)$ onto X^{*} and $J: X\to X^{**}$ is the canonical embedding, then*

i) T is hyperstonian,

ii) $L^{*}\circ J$ is an isometric isomorphism of X onto $N(T,B)$.

Also in ([6]) the converse of this theorem was proved. Namely, it was demonstrated that if T is hyperstonian, then $N(T,\mathcal{B})^{*}$ is congruent to $C(T)$. These considerations can be easily adapted to the case of Banach space ultrapowers. The following conclusion can be obtained:

Theorem 5. *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}$ be its ultrapower. Then its ultrapower $\ell_{\infty}(E)/N_{\mathcal{U}}$ is not a dual space.*

Proof. From our Representation Theorem for nonreflexive Banach space ultrapowers it follows that $\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}\cong C(\omega^{*})$. From this identification and from the fact that ω^{*} is not extremally disconnected ([14]) it is straightforward to see that the Banach space $\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}$ is not a dual space. $\square$

5. Geometry of nonreflexive Banach space ultrapower $\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}$

5.1 Extreme points of the unit ball $B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}$.

It will be shown that the unit ball $B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}$ (where E is any infinite dimensional nonsuperreflexive Banach space) has an abundance of extreme points. Even it can be proved that $B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}=\overline{co}\left(\partial_{e}\left(B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}\right)\right)$.
it will be demonstrated that every extreme point of the unit ball $B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}$ can be represented as the image (with respect to some quotient mapping) of an extreme point in $B_{\ell_{\infty}(E)}$ (cf. [12]).

Theorem 6 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space. If $q:\ell_{\infty}(E)\to\ell_{\infty}(E)/N_{\mathcal{U}}$ is the quotient mapping, then $\partial_{e}(B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}})=q(\partial_{e}(B_{\ell_{\infty}(E)}))$.*

Proof. From the Representation Theorem it is known that the spaces

$\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}$ and $C(\omega^{*})$ are congruent. Consequently, it must be prove that $\partial_{e}[B_{C(\beta\omega^{*})}]=\pi[\partial_{e}(B_{C(\beta\omega)})]$ (or isometrically isomorphic: $\partial_{e}[B_{\ell_{\infty}(E)/\mathcal{N}_{\mathcal{U}}}]=$

$\pi[\partial_e(B_{\ell_\infty(E)})])$ where $\pi : C(\beta\omega) \to C(\omega^*)$ is identified with the quotient mapping defined by $\pi(f) = f \upharpoonright \omega^*$ for every $f \in C(\beta\omega)$. Recall that if T is a compact Hausdorff space, then $|f(t)| = 1$ for all $t \in T$. Suppose that $\widehat{p}$ is an extreme point of $B_{C(\beta\omega^*)}$. Then $|\widehat{p}(t)| = 1$ for all $t \in \omega^*$. It is possible for each $t \in \omega^*$ to single out an open neighborhood V_t of t in ω^* such that $\widehat{p} \upharpoonright V_t$ is constant. Also it is possible to find out an open neighborhood $\widehat{V_t}$ of t in $\beta\omega$ such that $V_t = \widehat{V_t} \cap (\omega^*)$. From the fact that $\beta\omega = \bigcup \left\{ \widehat{V_t} : t \in \omega^* \} \cup \{\{n\} : n \in \omega \right\}$ is compact it can be deduced that there exists a finite family $\left\{ \widehat{V}_{t_1}, ..., \widehat{V}_{t_k}, \{n_1\}, ..., \{n_j\} \right\}$ covering $\beta\omega$. Also it can be claimed that $n_i \notin \widehat{V}_{t_l}$ for any i, l. Then it is possible to introduce the mapping $p : \beta\omega \to \mathbb{R}$ defined by:

$$
\begin{aligned}
p(t) &= \widehat{p} \upharpoonright V_{t_i} \text{ for } t \in \widehat{V}_{t_i},\ i = 1, ..., k \\
& \quad \text{or} \\
p(t) &= 1 \text{ for } t = n_i,\ i = 1, ..., j.
\end{aligned}
$$

It can be observed that the mapping p is continuous. Namely, if $t_0 \in \omega$, then $\{t\}$ is an open neighborhood of t_0. If $t_0 \in \omega^*$, then $t_0 \in V_{t_i} \subseteq \widehat{V}_{t_i}$ for some i such that $1 \leq i \leq k$. If it is assumed that the net $t_\delta \to t_0$ is in $\beta\omega$, then (t_δ) is eventually in $\widehat{V}_{t_i}$ and eventually

$$p(t_\delta) = \widehat{p} \upharpoonright V_{t_i}(t_\delta) = \widehat{p} \upharpoonright V_{t_i}(t_0) = p(t_0).$$

Therefore, it follows that $p(t_\delta) \to p(t_0)$ and p is continuous at t_0. From the fact that $p : \beta\omega \to \mathbb{R}$ it is obvious that $p \in C(\beta\omega)$ and $|p(t)| = 1$ for all $t \in \beta\omega$ and - consequently - $p \in \partial_e[B_{C(\beta\omega)}]$. It can be also observed that $\widehat{p} = p \upharpoonright \omega^* = \pi(p)$. Consequently, it is straightforward that $\partial_e[B_{C(\beta\omega \setminus \omega)}] \subseteq \pi(\partial_e[B_{C(\beta\omega)}])$. Undoubtedly, this inclusion can be reversed and the theorem is proved. □

Recall that a closed subspace M of a real normed linear space E is said to be *proximinal* in E if for each point $x \in E$ it is possible to find out the point $y \in M$ such that $\|x - y\| = \inf \{\|x - z\| : z \in M\}$. Then it is possible to state the following theorem (cf. [12]):

Theorem 7 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space, $\ell_\infty(E)$ be the $\ell_\infty-$sum of countably many copies of E and $N_{\mathcal{U}} = \left\{ (x_i) \in \ell_\infty(E) : \lim_{\mathcal{U}} \|x_i\| = 0 \right\}$ be the closed ideal in $\ell_\infty(E)$. Then the ideal $N_{\mathcal{U}}$ is proximinal in $\ell_\infty(E)$.*

Proof. Recall that $\mathcal{N}_{\mathcal{U}}$ is isometrically isomorphic to the ideal $I = \{f \in C(\beta\omega) : f(t) = 0$ for all $t \in \omega^*\}$ and $\ell_\infty(E)$ is congruent to the space $C(\beta\omega)$ (i.e., $\mathcal{N}_{\mathcal{U}} \cong I$ and $\ell_\infty(E) \cong C(\beta\omega)$, respectively) (Propositions 1 and 2). Then it must be shown that the closed ideal I is proximinal in the space $C(\beta\omega)$. Suppose that $f \in C(\beta\omega)$ and $F = f \upharpoonright \omega^* \in C(\omega^*)$. From Tietze's Extension Theorem it is possible to find out the function $h \in C(\beta\omega)$ such that $h \upharpoonright \omega^* = F = f \upharpoonright \omega^*$ and $\|h\| = \|F\| = \sup_{t \in \omega^*} |h(t)|$. It is clear that $f - h \in I$ and $\|f + I\| = \|h + I\| = \inf\{\|h - g\| : g \in I\} \leq \|h\| = \|h \upharpoonright \omega^*\| = \|f \upharpoonright \omega^*\|$. Also for any $g \in I$ it follows that $\|h - g\| \geq \|(h-g) \upharpoonright \omega^*\| = \|h \upharpoonright \omega^*\| = \|f \upharpoonright \omega^*\|$. Consequently, $\|f \upharpoonright \omega^*\| = \|f + I\|$. Suppose that $g_0 = f - h$, then $g_0 \in I$ and $\|f - g_0\| = \|h\| = \|f \upharpoonright \omega^*\| = \|f + I\|$ and $\|f - g_0\| = \inf\{\|f - g\| : g \in I\}$. Therefore, I is proximinal in $C(\beta\omega)$ and from the identification of $\mathcal{N}_{\mathcal{U}}$ with I it follows that the ideal $\mathcal{N}_{\mathcal{U}}$ is proximinal in $\ell_\infty(E)$. □

In order to prove our main corollary we are forced to refer to the theorem obtained by Godini (cf. [8]). Namely:

Theorem 8 (Godini). *If E is any real normed linear space, $M \subseteq E$ is a closed subspace and $r : E \to E/M$ is the quotient mapping, then the following conditions are equivalent:*
1/ $r(B_E) = B_{E/M}$,
2/ $r(B_E)$ is closed in E/M,
3/ M is proximinal in E.

Then it is possible to state the following corollary:

Corollary 9 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_\infty(E)/N_{\mathcal{U}}$ be its ultrapower. Then*
$B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}} = \overline{co}\left(\partial_e\left(B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}\right)\right)$.
Proof. It can be immediately seen that $B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}} = S_{B_{\ell_\infty(E)}} = r(\overline{co}(\partial_e(B_{\ell_\infty(E)}))) = \overline{co}(r(\partial_e(B_{\ell_\infty(E)}))) = \overline{co}(\partial_e(B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}))$. □

5.2 Smooth points of the unit ball $B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}$.

In the previous section it was indicated that the quotient mapping $q : \ell_\infty(E) \to \ell_\infty(E)/\mathcal{N}_{\mathcal{U}}$ for any infinite dimensional nonsuperreflexive Banach space E preserves extreme points of the unit ball $B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}$. But this does not hold for smooth points. It can be demonstrated that the unit ball $B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}$ has no smooth points (cf. [12]).

Theorem 10 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_\infty(E)/N_{\mathcal{U}}$ be its ultrapower. Then the unit ball $B_{\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}}$ has no smooth points.*

Proof. From the identification (under CH) of Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_{\mathcal{U}}$ with the space $C(\omega^*)$ it follows that it must be demonstrated that for the function $f \in C(\omega^*)$ such that $\|f\| = 1$ the set $A = \{t \in \omega^* : |f(t)| = \|f\| = 1\}$ is nonempty and contains more than one element. It can be observed that the set $A = \bigcap_{n=1}^{\infty} \left\{t \in \omega^* : |f(t)| > \|f\| - \frac{1}{n}\right\}$ is G_δ subset of ω^*. From ([14]) it is known that A has nonempty interior and - consequently - contains a clopen subset of the Stone-Cech remainder ω^*. From the fact that ω^* has no isolated points it is deducible that $card(A) \geq 2$. Then it is obvious that the function f is not a smooth point of the unit ball $B_{C(\omega^*)}$ and our theorem is proved. □

It should be observed that the smooth points of the unit ball $B_{\ell_\infty(E)} \cong B_{C(\beta\omega)}$ can be identified with the points of Fréchet differentiability of the norm of $\ell_\infty(E)$. It follows from the fact that if $f \in B_{C(\beta\omega)}$, $\|f\| = 1$ and f peaks at $t_0 \in \beta\omega$, then $t_0 \in \omega$ is an isolated point of the Stone-Cech compactification $\beta\omega$.

7. Complemented subspaces of $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$

It is known that any closed subspace M of any Banach space E is said to be *complemented* in E if the Banach space E can be written as a direct sum of M and a closed subspace N of E. Then the projection operator $P : E \to M$ (i.e., the mapping of E onto M along N) is continuous; M and N are termed *complementary subspaces* and E can be written as follows: $E = M \oplus N$. Rosenthal (cf. [7]) showed that if the space T is extremally disconnected and a Banach space E has no copy of the space ℓ_∞, then every bounded linear operator $U : C(T) \to E$ is weakly compact. It will be seen that Rosenthal's result can be easily generalized to the case of nonreflexive Banach space ultrapowers $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (cf. [12, 13]).

Theorem 11 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space. If M is an infinite dimensional complemented subspace of a Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$, then M contains a subspace which is isometrically isomorphic to the $\ell_\infty-$sum of countably many Banach spaces.*

Proof. In this proof it must be demonstrated that if the subspace M contains no $\ell_\infty-$sum of countably many Banach spaces E, then M is finite dimensional. Namely, define the continuous projection operator $P : \ell_\infty(E)/\mathcal{N}_\mathcal{U} \to M$ (i.e., the mapping of $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ onto M along its complement) and the quotient mapping $q : \ell_\infty(E) \to \ell_\infty(E)/\mathcal{N}_\mathcal{U}$. From the above mentioned Rosenthal's result and the facts that $\ell_\infty(E) \cong C(\beta\omega)$ and the space $\beta\omega$ is extremally disconnected it follows that the composition operator $U = P \circ q$ is weakly compact. Consequently, it can be seen that $U(B_{\ell_\infty(E)})$ is relatively weakly compact and from Theorem 3 it is deducible that $P(B_{\ell_\infty(E)/\mathcal{N}_\mathcal{U}}) = U(B_{\ell_\infty(E)})$ is also weakly compact. Hence, we obtain that the projection operator P is weakly compact and - basing on the relation of congruence $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \cong C(\omega^*)$ - it is observed that $P^2 = P$ is compact. Then it is straightforward to see that the subspace M is finite dimensional. $\Box$

Immediately we get the following result (cf. [12, 13]).

Corollary 12 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ be its ultrapower. Then $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ has no infinite dimensional complemented subspaces which are separable or reflexive.*

In the next theorem we are going to show that any nonreflexive Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is isometrically isomorphic to its square $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}$ with an adequate norm (cf. [12, 13]).

Theorem 13 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ be its ultrapower. If the square ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}$ has the norm $\|(x,y)\|_0 = \max(\|x\|, \|y\|)$ where $x, y \in \ell_\infty(E)/\mathcal{N}_\mathcal{U}$, then there exists an isometric isomorphism Z of $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}$ onto $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$.*

Proof. Let A and B be two nonempty disjoint clopen subsets of the Stone-Cech remainder ω^* such that $A \cup B = \omega^*$. Recall that the Parovicenco space ω^*

is totally disconnected and each nonempty clopen subset of ω^* has the following form: $cl_{\beta\omega}D\backslash\omega$ where $D \subseteq \omega$ is infinite. Also each such clopen subset is homeomorphic to ω^*. Then it is possible to find out the following homeomorphisms: $\phi : A \to \omega^*$ and $\phi : B \to \omega^*$. For two functions $f, g \in C(\omega^*)$ define the operator $Q(f, g) = h$ such that

$$\begin{aligned} h(t) &= f(\varphi(t)) \text{ if } t \in A \\ &\text{or} \\ h(t) &= g(\varphi(t)) \text{ if } t \in B. \end{aligned}$$

It is obvious that $h \in C(\omega^*)$, for assume that $t_\delta \to t$ and $t_\delta, t \in \omega^*$. If $t \in A$, then t_δ is eventually in A and the sequence t_δ is convergent to t in A. As the result of this presupposition it is obtained that $\varphi(t_\delta) \to \varphi(t)$ and $h(t_\delta) = (f \circ \varphi)(t_\delta) \to (f \circ \varphi)(t) = h(t)$. Analogously, if $t \in B$, then $h(t_\delta) \to h(t)$. Thus $h \in C(\omega^*)$ and the operator $Q : C(\omega^*) \times C(\omega^*) \to C(\omega^*)$ is linear. Now we want to show that Q is onto. Namely, for $h \in C(\omega^*)$ suppose that $f = h \circ \varphi^{-1}$ and $g = h \circ \varphi^{-1}$. Then it is obtained that $Q(f, g) = h$ and

$$\begin{aligned} \|Q(f,g)\| &= \sup\{|Q(f,g)(t)| : t \in \beta\omega\backslash\omega\} \\ &= \max\left(\sup\{|f(\varphi(t))| : t \in A\}, \sup\{|g(\varphi(t))| : t \in B\}\right) \\ &= \max\left(\|f\|, \|g\|\right) = \|(f,g)\|_0. \end{aligned}$$

Therefore, it can be easily observed that Q is an isometric isomorphism of $C(\omega^*) \times C(\omega^*)$ onto $C(\omega^*)$. □

From the above theorem it can be proven that any nonreflexive Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ can be represented as a direct sum of two closed subspaces which are isometrically isomorphic to $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (cf. [12, 13]).

Corollary 14 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ be its ultrapower. Then $\ell_\infty(E)/\mathcal{N}_\mathcal{U} = M \oplus N$ where M and N are closed subspaces of this ultrapower. Both subspaces M and N are isometrically isomorphic to the ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$.*

Proof. Suppose that Q is the operator from the proof of the previous theorem mapping $C(\omega^*) \times C(\omega^*)$ onto $C(\omega^*)$. Then these subspaces M and N can be represented as $M = Q(\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \{0\})$ and $N = Q(\{0\} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U})$. Consequently, M and N are isometrically isomorphic to the spaces $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \{0\}$ and $\{0\} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (respectively) and each of these spaces is isometrically isomorphic to the Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. From the fact that M and N are closed subspaces it follows that $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \{0\}$ and $\{0\} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}$ are complementary subspaces. Hence, it can be proved that $\ell_\infty(E)/\mathcal{N}_\mathcal{U} = Q(\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}) = Q(\ell_\infty(E)/\mathcal{N}_\mathcal{U} \times \{0\}) \oplus Q(\{0\} \times \ell_\infty(E)/\mathcal{N}_\mathcal{U}) = M \oplus N$. Therefore, the following formula is obtained $M \cong N \cong \ell_\infty(E)/\mathcal{N}_\mathcal{U}$. □

6. Concluding remarks

It should be stressed that the relation of isometric isomorphism between nonreflexive Banach space ultrapowers and the space of continuous, bounded and real-valued functions on the Parovicenko space (which occurs under CH), i.e., $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \cong C(\omega^*)$ is very helpful in proving theorems about these model-theoretical constructions. It was demonstrated that if the *Continuum Hypothesis* holds and E is any infinite dimensional nonsuperreflexive Banach space, then its ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is always congruent to the space $C(\omega^*)$. This identification allowed to answer several questions concerning the structure of nonreflexive Banach space ultrapowers.

In the future it is planned to show (using the above mentioned Representation Theorem) that all nonreflexive Banach space ultrapowers are (as Banach spaces) *primary*. Consequently, this fact will enable to demonstrate that every nonsuperreflexive Banach space can be represented - from the model-theoretical point of view - in the form of its primary ultrapower.

References

[1] Aksoy, A. G. and Khamsi, M. A., Nonstandard methods in fixed point theory, Springer - Verlag, New York, 1990.

[2] Albiac, F. and Kalton, N. J., Topics in Banach space theory, Springer - Verlag, New York 2006.

[3] Ben Yaacov, I., Berenstein, A., Henson, C. W., Usvyatsov, A.: Model theory for metric structures, [in:] Chatzidakis, Z., Pillay, A., and Wilkie, A., Model theory with applications to algebra and analysis. Vol. II, Cambridge University Press, Cambridge 2008, $315 - 427$.

[4] Bretagnolle, J., Dacunha-Castelle, D., Krivine, J.-L.: *Lois stables et espaces* L^p, *Ann. Inst. H. Poincaré Sect. B* **2** (1965/1966), $231 - 259$.

[5] Dacunha-Castelle, D., Krivine, J.-L.: Applications des ultraproduits à l'étude des espaces et des algèbres de Banach, *Studia Math.* **41** (1972), $315 - 334$.

[6] Dixmier, J.: Sur certaines espaces considerés par M. H. Stone, *Summa Brasil. Math.* **2** (1951), $151 - 182$.

[7] Diestel, J. and Uhl, J. J., Vector Measures, American Mathematical Society, Providence, 1977.

[8] Godini, G.: Characterizations of proximinal linear subspaces in normed linear spaces, *Rev. Roumaine Math. Pures Appl.* **18** (1973), $901 - 906$.

[9] Grothendieck, A.: Sur les applications lineare faiblement compactes d'espaces du type $C(K)$, *Can. J. Math.* **5** (1953), $129 - 173$.

[10] Heinrich, S.: Ultraproducts in Banach space theory, *J. Reine Angewandte Math.* **313** (1980), $72 - 104$.

[11] Henson, C. W., Iovino, J.: Ultraproducts in analysis, [in:] C. W. Henson, C. W., Iovino, J., Kechris, A. S., and Odell, E. W., Analysis and logic, Cambridge University Press, Cambridge 2003, $1 - 113$.

[12] Leonard, I. E., Whitfield, J. H. M.: A classical Banach space: ℓ_∞/c_0, *Rocky Mountain J. Math.* **13** $(1983), 531 - 539$.

[13] Matafune, G.: On the space ℓ_∞/c_0, *Rocky Mountain J. Math.* **17** (1987), $583 - 586$.

[14] Walker, R. C., The Stone-Cech compactification, Springer - Verlag, New York, 1974.

Piotr WILCZEK, Foundational Studies Center, ul. Na Skarpie 99/24, 61-163 Poznań, POLAND, edwil@mail.icpnet.pl