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Explicit Analytical Solutions of Radial Permeable Power Rate Flow

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Abstract: The research of different kinds of permeable non-Newtonian fluid flow is increasing day by day owing to the development of science, technology and production modes. It is most common to use power rate equation to describe such flows. However, this equation is nonlinear and very difficult to derive explicit exact analytical solutions. Generally, people can only derive approximate solutions with numerical methods. Recently, an advanced separating variables method which can derive exact analytical solutions easier is developed by Academician CAI Ruixian (the method of separating variables with addition). It is assumed that the unknown variable may be indicated as the sum of one-dimensional functions rather than the product in the common method of separating variables. Such method is used to solve the radial permeable power rate flow unsteady nonlinear equations on account of making the process simple. Four concise (no special functions and infinite series) exact analytical solutions is derived with the new method about this flow to develop the theory of non-Newtonian permeable fluid, which are exponential solution, two-dimensional function with time and radius, logarithmic solution, and double logarithmic solution, respectively. In addition, the method of separating variables with addition is developed and applied instead of the conventional multiplication one. It is proven to be promising and encouraging by the deducing. The solutions yielded will be valuable to the theory of the permeable power rate flow and can be used as standard solutions to check numerical methods and their differencing schemes, grid generation ways, etc. They also can be used to verify the accuracy, convergency and stability of the numerical solutions and to develop the numerical computational approaches.

Key words: non-Newtonian fluid, analytical solution, permeable power rate flow, method of separating variables with addition

1 Introduction

The research of different kinds of permeable non-Newtonian fluid flow has been increasing day by day owing to the development of rheology science and technology for almost half a century. It propels the development of corresponding disciplines such as oil industry, polymer chemistry, biofluid mechanics and so on, which are closely connected with the rheology of most non-Newtonian fluid.

To most research fields, which have strict subject content, the early traditional study approach is to derive some basic disciplines by lots of experiments and observations, then to find out possible mathematical equations and their analytical solutions. However, the numerical solutions are used commonly at present due to the difficulties to get analytical solutions and the rapid development of electronic science. But analytical solution still has its own important meaning in theory. In addition, analytical solution is also viewed as the benchmark solution to check the accuracy, convergence and stability of various numerical methods and to improve their differencing schemes, grid generation ways. So far, there have been publishing some papers about analytical solutions of rheology[1–7]. Refs. [1–7] investigate analytical solutions of unsteady rotational flow of viscoelastic fluid and couette flow of generalized second-order fluid. Nevertheless, they all contain infinite series and special functions and are very difficult to apply to practice.

The approximate or numerical solutions are more inconvenient to utilize[8]. Thus, an advanced method which can derive exact analytical solutions easier has been developed by CAI, et al[7, 9–13]. It is assumed that the unknown variable may be indicated as $f(x, y) = X(x)+Y(y)$, i.e., the sum of unknown one-dimensional functions, rather than the product of them, namely, $f(x, y) = X(x) \cdot Y(y)$ in the common method of separating variables. The new way (the method of separating variables with addition) is used in this paper to deduce explicit exact analytical solutions for the permeable power rate flow unsteady nonlinear equations and to develop related subjects.

As mentioned above, the basis structure of this paper is as follows: present the simplified governing equation, derive 4 different analytical solutions with the proposed new method in different cases.

2 Governing Equation

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The classical permeable power rate flow unsteady nonlinear radial equation without simplification (i.e., ignoring square of derivatives, linearizing the nonlinear equation) is expressed as follows[8]:

$$\frac{1}{r}\left(-\frac{\partial p}{\partial r}\right)+c_1\left(\frac{\partial p}{\partial r}\right)\left(-\frac{\partial p}{\partial r}\right)+\frac{1}{n}\left(-\frac{\partial^2 p}{\partial r^2}\right)= -\varphi c\left(\frac{\mu}{k}\right)^{\frac{1}{n}}\left(-\frac{\partial p}{\partial r}\right)^{\frac{n-1}{n}}\frac{\partial p}{\partial t}, \tag{1}$$

where r is radial coordinate; t is time coordinate; p is pressure (MPa); c_1 is liquid compressibility (MPa^{-1}), $c_1=(1/\rho)\bullet(\partial p/\partial\rho)$; n is rheological behavior index; ϕ is porosity; μ is viscosity (Pa • s); k is Newton fluid equivalent permeability (μm^2); c is total compressibility, $c= c_1+ c_\phi$, c_ϕ is pore compressibility (MPa^{-1}).

3 Solution Method

Owing to the complication and nonlinearity of the governing equations, commonly it is difficult to find out analytical solutions, most of which can only be resolved by numerical or linearizing method. The main motivation of deriving analytical solution here is to obtain some possible explicit analytical solutions, but not to find a specified solution for the given boundary conditions. Therefore, the boundary conditions are undetermined before deriving the explicit analytical solution and deduced from the solution afterward[7, 9–13].

Many unprecedented concise analytical solutions can be derived by the new method named method of separating variables with addition. The dependent variable is supposed to be the sum (instead of product in conventional method) of several one-dimensional functions. For example, let unknown function $F(x, y) = f(x)\times g(y)$ change into $F(x, y)= f(x)+g(y)$.

We can use the method of separating variables with addition to present the main equation as

$$p = p(r,t) = R(r) + T(t), \tag{2}$$

where $R(r)$ and $T(t)$ are functions of r and t, respectively.

A derivative equation of Eq. (1) can be obtained by substituting Eq. (2) into Eq. (1). Let $\phi c(\mu/k)^{1/n}$ be constant S (which is practicable in most cases) and n equals arbitrary positive constant, then Eq. (2) is rewritten as follows:

$$\left[\frac{R'}{r}+c_1(R')^2+\frac{R''}{n}\right](-R')^{\frac{1}{n}-1} = k_0 = ST'. \tag{3}$$

According to the method of separating variables with addition, both sides of Eq. (3) are constant, which means that Eq. (3) is a fundamental equation, and we can derive different analytical solutions with different type of Eq. (3). By the way, k_0 equals constant (k_i is also constant in the following equations).

3.1 Analytical solution with k_0=0, c_1=0

In this case, Eq. (3) can be simplified as follows:

$$\left(\frac{R'}{r}+\frac{R''}{n}\right)(-R')^{\frac{1}{n}-1} = 0 = ST'. \tag{4}$$

The value of T' in Eq. (4) can be derived easily as follows:

$$T' = k_1. \tag{5}$$

The meaningful solution of the left side in Eq. (4) can be obtained when $R'/r + R''/n = 0$ (if R'=0, the solution is meaningless owning to obtaining a constant).

The solution of the left side of Eq. (4) can be found by simple integration as follows:

$$R = \frac{k_1}{1-n}r^{1-n}. \tag{6}$$

Then, a general solution can be found as the sum of Eqs. (5) and (6):

$$p = \frac{k_1}{1-n}r^{1-n} + k_1 = k_1\left(1+\frac{r^{1-n}}{1-n}\right). \tag{7}$$

Eq. (7) represents that the pressure is directly proportional to constant k_1 and a simple function of r. Since pressure p should be positive, constant k_1 also has to be a positive number. In addition, n cannot equal 1 here, otherwise, p would be infinite (actually, n=1 is equivalent to Newton shearing condition but not permeable power rate flow).

It can be seen from Eq. (7) that a very simple exponential solution about permeable power rate flow is derived by the new method when S is constant, c_1=0. The solution is feasible in the form of concise analytical solution, but the performance of unsteady flow may not be presented. Nevertheless, it can be emphasized that no concise analytical solution of Eq. (1) is gained without applying the new method. Consequently, the same method will be used in the following derivation processes. The typical curves of Eq. (7) are presented in Fig. 1.

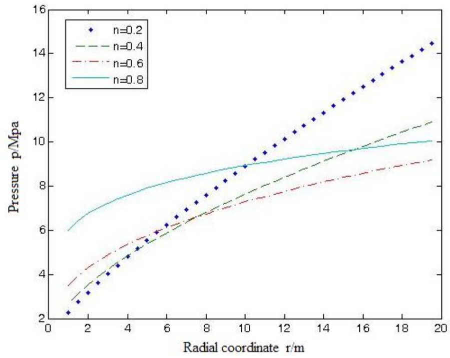

Fig. 1. Variation of pressure with different rheological behavior index

3.2 Analytical solution with unsteady flow

Different from abovementioned solution, the solution of this section is derived with $n=1$, $c_1=0$, but $k_0\neq0$, it can be expressed as follows:

$$R''+\frac{R'}{r}=k_0, \tag{8}$$

$$S_1T'=k_0, \tag{9}$$

where S_1 is the value of S with $n=1$.

Similar to the derivation of former paragraph, a solution can be found by substituting the derivative forms obtained by integrating Eq. (8) and Eq. (9) into Eq. (2) as follows:

$$p=\frac{k_0r^2}{4}+k_1\ln r+\frac{k_0t}{S_1}+k_2. \tag{10}$$

It is evident that the above solution is two-dimensional functions with time and radius. The function of time is linear and the function of radius is the sum of square and logarithmic.

The simple typical curves of Eq. (10) without time function are given in Fig. 2 with $k_0=4$, $k_1=k_2=1$.

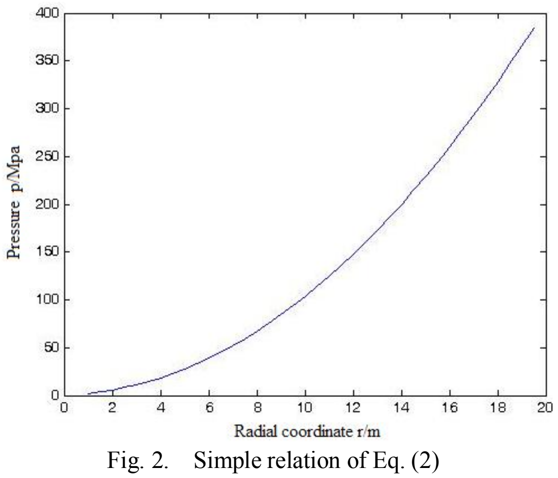

Fig. 2. Simple relation of Eq. (2)

3.3 Analytical solution with $k_0=0$, $c_1\neq0$

Similar to section 3.1, the solution is one-dimensional (T is constant), in which the pressure is function of radius only. The derivative differential equation is as follows:

$$R''+\frac{n}{r}R'+c_1n(R')^2=0. \tag{11}$$

According to the mathematical handbook, a solution of Eq. (11) can be expressed as follows:

$$R=\int\exp\left(-\int\frac{n}{r}\mathrm{d}r\right)\left[\int c_1n\exp\left(-\int\frac{n}{r}\mathrm{d}r\right)\mathrm{d}r+k_1\right]^{-1}\mathrm{d}r+k_2. \tag{12}$$

However, no any simple analytical solution can be found when n is arbitrary positive constant and $k_1\neq0$; but let $k_1=0$, Eq. (12) can be simplified as follows:

$$R=\frac{1-n}{c_1n}\ln r+k_2. \tag{13}$$

As mentioned in the last line of section 3, k_i is arbitrary constant. Therefore, the final solution can be presented as

$$p=\frac{1-n}{c_1n}\ln r+k_2. \tag{14}$$

It is noticeable that the solution in Eq. (14) is a simple solution with logarithm only. However, when positive constant n is equal to 1, it is meaningless for dependent variable p is constant.

The curve of Eq. (14) is presented in Fig. 3.

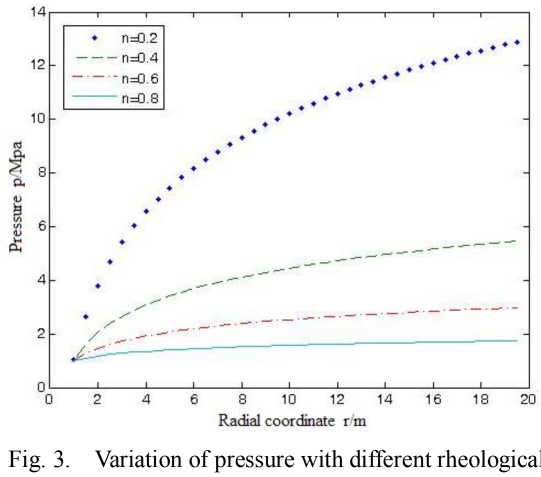

Fig. 3. Variation of pressure with different rheological behavior index

3.4 Special solution when $n=1$, $k_0=0$, $c_1\neq0$

Actually, a special solution can be obtained from Eq. (12) with $n=1$, $k_0=0$ and $c_1\neq0$ that

$$R=\frac{\ln(\ln r)}{c_1}+k_2, \tag{15}$$

then

$$p = \frac{\ln(\ln r)}{c_1} + k_2. \tag{16}$$

Compared with the previous section, the solution in Eq. (16) is an unusual double logarithmic solution when $n=1$, a rather special one. The curve of Eq. (16) is presented in Fig. 4.

4 Conclusions

(1) A governing equation is given to the classical unsteady nonlinear radial permeable power rate flow. Based on this equation, new mathematical method is applied to obtain possible analytical solution.

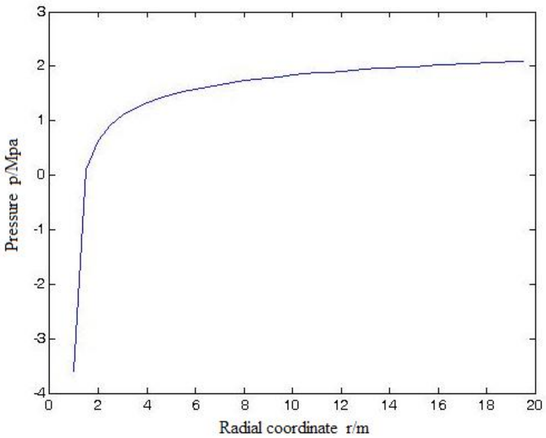

Fig. 4. Simple relation of Eq. (16)

(2) The method of separating variables with addition is developed and applied instead of the conventional multiplication one. It is proven to be effective to resolve some partial differential equations to which no explicit analytical solutions has been obtained so far.

(3) Four simple reasonable analytical solutions for the radial permeable power rate flow are given. The above-mentioned solutions will be valuable to the theory of the permeable power rate flow and can be used as standard benchmark solutions to check numerical methods and their differencing schemes, grid generation ways, etc.

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