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Dynamics Analysis of a Parallel Mill-turn Tool Spindle Head Driven by Dual-Linear Motors Using Extended Transfer Matrix Method

WU Wenjing LIU Qiang*

School of Mechanical Engineering and Automation, Beihang University, Beijing 100191

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Abstract: The hybrid dynamics of multi-rigid-body and multi-flexible-body system becomes the mainstream of multi-body dynamics. Currently there lacks a compact approach to model the hybrid dynamics, especially in modern machine tool application, due to the difficulty of solving the hybrid equations or the limitation of current software when dealing with the hybrid dynamics. The extended transfer matrix method (E-TMM), which extends elements in three-dimensional space with higher matrixes, is proposed to simplify the modeling process of the hybrid dynamics. The E-TMM modeling approaches of 3 basic elements including 3D vibrant rigid body, joint and flexible body are studied in details. A parallel mill-turn tool spindle head unit driven by dual-linear motors is chosen as a plant to demonstrate the E-TMM modeling process. By using E-TMM, the spindle head unit is simplified as a topological network consisting of the three types of element, i.e., 3D vibrant rigid body, joint and flexible body, including 11 rigid bodies, 14 joints and 1 3D-Timoshenko beam. Then the dynamic model of the system can be easily obtained by deducing the element-network by means of state vector transformation. The dynamic characteristics of the spindle head, such as natural frequencies, dynamic flexibility, etc. can be predicted by solving the obtained model. Experiment verification indicates that the E-TMM is valid with enough accuracy in the dynamic analysis of the parallel mill-turn tool spindle head. The E-TMM is capable of modeling the dynamics of machine tool structure with no requirements of deducing and solving the sophisticated differential equations. Moreover the E-TMM provides a simple and elegant tool for hybrid dynamic analysis in future dynamic design of machine tools.

Key words: NC machine tools, dynamics, modeling, transfer matrix method (TMM)

Nomenclature

$\boldsymbol{T}_K$	Transfer matrix of joint element	A	Cross-sectional area of the beam
$\boldsymbol{T}_R$	Transfer matrix of spatial vibrant rigid element	$\boldsymbol{E}_i$	Constant matrix during derivation
$\boldsymbol{T}_{Ti}$	Transfer matrix of 3-D Timoshenko beam element	m_Z	Beam bending moment in Z direction
$\boldsymbol{T}_F$	Transfer matrix of FEM Element with reduced DOF	q_y	Beam shearing force in Y direction
$\boldsymbol{T}_i$	Transfer matrix of element '*i*'.	ρ	Density of the beam
$\boldsymbol{T}_{I,i}$	Transfer matrix of element '*i*' with multiple inputs or outputs on input side.	θ_Z	Angular displacement in Z direction
$\boldsymbol{T}_{O,i}$	Transfer matrix of element '*i*' with multiple inputs or outputs on output side.	I_Z	Cross sectional moment of inertia
$\boldsymbol{Z}_{In}$	State vector of input point	E	Elastic modulus of the beam
$\boldsymbol{Z}_{Out}$	State vector of output point	G	Shear modulus of the beam
$\boldsymbol{Z}_{p,q}$	State vector of connection point between element '*p*' and element '*q*'; The base plate and the element end are regarded as element '*0*'.	$\boldsymbol{P}_{AB}$	The coordinates square matrix of point '*B*' relative to point '*A*'. While '*A*' or '*B*' is written as '$P_{i,j}$', it means the connection point of element '*i*' and '*j*'.
$\boldsymbol{Z}_{I,j}$	Input state vector of element '*i*' with multiple inputs or outputs	GA_S	Shearing rigidity of the beam GAS=GA/κ
$\boldsymbol{Z}_{O,i}$	Output state vector of element '*i*'. with multiple inputs or outputs	κ_s	Shape factor of the beam
$\boldsymbol{K}$	The equivalent stiffness matrix of joint element	$\boldsymbol{I}_{i\times i}$	Identity matrix of $i \times i$
$\boldsymbol{C}$	The equivalent damping matrix of joint element	$\boldsymbol{O}_{i\times j}$	Zero matrix of $i \times j$

1 Introduction

The hybrid dynamics of multi-rigid-body and multi-flexible-body system has become the mainstream of multi-body dynamics study during recent decades[1, 2]. Many applications has been found in the complex mechanical systems. In a machine tool, the elements such as the large-span transverse beam, the links in parallel mechanism are modeled as flexible bodies when the detail analysis is required.

Generally, the overall dynamical equations are implemented to describe the different motions of the hybrid systems. For instance, the ordinary differential equations (ODE) are used for modeling the motions of the rigid-body system, and the partial differential equations (PDE) are used for describing the motions of flexible-bodies and the algebraic equations or ordinary differential equations are used for the boundary conditions[3, 4]. In most cases, such complicated hybrid equations can not be practically applied in the real world. The Finite Element Method (FEM) software has a great advantage in dealing with single component, but it has a great limitation for multi-body systems[5]. ZAEH proposed a new method for simulation of machining performance by integrating finite element and multi-body simulation (MBS) for machine tools[6]. To predict the machining results exactly, large movements on flexible structures have to be calculated. With the specific integration of FEM and MBS for the domain of machine tools it is possible to predict the dynamic machine behavior. The simulation system is based on the relative nodal method for large deformation problems. Some other commercial software packages for multi-body dynamics analysis may be powerful in dealing with multi-body systems, but there is also the limitation when modeling the flexible bodies and joints.

Transfer matrix method (TMM) was first proposed by Prohl[7]. Subsequently, the effects of damping and stiffness of the fluid film bearing were included by Koenig[8], Guenther and Lovejoy[9]. Lund[10] achieved significant advances in the TMM by considering the effects of gyroscopic, internal friction and aerodynamic cross-coupling forces. The TMM, without the requirement of the overall dynamical equations of the system, demonstrates the better modeling flexibility when applied to the 1~2 dimensional dynamic systems such as the rotor, spindle and gear transmission chain .Bansal and Kirk[11] applied the TMM in modal analysis for calculating the damped natural frequencies and examining the stability of flexible rotors mounted on flexible bearing supports. Lund[12] presented a scheme for estimating the sensitivity of the critical speeds of a rotor to change the design factors. LEE[13] used TMM to analyze the steady-state responses of rotor-bearing systems with an unbalancing shaft.

TMM is also easier to program with less calculations so that it has found the successful engineering applications. RUI Xiaoting[14, 15] firstly applied TMM in the gun dynamics analysis.

The TMM uses a mixed form of the element force-displacement relationship and transfers the structural behavior parameters (state array) from one section to the other. Hence, the transfer matrix method produces a system of equations that are simpler in comparison to those produced by the stiffness method. Meanwhile, the TMM still has a great limitation when it applied to the hybrid dynamic systems with multi-dimensional multi-rigid /flexible bodies.

The work in this paper extends the TMM, so called extended Transfer Matrix Method(E-TMM), to meet the requirement for modeling the hybrid dynamic system especially for the machine tools. The paper is organized as follows: in section 2, the extended transfer matrix method is presented. Then, the detailed modeling process of the parallel Mill-turn tool spindle head driven by dual-linear motors is proposed in Section 3. Solution, simulation results and experiment verification are presented in Section 4. Some concluding remarks are given in Section 5.

2 Extended Transfer Matrix Method

The classical TMM utilizes a marching procedure, starting with the boundary conditions at one side of the system, and successively marching along the structure to the other side of the system. This method is particularly suitable for “chain linked” structures such as rotor systems. The state of the rotor system at a specific point is transferred between successive points through transfer matrices. The basic elements in classical TMM are of 1~2 degree of freedom (DOF) generally. Its application in machine tools is mainly concerned with the dynamic analysis of spindle assembly and the torsional vibration analysis of drive system[16].

As for the whole machine tool assembly, its structure is not only a “chain link” but also a complicated “network” especially for parallel/ hybrid machine tool. Meanwhile the actual vibration of the whole machine is always a multi-dimensional issue.

In order to use this method in the dynamic analysis of the whole machine tool assembly, we extend the basic elements with 6 DOF in three-dimensional space.

* Corresponding author. E-mail: qliusmea@buaa.edu.cn
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Each element's mathematical model is a matrix. The dynamic modeling of the mechanical system can be obtained as a higher dimensional matrix by deducing the complicated elements network with the idea of state vector transmission. It is easy to acquire the dynamic characteristics by solving the overall equation with higher dimensional matrix.

The basic elements including three-dimensional vibrant rigid body, joint and flexible body are represented as follows.

2.1 Joint element

In order to model the mechanical system systematically, we use a generic model, as shown in Fig. 1, to represent the typical joints, such as linear guide, bolt connection, bearing, etc. The equivalent dynamic model of the joint element effecting between different bodies is shown in Fig. 1.

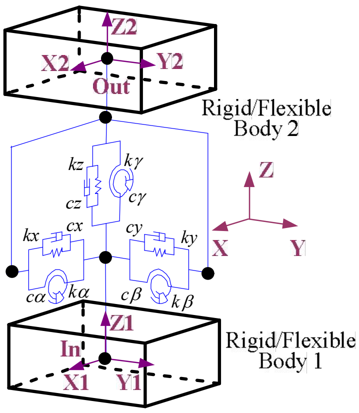

Fig. 1 Definition of joint element

In Fig. 1, the couplings between two bodies are simplified to 3 orthogonal linear-spring-dampers and 3 orthogonal angular-spring-dampers along the X-Y-Z axes respectively. Its action spots are point '*In*' on Body 1 and point '*Out*' on Body 2. In this equivalent dynamic model, all we should do is to set the values of the stiffness and damping of the spring-dampers in different joints. For a ball bearing, the value of one angular spring is near to zero. And for a linear guide, the value of one linear spring is near to zero. The actual joint can also be simplified by one or several joint elements according to your physical demand. The equivalent stiffness and damping in respective direction are

$$\boldsymbol{K} = (kx\ ky\ kz\ k\alpha\ k\beta\ k\gamma)^{\mathrm{T}}, \boldsymbol{C} = (cx\ cy\ cz\ c\alpha\ c\beta\ c\gamma)^{\mathrm{T}}.$$

The state vectors of input point '*In*' and output point '*Out*' in modal coordinate are

$$\boldsymbol{Z}_{\mathrm{In}} = (X\ Y\ Z\ A\ B\ G\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{\mathrm{In}}^{\mathrm{T}}$$

$$\boldsymbol{Z}_{\mathrm{Out}} = (X\ Y\ Z\ A\ B\ G\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{\mathrm{Out}}^{\mathrm{T}}$$

According to the power balance equation, the mathematical model of joint element can be obtained as

$$\boldsymbol{Z}_{Out} = \boldsymbol{T}_K \boldsymbol{Z}_{In} \tag{1}$$

where $\boldsymbol{T}_K$ is the transfer matrix of joint element and

$$\boldsymbol{T}_{\mathrm{K}} = \begin{pmatrix} \boldsymbol{I}_{6\times6} & \boldsymbol{K} \\ \boldsymbol{O}_{6\times6} & \boldsymbol{I}_{6\times6} \end{pmatrix},$$

$$\boldsymbol{K} = \mathrm{diag}(\frac{-1}{kx+\mathrm{j}\omega cx}\ \frac{-1}{ky+\mathrm{j}\omega cy}\ \frac{-1}{kz+\mathrm{j}\omega cz}\ \cdot \frac{-1}{k\alpha+\mathrm{j}\omega c\alpha}\ \frac{-1}{k\beta+\mathrm{j}\omega c\beta}\ \frac{-1}{k\gamma+\mathrm{j}\omega c\gamma})$$

2.2 Spatial vibrant rigid body element

As shown in Fig. 2, it is a spatial vibrant rigid body element with multiple inputs and multiple outputs. Other elements are connected to the rigid body via points 'In1'~'InM' and points 'Out1'~'OutN'. Define the state vectors of input point and output point in modal coordinate as

$$\boldsymbol{Z}_{In} = (X\ Y\ Z\ A\ B\ C\ Fx_1\ Fy_1\ Fz_1\ Tx_1\ Ty_1\ Tz_1 \cdots Fx_i\ Fy_i\ Fz_i\ Tx_i\ Ty_i\ Tz_i \cdots Fx_m\ Fy_m\ Fz_m\ Tx_m\ Ty_m\ Tz_m)_{In}^{T}$$

$$\boldsymbol{Z}_{Out} = (X\ Y\ Z\ A\ B\ C\ Fx_1\ Fy_1\ Fz_1\ Tx_1\ Ty_1\ Tz_1 \cdots Fx_i\ Fy_i\ Fz_i\ Tx_i\ Ty_i\ Tz_i \cdots Fx_n\ Fy_n\ Fz_n\ Tx_n\ Ty_n\ Tz_n)_{Out}^{T}$$

Considering a single input single output element, the state vectors of input point and output point are

$$\boldsymbol{Z}_{In} = (X\ Y\ Z\ A\ B\ C\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{In}^{T}$$

$$\boldsymbol{Z}_{Out} = (X\ Y\ Z\ A\ B\ C\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{Out}^{T}.$$

The mass is m. The inertia matrix is J_I relative to the point '*In1*' in the connected coordinate with point '*In1*' as origin. The coordinates of point '*In1*', '*Out1*' and mass centre '*C*' is shown in Fig. 2. Then the output of the spatial vibrant rigid body element, according to the law of mass centre motion and the law of momentum moment relative to mass center, can be represented as Eq.(2).

$$\boldsymbol{Z}_{Out} = \boldsymbol{T}_R \boldsymbol{Z}_{In} \tag{2}$$

Where

T_R is the transfer matrix of Spatial vibrant rigid body element and

$$T_R = \begin{pmatrix} I_3 & -P_{IO} & O_{3\times3} & O_{3\times3} \\ O_{3\times3} & I_3 & O_{3\times3} & O_{3\times3} \\ m\omega^2 I_3 & -m\omega^2 P_{IC} & I_3 & O_{3\times3} \\ m\omega^2 P_{CO} & -\omega^2(mP_{IO}P_{IC} + J_I) & P_{IO} & I_3 \end{pmatrix}$$

P_{IO} is the coordinates square matrix of output point relative to input point. P_{IC} is the coordinates square matrix of mass centre relative to input point.

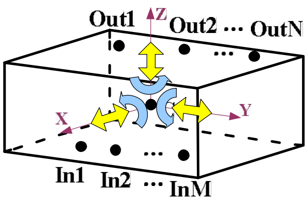

Fig. 2 Definition of Spatial vibrant rigid body element

The model of spatial vibrant rigid body with single input and multiple outputs, multiple inputs and single output, multiple inputs and multiple outputs, can also be acquired by the same method according to your requirement. In those cases, the state vectors of the element may be different.

2.3 Flexible body element

2.3.1 3-D Timoshenko beam

The theory of Timoshenko beam considers the moment of inertia caused by beam's bending deformation and the shearing deformation simultaneously. It greatly improves the previous theories of beam dynamics. When the beam's modal order is not very high and the beam is not slender enough, its precision of dynamic parameters can greatly be improved[17][18]. Thus the Timoshenko beam is always used to model the cutting tool in the dynamic analysis of machine tools.

In this paper, we assume the cutting tool as a Timoshenko beam only with the transverse vibration along X direction, ignoring the longitudinal and torsional vibrations.

As shown in Fig. 3, it is the projection view of a finite Timoshenko beam in the plane of *XY* and *XZ*. Based on the force and torque equilibrium equations, constitutive equations of bent beam and beam deformation equations caused by shearing force, the relationship between Z_{Out} and Z_{In} of 3-D Timoshenko beam can be written as

$$Z_{Out} = T_{Ti} Z_{In} \tag{3}$$

where T_{Ti} is the transfer matrix of 3-D Timoshenko beam element.

The detail derivation of T_{Ti} is shown in the appendix.

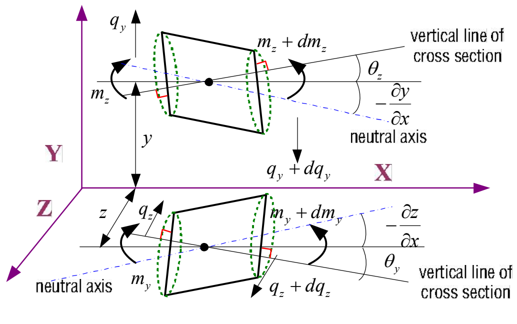

Fig. 3 Forces, moments, and torques acting on a finite Timoshenko beam

2.3.2 FEM Element with reduced DOF

For other flexible parts in machine tools, we use FEM Element with reduced DOF to describe. It has been verified by plenty of experiments that there are only a few lower order modal taking effect in actual mechanism. So when analyzing the dynamic performance of a mechanism, it is enough to get several lower order modals so long as it can satisfy the engineering precision requirement [19].

In order to get the model of FEM DOF reduced element, many existing FEM softwares can be used like ANSYS, Nastran, ect. Choosing different DOF condensation methods like R.J. Guyan static condensation method, P.Mario dynamic condensation method or modal condensation method, according to different requirements, the FEM DOF reduced element model can be easily obtained as a stiffness matrix, mass matrix and damping matrix. In modal coordinate we have

$$(-M\omega^2 + jC\omega + K)Q = F \tag{4}$$

Assuming the input and output state vectors as Z_{In} and Z_{Out}, we have

$$T_F(E_1 Z_{In} + E_2 Z_{Out}) = E_3 Z_{In} + E_4 Z_{Out} \tag{5}$$

where $\boldsymbol{T}_F$ is the transfer matrix of FEM Element with reduced DOF and

$$\boldsymbol{T}_{\mathrm{F}} = -\boldsymbol{M}\omega^2 + \mathrm{j}\boldsymbol{C}\omega + \boldsymbol{K} \,,$$

$\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ and $\boldsymbol{E}_4$ are constant adjusting matrixes.

Thus we get the FEM Element with reduced DOF that is described by main DOF. However, the new model is still a physical model, of which the condensation mass and stiffness are still equivalent physical parameters. Consequently the FEM Element with reduced DOF can be easily applied in E-TMM.

In the modeling process of the following section we mainly use the 3-D Timoshenko beam element, spatial vibrant rigid body element and joint element.

3 Dynamic modeling of spindle head

3.1 Mechanism of spindle head unit

A novel mill-turn tool spindle head driven by dual linear motors, as shown in Fig. 4, is designed for a CNC machine tool as a test bed in lab[20]. It is a functional unit with one translation and one rotation using two asymmetrical parallel linear motors. It is characterized by a fixed platform, a moving platform (the spindle and cutting tool in our case), and two kinematic chains named main kinematic chain and secondary kinematic chain linking between the fixed platform and the moving platform. The main kinematic chain is a PR assembly, which has one prismatic pair and one revolute pair. The secondary kinematic chain is a PRR assembly, which has one prismatic pair implemented by linear guide system and two revolute pairs implemented by knuckle bearings. For its simple kinematic chain, high stiffness, high precision, and controllability, it can be equipped in the turning-milling composite machine tools.

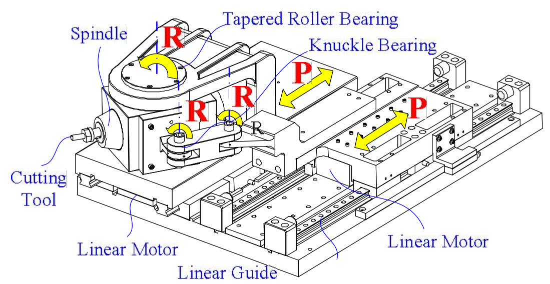

Fig. 4 A novel spindle head driven by dual linear motors

3.2 Elements classification and element network topology

As shown in Fig. 5, elements 2 and 4 are primary sections of linear motors, element 11 is the electric spindle, and element 26 is the tool. Element 11 is fixed with element 8 via a clamp. Element 9 and 10 are installed in element 1 by tapered roller bearings. Both ends of link element 6 are connected with element 5 and element 7 via knuckle bearings. Element 5 is fixed with element 3 by bolts, while Element 2 is fixed with element 1 and element 4 is fixed with 3 by bolted connection. Element 1 and element 3 are connected with element 0 which is the base plate by linear guide systems. Therefore, a simplified model of the mechanism is shown in Figure 6.

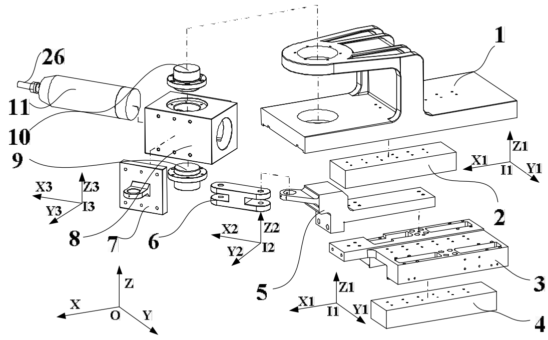

Fig. 5 Moving parts assembly and coordinate systems

It can be clearly detected from Fig.6 that the simplified model of the spindle head is not only a simple “chain linked” structure but also a complicated network with chain, branch and loop included. According to the characteristic of the spindle head, elements 1~11 are 3-D vibrant rigid bodies, elements 12~25 are joints, and element 26 is 3-D Timoshenko beam. The relationship between the connection points of the elements can represented by using a 12×1 state vector, including 6 position variables and 6 force variables.

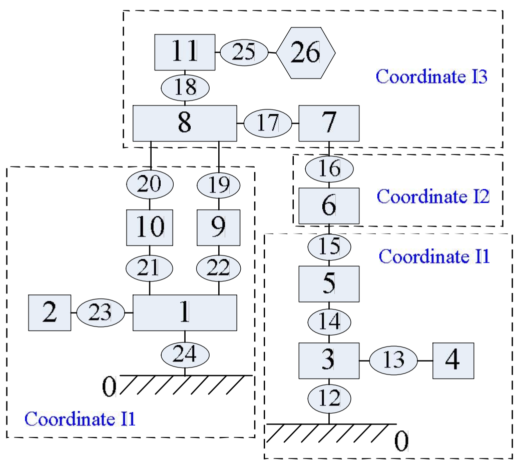

Fig. 6 Topology network of the elements

3.3 Elements state vectors and transfer equations

With regard to the element with one input and one output, the state vector has the same form as

$$\boldsymbol{Z} = (X\ Y\ Z\ A\ B\ C\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)^T$$

As for the element with multi-inputs and multi-outputs, such as element 1, 3 and 8, the state vectors are different.

In Fig.6, it shows clearly that element 3 has two inputs from element 12 and 13, and one output to element 14. Its state vector $\boldsymbol{Z}_{I,3}$ can be written as :

$$\boldsymbol{Z}_{I,3} = \boldsymbol{E}_1\boldsymbol{Z}_{3,12} + \boldsymbol{E}_2\boldsymbol{Z}_{3,13} \tag{6}$$

where $\boldsymbol{E}_1 = \begin{pmatrix} \boldsymbol{I}_{12\times12} \\ \boldsymbol{O}_{6\times12} \end{pmatrix}$,

$$\boldsymbol{E}_2 = \begin{pmatrix} \boldsymbol{O}_{12\times6} & \boldsymbol{O}_{12\times6} \\ \boldsymbol{O}_{6\times6} & \boldsymbol{I}_6 \end{pmatrix}.$$

For element 8, it has three inputs and one output. Define the state vector as

$$\boldsymbol{Z}_{I,8} = \boldsymbol{E}_3\boldsymbol{Z}_{8,17} + \boldsymbol{E}_4\boldsymbol{Z}_{8,19} + \boldsymbol{E}_5\boldsymbol{Z}_{8,20} \tag{7}$$

Where $\boldsymbol{E}_3 = \begin{pmatrix} \boldsymbol{I}_{12\times12} \\ \boldsymbol{O}_{12\times12} \end{pmatrix}$,

$$\boldsymbol{E}_4 = \begin{pmatrix} \boldsymbol{O}_{12\times6} & \boldsymbol{O}_{12\times3} & \boldsymbol{O}_{12\times3} \\ \boldsymbol{O}_{3\times6} & \boldsymbol{R}_{31} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times6} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{31} \\ \boldsymbol{O}_{6\times6} & \boldsymbol{O}_{6\times3} & \boldsymbol{O}_{6\times3} \end{pmatrix},$$

$$\boldsymbol{E}_5 = \begin{pmatrix} \boldsymbol{O}_{18\times6} & \boldsymbol{O}_{18\times3} & \boldsymbol{O}_{18\times3} \\ \boldsymbol{O}_{3\times6} & \boldsymbol{R}_{31} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times6} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{31} \end{pmatrix},$$

$$\boldsymbol{R}_{31} = \begin{pmatrix} \cos\theta_{31} & -\sin\theta_{31} & 0 \\ \sin\theta_{31} & \cos\theta_{31} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For element 1, it has two inputs and two outputs. Define the state vector as

$$\boldsymbol{Z}_{I,1} = \boldsymbol{E}_6\boldsymbol{Z}_{1,24} + \boldsymbol{E}_7\boldsymbol{Z}_{1,23} \tag{8}$$

$$\mathbf{Z}_{O,1} = \mathbf{E}_6\mathbf{Z}_{1,22} + \mathbf{E}_7\mathbf{Z}_{1,21} \tag{9}$$

where $\boldsymbol{E}_6 = \boldsymbol{E}_1$, $\boldsymbol{E}_7 = \boldsymbol{E}_2$.

The transfer equations of each element are as follows:

$$\begin{cases} \boldsymbol{T}_{O,1}\boldsymbol{Z}_{O,1} = \boldsymbol{T}_{I,1}\boldsymbol{Z}_{I,1} \\ \boldsymbol{Z}_{2,23} = \boldsymbol{T}_2\boldsymbol{Z}_{2,0} \\ \boldsymbol{Z}_{3,14} = \boldsymbol{T}_3\boldsymbol{Z}_{I,3} \\ \boldsymbol{Z}_{4,13} = \boldsymbol{T}_4\boldsymbol{Z}_{4,0} \\ \boldsymbol{Z}_{5,15} = \boldsymbol{T}_5\boldsymbol{Z}_{5,14} \\ \boldsymbol{Z}_{6,16} = \boldsymbol{T}_6\boldsymbol{E}_8\boldsymbol{Z}_{6,15} \\ \boldsymbol{Z}_{7,17} = \boldsymbol{T}_7\boldsymbol{E}_9\boldsymbol{Z}_{7,16} \\ \boldsymbol{Z}_{8,18} = \boldsymbol{T}_8\boldsymbol{Z}_{I,8} \\ \boldsymbol{Z}_{9,19} = \boldsymbol{T}_9\boldsymbol{Z}_{9,22} \\ \boldsymbol{Z}_{10,20} = \boldsymbol{T}_{10}\boldsymbol{Z}_{10,21} \\ \boldsymbol{Z}_{11,0} = \boldsymbol{T}_{11}\boldsymbol{Z}_{11,18} \\ \boldsymbol{Z}_{3,12} = \boldsymbol{T}_{12}\boldsymbol{Z}_{12,0} \\ \boldsymbol{Z}_{3,13} = \boldsymbol{T}_{13}\boldsymbol{Z}_{4,13} \end{cases}, \begin{cases} \boldsymbol{Z}_{5,14} = \boldsymbol{T}_{14}\boldsymbol{Z}_{3,14} \\ \boldsymbol{Z}_{6,15} = \boldsymbol{T}_{15}\boldsymbol{Z}_{5,15} \\ \boldsymbol{Z}_{7,16} = \boldsymbol{T}_{16}\boldsymbol{Z}_{6,16} \\ \boldsymbol{Z}_{8,17} = \boldsymbol{T}_{17}\boldsymbol{Z}_{7,17} \\ \boldsymbol{Z}_{11,18} = \boldsymbol{T}_{18}\boldsymbol{Z}_{8,18} \\ \boldsymbol{Z}_{8,19} = \boldsymbol{T}_{19}\boldsymbol{Z}_{9,19} \\ \boldsymbol{Z}_{8,20} = \boldsymbol{T}_{20}\boldsymbol{Z}_{10,20} \\ \boldsymbol{Z}_{10,21} = \boldsymbol{T}_{21}\boldsymbol{Z}_{1,21} \\ \boldsymbol{Z}_{9,22} = \boldsymbol{T}_{22}\boldsymbol{Z}_{1,22} \\ \boldsymbol{Z}_{1,23} = \boldsymbol{T}_{23}\boldsymbol{Z}_{2,23} \\ \boldsymbol{Z}_{1,24} = \boldsymbol{T}_{24}\boldsymbol{Z}_{24,0} \\ \boldsymbol{Z}_{25,26} = \boldsymbol{T}_{25}\boldsymbol{Z}_{11,25} \\ \boldsymbol{Z}_{26,0} = \boldsymbol{T}_{26}\boldsymbol{Z}_{25,26} \end{cases} \tag{10}$$

where $\boldsymbol{T}_{I,1}$ and $\boldsymbol{T}_{O,1}$ are the transfer matrix of rigid body with two inputs and two outputs; $\boldsymbol{T}_3$ is the transfer matrix of rigid body with two inputs and one output; $\boldsymbol{T}_8$ is the transfer matrix of rigid body with three inputs and one output ; $\boldsymbol{T}_2$ $\boldsymbol{T}_4$ $\boldsymbol{T}_5$ $\boldsymbol{T}_6$ $\boldsymbol{T}_7$ $\boldsymbol{T}_9$ $\boldsymbol{T}_{10}$ and $\boldsymbol{T}_{11}$ are the transfer matrixes of rigid bodies with one input and one output; $\boldsymbol{T}_{12}$~ $\boldsymbol{T}_{24}$ are the transfer matrixes of joints; $\boldsymbol{T}_{26}$ is the transfer matrix of 3-D Timoshenko beam.

In Eq.(10), the $\mathbf{E}_8$ and $\mathbf{E}_9$ are defined as

$$\boldsymbol{E}_8 = \begin{pmatrix} \boldsymbol{R}_{21} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{21} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{21} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{21} \end{pmatrix},$$

$$\boldsymbol{E}_9 = \begin{pmatrix} \boldsymbol{R}_{32} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{32} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{32} & \boldsymbol{O}_{3\times3} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_{32} \end{pmatrix},$$

$$\boldsymbol{R}_{21} = \begin{pmatrix} \cos\theta_{21} & -\sin\theta_{21} & 0 \\ \sin\theta_{21} & \cos\theta_{21} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{R}_{32} = \begin{pmatrix} \cos\theta_{32} & -\sin\theta_{32} & 0 \\ \sin\theta_{32} & \cos\theta_{32} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\boldsymbol{R}_{21}$ and $\boldsymbol{R}_{32}$ are the coordinate transformation matrixes.

For elements with multiple inputs and multiple outputs, we have

$$T_{O,1}(E_6Z_{1,22}+E_7Z_{1,21})=T_{I,1}(E_6Z_{1,24}+E_7Z_{1,23}) \quad (11)$$

$$Z_{3,14}=T_3(E_1Z_{3,12}+E_2Z_{3,13}) \quad (12)$$

$$Z_{8,18}=T_8(E_3Z_{8,17}+E_4Z_{8,19}+E_5Z_{8,20}) \quad (13)$$

3.4 Equations of geometrical harmony

For element 3, it has the geometrical relationship as

$$\begin{aligned}&\left(I_{6\times6}\ O_{6\times12}\right)Z_{I,3}=\left(I_{6\times6}\ O_{6\times6}\right)Z_{3,12}\\&=\begin{pmatrix}I_{3\times3} & -P_{P_{3,12}P_{3,13}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\left(I_{6\times6}\ O_{6\times6}\right)Z_{3,13}\end{aligned}$$

We can use Eq.(10) to obtain

$$E_{10}Z_{12,0}=E_{11}Z_{4,0} \quad (14)$$

where $E_{10}=\left(I_{6\times6}\ O_{6\times6}\right)T_{12}$,

$$E_{11}=\begin{pmatrix}I_{3\times3} & -P_{P_{3,12}P_{3,13}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\left(I_{6\times6}\ O_{6\times6}\right)T_{13}T_4$$

In the same way, for element 8 we have

$$E_{12}Z_{1,22}=E_{13}Z_{1,21}=E_{14}Z_{12,0}+E_{15}Z_{4,0} \quad (15)$$

where

$$E_{12}=\begin{pmatrix}I_{3\times3} & -P_{P_{8,17}P_{8,19}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\begin{pmatrix}R_{31} & O_{3\times3} & O_{3\times6}\\ O_{3\times3} & R_{31} & O_{3\times6}\end{pmatrix}T_{19}T_9T_{22},$$

$$E_{13}=\begin{pmatrix}I_{3\times3} & -P_{P_{8,17}P_{8,20}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\begin{pmatrix}R_{31} & O_{3\times3} & O_{3\times6}\\ O_{3\times3} & R_{31} & O_{3\times6}\end{pmatrix}T_{20}T_{10}T_{21},$$

$$E_{14}=\left(I_{6\times6}\ O_{6\times6}\right)T_{17}T_7E_9T_{16}T_6E_8T_{15}T_5T_{14}T_3E_1T_{12}$$

$$E_{15}=\left(I_{6\times6}\ O_{6\times6}\right)T_{17}T_7E_9T_{16}T_6E_8T_{15}T_5T_{14}T_3E_2T_{13}T_4$$

For element 1, we have

$$E_{16}Z_{1,22}=E_{17}Z_{1,21} \quad (16)$$

where $E_{16}=\begin{pmatrix}I_{3\times3} & -P_{P_{1,24}P_{1,22}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\left(I_{6\times6}\ O_{6\times6}\right)$,

$$E_{17}=\begin{pmatrix}I_{3\times3} & -P_{P_{1,24}P_{1,21}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\left(I_{6\times6}\ O_{6\times6}\right)$$

In addition, for the inputs of element 1 we have

$$E_{18}Z_{24,0}=E_{19}Z_{2,0} \quad (17)$$

where $E_{18}=\left(I_{6\times6}\ O_{6\times6}\right)T_{24}$,

$$E_{19}=\begin{pmatrix}I_{3\times3} & -P_{P_{1,24}P_{1,23}}\\ O_{3\times3} & I_{3\times3}\end{pmatrix}\left(I_{6\times6}\ O_{6\times6}\right)T_{23}T_2$$

3.5 Establishment of the system transfer equation

Combining Eq.(6)~(13), we have

$$\begin{cases}U_1Z_{1,22}+U_2Z_{1,21}=U_3Z_{24,0}+U_4Z_{2,0}\\ Z_{26,0}=U_5Z_{12,0}+U_6Z_{4,0}+U_7Z_{1,22}+U_8Z_{1,21}\end{cases} \quad (18)$$

where

$U_1=T_{O,1}E_6$
$U_2=T_{O,1}E_7$
$U_3=T_{I,1}E_6T_{24}$
$U_4=T_{I,1}E_7T_{23}T_2$
$U_5=T_{26}T_{25}T_{11}T_{18}T_8E_3T_{17}T_7E_9T_{16}T_6E_8T_{15}T_5T_{14}T_3E_1T_{12}$
$U_6=T_{26}T_{25}T_{11}T_{18}T_8E_3T_{17}T_7E_9T_{16}T_6E_8T_{15}T_5T_{14}T_3E_2T_{13}T_4$
$U_7=T_{26}T_{25}T_{11}T_{18}T_8E_4T_{19}T_{22}$
$U_8=T_{26}T_{25}T_{11}T_{18}T_8T_5T_{20}T_{10}T_{21}$

Rewrite the equations above into the matrix form, we have

$$\begin{pmatrix}U_1 & U_2\\ U_7 & U_8\end{pmatrix}\begin{pmatrix}Z_{1,22}\\ Z_{1,21}\end{pmatrix}=\begin{pmatrix}U_3 & O_{12\times12}\\ U_4 & O_{12\times12}\\ O_{12\times12} & I_{12\times12}\\ O_{12\times12} & -U_5\\ O_{12\times12} & -U_6\end{pmatrix}^T\begin{pmatrix}Z_{24,0}\\ Z_{2,0}\\ Z_{11,0}\\ Z_{12,0}\\ Z_{4,0}\end{pmatrix} \quad (19)$$

From Eq.(14), we have

$$\begin{pmatrix}Z_{1,22}\\ Z_{1,21}\end{pmatrix}=\begin{pmatrix}E_{20} & E_{25}\\ E_{21} & E_{26}\\ E_{22} & E_{27}\\ E_{23} & E_{28}\\ E_{24} & E_{29}\end{pmatrix}^T\begin{pmatrix}Z_{24,0}\\ Z_{2,0}\\ Z_{11,0}\\ Z_{12,0}\\ Z_{4,0}\end{pmatrix} \quad (20)$$

where

$$\begin{pmatrix} E_{20} & E_{25} \\ E_{21} & E_{26} \\ E_{22} & E_{27} \\ E_{23} & E_{28} \\ E_{24} & E_{29} \end{pmatrix}^T = \begin{pmatrix} U_1 & U_2 \\ U_7 & U_8 \end{pmatrix}^{-1} \begin{pmatrix} U_3 & O_{12\times12} \\ U_4 & O_{12\times12} \\ O_{12\times12} & I_{12\times12} \\ O_{12\times12} & -U_5 \\ O_{12\times12} & -U_6 \end{pmatrix}^T$$

Substituting Eq.(20) into Eq.(15), Eq.(16) and combining Eq.(14), Eq.(17), we have

$$\begin{pmatrix} E_{16}E_{20}-E_{17}E_{25} & E_{16}E_{21}-E_{17}E_{26} & E_{16}E_{22}-E_{17}E_{27} & E_{16}E_{23}-E_{17}E_{28} & E_{16}E_{24}-E_{17}E_{29} \\ E_{12}E_{20}-E_{13}E_{25} & E_{12}E_{21}-E_{13}E_{26} & E_{12}E_{22}-E_{13}E_{27} & E_{12}E_{23}-E_{13}E_{28} & E_{12}E_{24}-E_{13}E_{29} \\ E_{12}E_{20} & E_{12}E_{21} & E_{12}E_{22} & E_{12}E_{23}-E_{14} & E_{12}E_{24}-E_{15} \\ E_{18} & -E_{19} & O_{6\times12} & O_{6\times12} & O_{6\times12} \\ O_{6\times12} & O_{6\times12} & O_{6\times12} & E_{10} & -E_{11} \end{pmatrix} \begin{pmatrix} Z_{24,0} \\ Z_{2,0} \\ Z_{11,0} \\ Z_{12,0} \\ Z_{4,0} \end{pmatrix} = O_{30\times1} \quad (21)$$

Thus, we get the mathematical model of the mechanic system in the form of matrix, as shown in Eq.(21). When substituting the boundary conditions and inputting the parameters of each element into the Eq.(21), it is easy to acquire the dynamic characters of the spindle head.

4 Solution and analysis of spindle head dynamics

4.1 Acquisition of element parameters

The parameters of all elements must be acquired before solving the equation. For rigid body element and Timoshenko beam element, their parameters can be easily acquired in conceptual design stage based on the geometries and materials. While for joint element, its parameters are generally acquired by experimental identification, but when it is difficult to perform the identification before the physical machine is established, the empirical values will be adopted.

As for rigid body, its parameters including mass, inertia matrix, mass centre, input point and output point, can be acquired from 3D model. Table 1 shows the parameters of element 8 calculated from CAD software.

Table 1 parameters of element 8

Mass m/ kg	24.07344289
Inertia matrix J /(kg·m^2)	$\begin{pmatrix} 0.14147845 & -0.00397908 & -0.00000009 \\ -0.00397908 & 0.19613529 & -0.00000012 \\ -0.00000009 & -0.00000012 & 0.21549826 \end{pmatrix}$
Input point P_I/ m	(0,0.085,0)
Mass centre P_C / m	(-0.01598639, 0.01156835, 0)
Output point P_O / m	(0,0,0)

As for Timoshenko beam element, the required parameters include length, density, cross sectional area, second moment of area (Iz, Iy), elastic modulus, shear modulus and shape factor. As shown in Table 2, it is the parameters of Timoshenko beam element 26 in the spindle head unit.

Table 2 Parameters of Timoshenko beam element

length l / m	0.1
density ρ / (kg·m^{-3})	7850
cross sectional area s / m^2	7.854×10^{-5}
second moment of area I_z /m^4	3.14×10^{-8}
second moment of area I_y/m^4	3.14×10^{-8}
elastic modulus E / Pa	2.0×10^{11}
shear modulus G / Pa	8.0×10^{10}
shape factor κ_s	0.9

Fig 7 shows the real part and imaginary part of the normal transfer function from the identification test of the linear guide joint near element 3. The identified parameters are listed in Table 3.

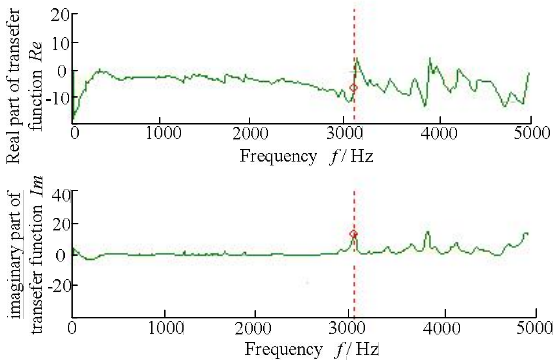

Fig.7 parameter testing of linear guide joint

Table 3 results of linear guide joint testing

Tangential stiff K_Y/ (N·m^{-1})	Normal stiff K_Z/ (N·m^{-1})	Tangential damping C_Y/ (N·s·m^{-1})	Normal damping C_Z/ (N·s·m^{-1})
4.94×10^7	2.63×10^8	115	360

4.2 Simulation results

According to the parameters above together with the boundary conditions, solving the Eq.(21) can easily acquire the plot of dynamic flexibility. As shown in Fig. 8, it is the original dynamic flexibility on cutting tool nose in *Z* direction. The cross dynamic flexibility plots of element 4's driving force (linear motor primary) versus the deformation on tool tip in *X*, *Y*, *Z* directions are illustrated in Fig. 9. The dynamic flexibilities on cutting tool nose are main dynamic characters of machine tools. Using the E-TMM, we can easily obtain the dynamic flexibility of any point in the mechanism versus the tool nose or other point in any directions. And these dynamic characters are worthwhile to determine the machining process optimization of machine tools.

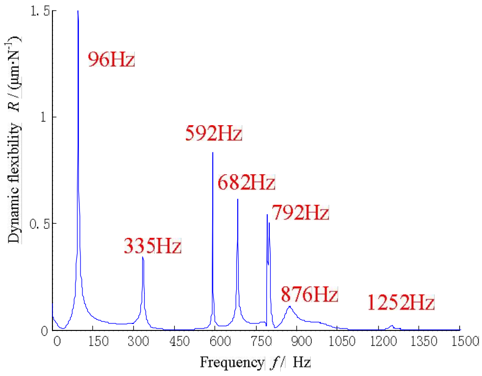

Fig. 8 original dynamic flexibility on tool tip

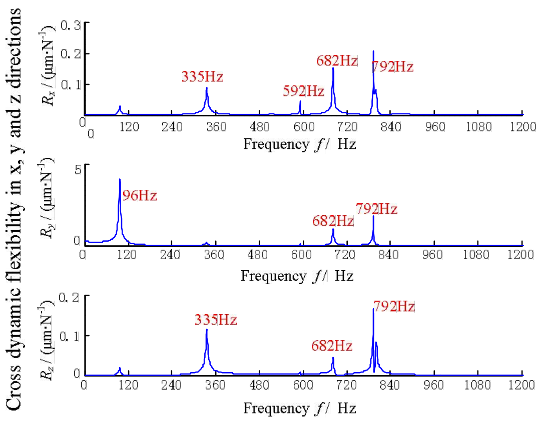

Fig. 9 cross dynamic flexibility on tool tip

In addition, using the Eq.(21) and given boundary conditions, we can also easily acquire the natural frequencies, principal modes of vibration and static deformation of any part in condition of some static forces.

4.3 Experiment verification

In order to verify the validity of this modeling method, some experiments are conducted. As shown in Fig. 10, the hammer test was implemented to make a comparison of natural frequency values between measurement and simulation. The impacting force and the acceleration were collected to calculate the frequency response function (FRF) and the natural frequencies.

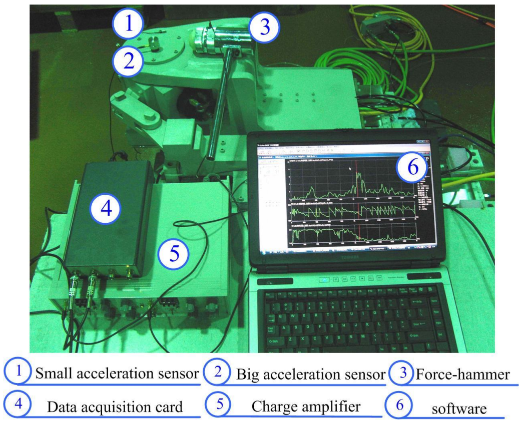

Fig. 10 measurement equipments

As shown in Fig.11, it is the real frequency characteristic and imaginary frequency characteristic of the FRF from the measurement. Fig.11(a) is the FRF when applying the force impact on element 4 in X direction and measuring the acceleration output on element 11 in Z direction. Fig.11 (b) is the FRF of when applying the force impact on element 4 in X direction and measuring the acceleration output on element 26 in Y direction.

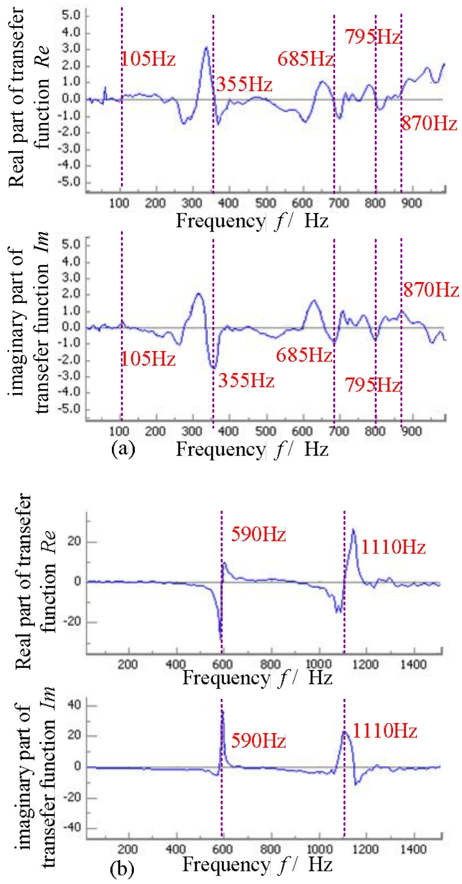

Fig. 11 FRFs of measurement

According to the FRF, the natural frequencies are listed in Table 4. It is indicated that the natural frequencies of simulation and measurement are approximately identical.

Table 4 Natural frequencies Comparison of measurement and simulation

order n	1	2	3	4	5	6	7
natural frequency of simulation f_s/Hz	96	335	592	682	792	876	1252
natural frequency of measurement f_m/Hz	105	355	590	685	795	870	1110
Error e/%	8.6	5.6	0.3	0.4	0.4	0.7	12.8

5 Conclusions

(1) The E-TMM is proposed to model and analyze the complicated mechanisms with hybrid rigid-bodies and flexible-bodies. A dynamic model of a new type spindle head unit with 11 rigid bodies, 14 joints and a Timoshenko beam, is obtained successfully using the E-TMM.

(2) Investigations have been carried out to study the dynamic flexibility on cutting tool tip. Some simulation based on the dynamic model was compared with the experimental tests. The results demonstrate the E-TMM model is valid with adequate accuracy. For further investigation, more characteristics, such as principal modes of vibration and static deformation, can also be predicted based on the dynamic model and some boundary conditions.

(3) The E-TMM proposed in this paper is capable of modeling the dynamics of machine tool structure with no requirements of deducing and solving the sophisticated differential equations. So the E-TMM provides a simple and elegant tool for dynamic analysis in future dynamic design of machine tools, especially in multi-rigid-body and multi-flexible-body system.

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Biographical notes

WU Wenjing, born in 1982, is currently a PhD candidate in School of Mechanical Engineering and Automation, Beihang University, China. His research interests include dynamic analysis and optimal design of modern machine tools.

Tel: +86-10-82316298; E-mail: guaizai_ren@163.com

LIU Qiang, born in 1963, is currently a professor in Beihang University, China. He got his Ph.D from Beihang University in 2000. His research interests include CNC machining process modeling and optimization, dynamic analysis machine tools, high performance motion control, etc.

Tel: +86-10-82316699; E-mail: qliusmea@buaa.edu.cn

Appendix

Detailed derivation of T_{Ti} is as follows.

As shown in Fig. 3，in the plane of XY, basic equations can be obtained as follows

$$\frac{\partial m_z(x,t)}{\partial x} = q_y(x,t) + \rho I_z \frac{\partial^2 \theta_z(x,t)}{\partial t^2} \tag{1}$$

$$\frac{\partial q_y(x,t)}{\partial x} = -\rho A \frac{\partial^2 y(x,t)}{\partial t^2} \tag{2}$$

$$\frac{\partial \theta_z(x,t)}{\partial x} = \frac{m_z(x,t)}{EI_z} \tag{3}$$

$$\frac{\partial y}{\partial x} = \theta_z - \frac{q_y(x,t)}{GA_s} \tag{4}$$

Then the transverse vibration equation in XY plane of the Timoshenko beam can be derived in Eq.(5) based on the equations above.

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI_z}\frac{\partial^2 y}{\partial t^2} - \frac{\rho I_z}{EI_z}\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho A}{GA_s}\left(\frac{\rho I_z}{EI_z}\frac{\partial^4 y}{\partial t^4} - \frac{\partial^4 y}{\partial x^2 \partial t^2}\right) = 0 \tag{5}$$

Displacement of free vibration can be expressed as

$$y(x,t) = Y(x)e^{i\omega t} \tag{6}$$

Substitute Eq.(6) into Eq.(5), we can get

$$\frac{d^4 Y}{dx^4} + \left(\frac{\rho k_s \omega^2}{G} + \frac{\rho\omega^2}{E}\right)\frac{d^2 Y}{dx^2} - \frac{\rho A\omega^2}{EI_z}\left(1 - \frac{\rho I_z \omega^2}{GA_s}\right)Y = 0 \tag{7}$$

The general solution of Eq.(7) is

$$Y(x) = A_1 \cosh\lambda_1 x + A_2 \sinh\lambda_1 x + A_3 \cos\lambda_2 x + A_4 \sin\lambda_2 x$$

where

$$\lambda_{1,2} = \sqrt{\sqrt{\beta + \frac{1}{4}(\sigma-\tau)^2} \mp \frac{1}{2}(\sigma+\tau)},$$

$$\beta = \frac{\rho A\omega^2}{EI_z}, \quad \sigma = \frac{\rho k_s \omega^2}{G}, \quad \tau = \frac{\rho\omega^2}{E}.$$

Considering Eq.(1~4), we get

$$\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_x = \boldsymbol{B}(x)a \tag{8}$$

where

$$\boldsymbol{B}(x) = \begin{pmatrix} \cosh\lambda_1 x & \sinh\lambda_1 x & \cos\lambda_2 x & \sin\lambda_2 x \\ \frac{\sigma+\lambda_1^2}{\lambda_1}\sinh\lambda_1 x & \frac{\sigma+\lambda_1^2}{\lambda_1}\cosh\lambda_1 x & \frac{\sigma-\lambda_2^2}{\lambda_2}\sin\lambda_2 x & -\frac{\sigma-\lambda_2^2}{\lambda_2}\cos\lambda_2 x \\ C_1\cosh\lambda_1 x & C_1\sinh\lambda_1 x & C_2\cos\lambda_2 x & C_2\sin\lambda_2 x \\ \frac{\rho A\omega^2}{\lambda_1}\sinh\lambda_1 x & \frac{\rho A\omega^2}{\lambda_1}\cosh\lambda_1 x & \frac{\rho A\omega^2}{\lambda_2}\sin\lambda_2 x & -\frac{\rho A\omega^2}{\lambda_2}\cos\lambda_2 x \end{pmatrix},$$

$$\boldsymbol{a} = \left(A_1, A_2, A_3, A_4\right)^T, C_1 = EI_z(\sigma+\lambda_1^2), \quad C_2 = EI_z(\sigma-\lambda_2^2).$$

From Eq.(8), we can get

$$\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_0 = \boldsymbol{B}(0)\boldsymbol{a}, \quad \begin{bmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{bmatrix}_l = \mathbf{B}(l)\mathbf{a}$$

That is

$$\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_l = \boldsymbol{B}(l)\boldsymbol{B}^{-1}(0)\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_0$$

Let be

$$\boldsymbol{B}(l)\boldsymbol{B}^{-1}(0) = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix}$$

Then

$$\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_l = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix}\begin{pmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{pmatrix}_0 \tag{9}$$

Similarly in the plane XZ we can get

$$\begin{pmatrix} Z \\ \Theta_y \\ M_y \\ Q_z \end{pmatrix}_l = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}\begin{pmatrix} Z \\ \Theta_y \\ M_y \\ Q_z \end{pmatrix}_0 \tag{10}$$

Based on the assumption in section 2.3, we have

$$\begin{pmatrix} X \\ A \\ T_x \\ F_x \end{pmatrix}_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ A \\ T_x \\ F_x \end{pmatrix}_0 \tag{11}$$

Considering

$$\begin{aligned} \boldsymbol{Z}_{In} &= (X\ Y\ Z\ A\ B\ C\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{In}^T \\ &= (X\ Y\ Z\ A\ \theta_y\ \theta_z\ Fx\ Qy\ Qz\ Tx\ My\ Mz)_0^T \\ \boldsymbol{Z}_{Out} &= (X\ Y\ Z\ A\ B\ C\ Fx\ Fy\ Fz\ Tx\ Ty\ Tz)_{In}^T \\ &= (X\ Y\ Z\ A\ \theta_y\ \theta_z\ Fx\ Qy\ Qz\ Tx\ My\ Mz)_l^T \end{aligned}$$

and Eq.(9~11), we have

$$\boldsymbol{Z}_{Out} = \boldsymbol{T}_{Ti}\boldsymbol{Z}_{In}$$

where

$$\boldsymbol{T}_{Ti}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & u_{11} & 0 & 0 & 0 & u_{12} & 0 & u_{14} & 0 & 0 & 0 & u_{13}\\
0 & 0 & v_{11} & 0 & v_{12} & 0 & 0 & 0 & v_{14} & 0 & v_{13} & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & v_{21} & 0 & v_{22} & 0 & 0 & 0 & v_{24} & 0 & v_{23} & 0\\
0 & u_{21} & 0 & 0 & 0 & u_{22} & 0 & u_{24} & 0 & 0 & 0 & u_{23}\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & u_{41} & 0 & 0 & 0 & u_{42} & 0 & u_{44} & 0 & 0 & 0 & u_{43}\\
0 & 0 & v_{41} & 0 & v_{42} & 0 & 0 & 0 & v_{44} & 0 & v_{43} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & v_{31} & 0 & v_{32} & 0 & 0 & 0 & v_{34} & 0 & v_{33} & 0\\
0 & u_{31} & 0 & 0 & 0 & u_{32} & 0 & u_{34} & 0 & 0 & 0 & u_{33}
\end{pmatrix}$$