

# Quaternion-Octonion $SU(3)$ Flavor Symmetry

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## Abstract

Starting with the quaternionic formulation of isospin  $SU(2)$  group, we have derived the relations for different components of isospin with quark states. Extending this formalism to the case of  $SU(3)$  group we have considered the theory of octonion variables. Accordingly, the octonion splitting of  $SU(3)$  group have been reconsidered and various commutation relations for  $SU(3)$  group and its shift operators are also derived and verified for different iso-spin multiplets i.e.  $I$ ,  $U$  and  $V$ - spins.

**Keywords:**  $SU(3)$ , Quaternions, Octonions and Gell Mann matrices.

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# 1 Introduction

According to celebrated Hurwitz theorem [1] there exists four - division algebra consisting of  $\mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers),  $\mathbb{H}$  (quaternions) and  $\mathcal{O}$  (octonions). All four algebras are alternative with antisymmetric associators. Quaternions were very first example of hyper complex numbers introduced by Hamilton [2, 3] and octonions by Caley [4] and Graves [5] having the significant impacts on mathematics and physics. Quaternions are also described in terms of Pauli spin - matrices for the non - Abelian gauge theory in order to unify [6] electromagnetism and weak forces within the electroweak  $SU(2) \times U(1)$  sector of standard model. Yet another complex system (i.e, Octonion) [7, 8, 9, 10, 11] also plays an important role in understanding the physics beyond the strong interaction between color degree of freedom of quarks and their interaction. A detailed introduction on the various aspects and applications of Exceptional, Jordan, Division, Clifford and non commutative as well as non associative algebras related to octonions has recently been discussed by Castro[12] by extending the octonionic geometry (gravity) developed earlier by Marques-Oliveira [13, 14]. Recently we have also developed [15] the quaternionic formulation of Yang – Mill’s field equations and octonion reformulation of quantum chromo dynamics ( $QCD$ ) where the resemblance between quaternions and  $SU(2)$  and that of octonions and  $SU(3)$  gauge symmetries has been discussed. The color group  $SU(3)_C$  is embedded with in the octonionic structure of the exceptional groups while the  $SU(3)$  flavor group has been discussed in terms of triality property of the octonion algebra. Keeping in view, the utility of quaternions and octonions, in this paper, we have made an attempt to discuss the quaternionic formulation of isospin  $SU(2)$  group and octonion reformulation of  $SU(3)$  flavor group (parallel to the  $SU(3)$  colour group [15]). Accordingly, we have derived the relations for different components of isospin with quark states. Extending this formalism to the case of  $SU(3)$  flavor group, the octonion splitting of  $SU(3)$  group have been reconsidered and various commutation relations for  $SU(3)$  group and its shift operators are also derived and verified in terms of different iso-spin multiplets i.e.  $I$ ,  $U$  and  $V$ - spins.

## 2 Quaternionic representation of isospin $SU(2)$ group

Let us consider the standard Lie algebra in terms of the quaternions by adopting the basic formula of the isotopic formalism for nucleons. Accordingly we may write a spinors  $\psi_a$  and  $\psi_b$  as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}; \quad (1)$$

whereas the adjoint spinor is described as

$$\bar{\psi} = \left( \bar{\psi}_a \quad \bar{\psi}_b \right); \quad (2)$$

where

$$\psi = (\psi_0 + e_1\psi_1) + e_2(\psi_2 - e_1\psi_3) = \psi_a + e_2\psi_b. \quad (3)$$

In equations (1) and (2),  $\psi_a = (\psi_0 + e_1\psi_1)$  and  $\psi_b = (\psi_2 - e_1\psi_3)$  are considered over the field of a quaternion described in terms of the four dimensional representations of real numbers and two dimensional representations of complex numbers. Accordingly, under  $SU(2)$  gauge symmetry, the quaternion spinor transforms as [15, 16],

$$\psi \longmapsto \psi' = U \psi \quad (4)$$

where  $U$  is  $2 \times 2$  unitary matrix and satisfies

$$U^\dagger U = UU^\dagger = UU^{-1} = U^{-1}U = 1. \quad (5)$$

On the other hand, the quaternion conjugate spinor transforms as

$$\bar{\psi} \longmapsto \bar{\psi}' = \bar{\psi} U^{-1}. \quad (6)$$

Hence, the combination  $\psi\bar{\psi} = \bar{\psi}\psi = \psi\bar{\psi}' = \bar{\psi}'\psi$  is an invariant quantity. So, any unitary matrix may then be written as

$$U = \exp\left(i\hat{H}\right); \quad (7)$$

where  $\hat{H}$  is Hermitian  $\hat{H}^\dagger = \hat{H}$ . Thus, we may express the Hermitian  $2 \times 2$  matrix in terms of four real numbers,  $a_1, a_2, a_3$ , and  $\theta$  as

$$\hat{H} = \theta\hat{1} + \tau_j a_j = \theta 1 + i e_j a_j; \quad (8)$$

where  $\hat{1}$  is the  $2 \times 2$  unit matrix,  $\tau_j$  are well known  $2 \times 2$  Pauli - spin matrices and  $e_1, e_2, e_3$  are the quaternion units related to Pauli - spin matrices as

$$e_0 = \hat{1}; \quad e_j = -i\tau_j. \quad (9)$$

Here  $e_j$  ( $\forall j = 1, 2, 3$ ), the basis elements of quaternion algebra, satisfy the following multiplication rule

$$e_j e_k = -\delta_{jk} + \epsilon_{jkl} e_l \quad (\forall j, k, l = 1, 2, 3). \quad (10)$$

where  $\delta_{jk}$  and  $\epsilon_{jkl}$  are respectively known as usual Kronecker delta and three index Levi - Civita symbols respectively. We may now describe the  $SU(2)$  isospin in terms of quaternions as

$$I_a = \frac{ie_a}{2} \quad (\forall a = 1, 2, 3) \quad (11)$$

and

$$I_{\pm} = \frac{i}{2} (e_1 \pm ie_2). \quad (12)$$

So, we may write the quaternion basis elements in terms of  $SU(2)$  isospin as

$$e_1 = \frac{1}{i} (I_+ + I_-); \quad e_2 = \frac{1}{i} (I_+ - I_-); \quad e_3 = \frac{1}{i} (I_3); \quad (13)$$

which satisfy the following commutation relation

$$[I_+, I_-] = ie_3; \quad [I_3, I_{\pm}] = \pm \frac{i}{2} (e_1 \pm ie_2). \quad (14)$$

Here  $SU(2)$  group acts upon the fundamental representation of  $SU(2)$  doublets of up ( $u$ ) and down( $d$ ) quark spinors

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (15)$$

So, we get for up quarks

$$I_+ |u\rangle = \frac{i(e_1 + ie_2)}{2} |u\rangle = 0; \quad I_- |u\rangle = \frac{i(e_1 - ie_2)}{2} |u\rangle = \frac{1}{2} |d\rangle; \quad I_3 |u\rangle = \frac{ie_3}{2} |u\rangle = \frac{1}{2} u |u\rangle; \quad (16)$$

and for down quarks we have

$$I_+ |d\rangle = \frac{i(e_1 + ie_2)}{2} |d\rangle = \frac{1}{2} |u\rangle; \quad I_- |d\rangle = \frac{i(e_1 - ie_2)}{2} |d\rangle = 0; \quad I_3 |d\rangle = \frac{ie_3}{2} |d\rangle = -\frac{1}{2} |d\rangle. \quad (17)$$

Conjugates of equations(16) and (17) are now be described as

$$\langle d| I_+ = \langle d| \frac{i(e_1 + ie_2)}{2} = 0; \quad \langle d| I_- = \langle d| \frac{i(e_1 - ie_2)}{2} = \frac{1}{2} \langle u|; \quad \langle d| I = \langle d| \frac{ie_3}{2} = \frac{1}{2} \langle d|; \quad (18)$$

and

$$\langle u| I_+ = \langle u| \frac{i(e_1 + ie_2)}{2} = \frac{1}{2} \langle d|; \quad \langle u| I_- = \langle u| \frac{i(e_1 - ie_2)}{2} = 0; \quad \langle u| I_3 = \langle u| \frac{ie_3}{2} = \frac{1}{2} \langle u|; \quad (19)$$

where  $e_1$ ,  $e_2$  and  $e_3$  are defined in (11)and (12). The effect of quaternion operator on up  $|u\rangle$  and down  $|d\rangle$  quarks states leads to

$$\begin{aligned} ie_1 |u\rangle &= |d\rangle; & ie_1 |d\rangle &= |u\rangle; \\ e_2 |u\rangle &= |d\rangle; & e_2 |d\rangle &= -|u\rangle \\ ie_3 |u\rangle &= |u\rangle; & ie_3 |d\rangle &= -|d\rangle. \end{aligned} \quad (20)$$

So, we may write

$$e_1 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ d \end{pmatrix}; \quad (21)$$

$$e_2 \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} d \\ -u \end{pmatrix}; \quad (22)$$

$$e_3 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ -d \end{pmatrix}; \quad (23)$$

transform a neutron (down quark) state into a proton ( up quark) state or vice versa. Only  $e_2$  gives real doublets of up and down quarks.

### 3 Octonions and Gellmann $\lambda$ Matrices

Octonions describe the widest normed algebra after the algebra of real numbers, complex numbers and quaternions. The octonions are non associative and non commutative normed division algebra over the algebra of real numbers. A set of octets ( $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ ) are known as the octonion basis elements and satisfy the following multiplication rules

$$\begin{aligned} e_0 &= 1; e_0 e_A = e_A e_0 = e_A; \\ e_A e_B &= -\delta_{AB} e_0 + f_{ABC} e_C. (\forall A, B, C = 1, 2, \dots, 7). \end{aligned} \quad (24)$$

The structure constants  $f_{ABC}$  is completely antisymmetric and takes the value 1 for the following combinations,

$$\begin{aligned} f_{ABC} &= +1; \\ \forall (ABC) &= (123), (471), (257), (165), (624), (543), (736). \end{aligned} \quad (25)$$

The relation between Gell - Mann  $\lambda$  matrices and octonion units are given as [15]

$$\frac{[e_{a+3}, e_7]}{[\lambda_{a+3}, \lambda_7]} = \frac{e_a}{2i\lambda_a}; \quad (26)$$

$$\frac{[e_7, e_a]}{[\lambda_7, \lambda_a]} = \frac{e_{a+3}}{2i\lambda_{a+3}}; \quad (27)$$

$$\frac{[e_a, e_{a+3}]}{[\lambda_a, \lambda_{a+3}]} = \frac{e_7}{2i\lambda_7}; \quad (28)$$

where  $a = 1, 2, 3$ . So, we have may describe [15] the following relationship between Gell - Mann  $\lambda$  matrices and octonion units as ,

$$\begin{aligned} \lambda_1 &= -ie_1 k_1; \lambda_2 = -ie_2 k_2; \lambda_3 = -ie_3 k_3; \lambda_4 = -ie_4 k_4; \\ \lambda_5 &= -ie_5 k_5; \lambda_6 = -ie_6 k_6; \lambda_7 = -ie_7 k_7; \end{aligned} \quad (29)$$

and  $\lambda_8$  are also related with  $e_3$  as

$$\lambda_8 = -ik_8 e_3 \text{ where } k_8 = \frac{8}{\sqrt{3}}; \quad (30)$$

where  $k_a = -1$  ( $\forall a = 1, 2, 3, \dots, 7$ ) are proportionality constants.

## 4 Octonion Formulation of $SU(3)$ Flavor Group

The Lie algebra of  $SU(3)$  exhibits most of the features of the larger Lie algebras. The elements of  $SU(3)$  group may be obtained in terms of  $3 \times 3$  Hermitian Gell Mann  $\lambda$  Hermitian matrices related to octonions where first three matrices describe the familiar isotopic spin generators from the  $SU(2)$  subgroup of  $SU(3)$ . These generators connect up ( $u$ ) and down quarks ( $d$ ). The fourth and fifth generators and the sixth and seventh generators are denoted as the  $V$  - spin and the  $U$  - spin.  $V$  - spin connects the up ( $u$ ) and strange quarks ( $s$ ) while  $U$  - spin connects the down ( $d$ ) and strange quarks ( $s$ ). The eighth generator is diagonal in nature responsible for hypercharge. The  $I$ ,  $U$  and  $V$  - spin algebra fulfills the angular momentum algebra and turn out to be the sub algebras of  $SU(3)$ . Hence, the  $SU(3)$  multiplets are constructed in form of  $I$ - multiplets,  $V$ - multiplets and an  $U$ - multiplets. The  $I$ - spin,  $U$ - spin and  $V$ -spin algebra are closely related and are the elements of sub algebra of  $SU(3)$ . As we may define  $SU(3)$  multiplets as

$$\begin{aligned} I_1 &= \frac{1}{2}\lambda_1 = \frac{ie_1}{2}; & I_2 &= \frac{1}{2}\lambda_2 = \frac{ie_2}{2}; & I_3 &= \frac{1}{2}\lambda_3 = \frac{ie_3}{2} \text{ (I - Spin)} \\ V_1 &= \frac{1}{2}\lambda_4 = \frac{ie_4}{2}; & V_2 &= \frac{1}{2}\lambda_5 = \frac{ie_5}{2}; & V_3 &= \frac{ie_3}{4} (8\sqrt{3} + 1) \text{ (U - Spin)} \\ U_1 &= \frac{1}{2}\lambda_6 = \frac{ie_6}{2}; & U_2 &= \frac{1}{2}\lambda_7 = \frac{ie_7}{2}; & U_3 &= \frac{ie_3}{4} (8\sqrt{3} - 1) \text{ (V - Spin)} \end{aligned} \quad (31)$$

along with the hyper charge is written in terms of octonions as follows

$$Y = \frac{1}{\sqrt{3}}\lambda_8 = -\frac{8ie_3}{\sqrt{3}}. \quad (32)$$

Here  $I_1, I_2$  and  $I_3$  contain the  $2 \times 2$  isospin operators (i. e. quaternion units).  $U_3, V_3, I_3$  and  $Y$  are linearly independent generators and are simultaneously diagonalized. It will be noted that  $\lambda_1, \lambda_2, \lambda_3$  agree with  $\sigma_1, \sigma_2, \sigma_3$ . The matrices  $V_1, V_2, U_3$  and  $U_2$  connect the nucleon and  $\Lambda$  and changes strangeness by one unit and isotopic spin by a half unit.. The operators  $I_{\pm}, U_{\pm}, V_{\pm}$  are also the shift operators known as raising ( $I_+, U_+, V_+$ ) and lowering ( $I_-, U_-, V_-$ ) operators.  $I_{\pm}, U_{\pm}, V_{\pm}$  are called as ladder operators. The complexified variants contain the third operators  $T_{\pm}, U_{\pm}, V_{\pm}$  which characterizes the states of  $SU(3)$  multiplets. The operators  $I_{\pm}, U_{\pm}, V_{\pm}$  are defined as

$$I_{\pm} = I_x \pm iI_y = \frac{1}{2}(\lambda_1 \pm i\lambda_2) = \frac{i}{2}(e_1 \pm ie_2). \quad (33)$$

$$V_{\pm} = V_x \pm iV_y = \frac{1}{2}(\lambda_4 \pm i\lambda_5) = \frac{i}{2}(e_4 \pm ie_5). \quad (34)$$

$$U_{\pm} = U_x \pm iU_y = \frac{1}{2}(\lambda_6 \pm i\lambda_7) = \frac{i}{2}(e_6 \pm ie_7). \quad (35)$$

With the help of shift operators and their properties ,we may derive the quark states of these multiplets as  $|q_1\rangle|q_2\rangle|q_3\rangle$ .So the quark states of  $I$  , $U$  and  $V$  spin are described as

$$I_- |q_1 \rangle = |q_2 \rangle; \quad I_+ |q_2 \rangle = |q_1 \rangle. \quad (36)$$

$$U_- |q_2 \rangle = |q_3 \rangle; \quad U_+ |q_3 \rangle = |q_2 \rangle. \quad (37)$$

$$V_- |q_1 \rangle = |q_3 \rangle; \quad V_+ |q_3 \rangle = |q_1 \rangle. \quad (38)$$

Thus,the operators  $I_{\pm}$ ,  $U_{\pm}$ ,  $V_{\pm}$  are viewed as operators which transforms one flavor into another flavor of quarks

$$\begin{aligned} I_{\pm} (I_3) &\longmapsto I_3 \pm 1; \\ V_{\pm} (V_3) &\longmapsto V_3 \pm 1; \\ U_{\pm} (U_3) &\longmapsto U_3 \pm 1. \end{aligned} \quad (39)$$

It means the action of  $I_{\pm}$  , $U_{\pm}$  and  $V_{\pm}$  shifts the values of  $I_3$  , $V_3$  and  $U_3$  by  $\pm 1$ .



## 5 Commutation relations for Octonion Valued Shift Operators

The  $I$ ,  $U$  and  $V$  - spin algebras are closed. Let us obtain the commutation relations of shift operators  $I_{\pm}$ ,  $U_{\pm}$ ,  $V_{\pm}$  [16] for  $SU(3)$  group. Using equations (24), (25), (34) and (35), we get

$$\begin{aligned}
 [U_3, U_{\pm}] &= \pm \left[ \frac{i}{2} (e_6 \pm ie_7) \right] = \pm U_{\pm}; \\
 [V_3, V_{\pm}] &= \pm \left[ \frac{i}{2} (e_4 \pm ie_5) \right] = \pm V_{\pm}; \\
 [I_3, I_{\pm}] &= \pm \left[ \frac{i}{2} (e_1 \pm ie_2) \right] = \pm I_{\pm}.
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 [U_+, U_-] &= \frac{ie_3}{2} (8\sqrt{3} - 1) = 2U_3; \\
 [V_+, V_-] &= \frac{ie_3}{2} (8\sqrt{3} + 1) = 2V_3; \\
 [I_+, I_-] &= ie_3 = 2I_3.
 \end{aligned} \tag{41}$$

$$\begin{aligned}
 [Y, U_{\pm}] &= \pm \left[ \frac{i}{2} (e_6 \pm ie_7) \right] = \pm U_{\pm}; \\
 [Y, V_{\pm}] &= \pm \left[ \frac{i}{2} (e_4 \pm ie_5) \right] = \pm V_{\pm}; \\
 [Y, I_{\pm}] &= \pm \left[ \frac{i}{2} (e_1 \pm ie_2) \right] = \pm I_{\pm}.
 \end{aligned} \tag{42}$$

$$[I_+, V_+] = [I_+, U_+] = [U_+, V_+] = 0; \tag{43}$$

$$\begin{aligned}
 [I_+, V_-] &= - \left[ \frac{i}{2} (e_6 - ie_7) \right] = -U_-; \\
 [I_+, U_+] &= \left[ \frac{i}{2} (e_4 + ie_5) \right] = V_+; \\
 [U_+, V_-] &= \frac{i}{2} (e_1 - ie_2) = I_-.
 \end{aligned} \tag{44}$$

Accordingly, we may write the hypercharge as

$$Y = \frac{1}{\sqrt{3}}\lambda_8 = -\frac{8ie_3}{\sqrt{3}} = \frac{2}{3}(U_3 + V_3) = \frac{2}{3}(2U_3 + I_3) = \frac{2}{3}(2V_3 - I_3). \quad (45)$$

and the term hyper charge  $Y$  commutes with third component of  $I$ ,  $U$  and  $V$  - spin multiplets of  $SU(3)$  flavor group

$$[Y, I_3] = [Y, U_3] = [Y, V_3] = 0 \quad (46)$$

Under the following dependency conditions,

$$I_3 = \frac{ie_3}{2} = V_3 - U_3 \quad (47)$$

and it also

$$\begin{aligned} I_+ &= \frac{i}{2}(e_1 \pm ie_2) = (I_-)^\dagger \\ V_+ &= \frac{i}{2}(e_4 \pm ie_5) = (V_-)^\dagger \\ U_+ &= \frac{i}{2}(e_6 \pm ie_7) = (U_-)^\dagger \end{aligned} \quad (48)$$

The commutation relation between the third components of  $T, U$  and  $V$  with  $T_\pm$  are given as

$$\begin{aligned} [I_3, I_\pm] &= \pm \left[ \frac{i}{2}(e_1 \pm ie_2) \right] = \pm I_\pm; \\ [U_3, I_\pm] &= \mp \frac{1}{2} \left[ \frac{i}{2}(e_1 \pm ie_2) \right] = \mp \frac{1}{2} I_\pm; \\ [V_3, I_\pm] &= \pm \frac{1}{2} \left[ \frac{i}{2}(e_1 \pm ie_2) \right] = \pm \frac{1}{2} I_\pm; \end{aligned} \quad (49)$$

The octonions are related to raising  $I_+, U_+, V_+$  and lowering  $I_-, U_-, V_-$  operators as

$$\begin{aligned} e_1 &= i(I_+ + I_-); \quad e_2 = (I_+ - I_-); \quad e_3 = 2I_3; \\ e_4 &= i(V_+ + V_-); \quad e_5 = (V_+ - V_-); \\ e_6 &= i(U_+ + U_-); \quad e_7 = (I_+ + I_-). \end{aligned} \quad (50)$$

The commutation relations between  $I_+$  and  $I_-$ ,  $U_+$  and  $U_-$  and  $V_+$  and  $V_-$  are described as

$$\begin{aligned}
[I_+, I_-] &= ie_3 = 2I_3; \\
[U_+, U_-] &= \frac{ie_3}{2} (8\sqrt{3} - 1) = 2U_3; \\
[V_+, V_-] &= -\frac{ie_3}{4} (8\sqrt{3} + 1) = 2V_3.
\end{aligned} \tag{51}$$

## 6 Discussion and Conclusion

In the foregoing analysis, we have described the resemblance between real Lie algebra  $SU(2)$  and the quaternion formalism of isospin  $SU(2)$  group by using the isomorphism between quaternion basis elements and Pauli matrices. On the same footing, we have established the connection between octonion basis elements and Gell Mann  $\lambda$  matrices of  $SU(3)$  flavor group. Accordingly, we have discussed the various sub algebras of  $SU(3)$  flavor group constructed in terms three  $SU(2)$  algebras of  $I$ ,  $U$  and  $V$  -spin multiplets. It therefore appears that the  $SU(3)$  flavor group may represent a new fundamental symmetry using octonions for classifying hadrons within  $SU(3)$  multiplets. The isospin symmetries are good approximation to simplify the interaction among hadrons. The motivation behind the present theory was to develop a simple compact and consistent algebraic formulation of  $SU(2)$  and  $SU(3)$  symmetries in terms of normed algebras namely quaternions and octonions. So, we have described the compact simplified notations instead of using the Pauli and Gellmann matrices respectively got  $SU(2)$  and  $SU(3)$  groups. Octonion representation of  $SU(3)$  flavor group directly establishes the one to one mapping between the non-associativity and the theory of strong interactions. It also shows that the theory of hadrons (or quark-colour etc.) has the direct link with non- associativity (octonions) while the isotopic spin leads to non commutativity (quaternions). So, we have derived and verified the commutation relations of shift operators and generators of  $SU(3)$  flavour group using octonions. As such, normed algebras namely the algebra of complex numbers, quaternions and octonions play an important role for the physical interpretation of quantum electrodynamics (QED), standard model of EW interactions and quantum chromo dynamics (QCD).

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