Conditions for the feasibility of multiple rolling for mechanical systems with multiple contact points

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Abstract

We illustrate a theoretical procedure determining necessary conditions for which simultaneous pure rolling kinetic constraints acting on a mechanical system can be fulfilled. We also analyze the sufficiency of these conditions by generalizing to this case a well known and usually accepted assumption on the behavior of pure rolling constraint. We present in detail the application of the procedure to some significative mechanical systems.

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Introduction

Mechanical systems subject to rolling kinetic constraints are one of the most studied argument of Classical Mechanics, especially for its wideness of applicability in several branches of Mechanical Sciences: Contact Mechanics, Tribology, Wear, Robotics, Ball Bearing Theory and Control Theory applied to moving engines and vehicles are only some of the important fields where the results about pure rolling constraint can be fruitfully used.

It is well known that, when a mechanical system moves in contact with an assigned rough surface, the effective fulfilment of the kinetic conditions determined by the rolling without sliding requirement of the system on the surface depends on the behavior, with respect to the considered law of friction, of the reaction forces acting on the system in the contact points. For example, the roll of a disk on a rough straight line, considering the Coulomb's law of friction, can happen only if the contact force lie inside the friction cone (see Example 1 below).

However, even in the simplest case of a mechanical system formed by a single rigid body, in the case of multiple contact points between the rigid body and the rough surface, it could be an hard task to obtain sufficient information about the contact reactions in order to determine if the laws of friction are satisfied or not during the motion. In fact the most common methods to determine information about the reactions, starting from the simple application of linear and angular momenta equations (see e.g. [1, 2]) to most refined techniques such as lagrangian multipliers in lagrangian mechanics (see e.g. [3]) or deep analyses of the contact between the system and the surface (see e.g. [4]), have a global character. Then these methods, for their very nature, can determine only a reactive force system equivalent to the real one but, in the general case, these methods cannot determine the single reactive forces in the contact points. The problem becomes even more complicated in case of multibody system, due to the presence of the internal reactions in the link between the parts of the system.

In this paper we consider the motion of a mechanical system having two or more distinct contact points with one or more assigned rough surfaces, and we determine necessary conditions for which in all the contact points the pure rolling kinetic constraint can hold. We also analyze the sufficiency of these conditions by generalizing to this case a well known and usually accepted assumption on the behavior of pure rolling constraint. Moreover, we briefly discuss the possible behaviors of the system when the necessary conditions are not fulfilled.

The procedure to determine if the rolling condition can be fulfilled can be applied both to systems formed by a single rigid body and to multibody systems. It is essentially based on the application of linear and angular momenta equations to the (parts forming the) mechanical system, and therefore it gives an underdetermined system in the unknown single contact reactions. Nevertheless, we show that the lack of complete knowledge of the single contact reactions is not an obstacle to determine the feasibility of the rolling conditions.

It is however important to remark that, although the procedure has a very simple and unassailable theoretic foundation, its effective application to general systems could present insurmountable difficulties. This is essentially due to the fact that the general procedure explicitly requires the knowledge of the motion law of the system, and in the general case the explicit time–dependent expression of the motion cannot be obtained because of complications determined by the geometry of the system itself and/or by the integrability of the equations of motion. Nevertheless there are several significative cases where the procedure can be explicitly performed. In the paper, we illustrate three examples with rising complication: the well known case of a disk falling in contact with an inclined plane (that is presented only to point out some key points of the general procedure); the case of a system formed by a non–coupled pair of disks connected with a bar and moving on the horizontal plane; the case of a heavy sphere falling in contact with a guide having the form of a V–groove non symmetric with respect to the vertical axis and inclined with respect to the horizontal.

The main content of this paper can be approached starting from a very standard background knowledge, essentially focused to the linear and angular equations of motion for a mechanical system, the so called Cardinal Equations, and the basic theory of pure rolling conditions and kinetic constraints. On the other hand, the list of possible references involving theory and application of pure rolling constraint is almost endless. Therefore we chose to cite only a very limited list of references sufficient to make the paper self–consistent: the classical book of Levi–Civita and Amaldi [1] and the book of Goldstein [2] for the Cardinal Equations and the basic concepts about pure rolling conditions; the book of Neimark and Fufaev [5] and the paper of Massa and Pagani [6] for the behavior of systems subject to kinetic constraints. The interested reader can find in the wide but not exhaustive lists of references of [7, 8] as a useful starting point to delve in the expanse of the material related to this argument.

The paper is divided in four sections. Section 1 contains a very brief preliminary description of the well known analysis of the rolling condition for a disk in contact with an inclined plane. This remind is motivated by some useful affinities with the general procedure for generic systems. Section 2 contains the discussion of the general case, and the determination of the necessary conditions for pure rolling conditions simultaneously hold. Section 3 presents the example of the system formed by the non–coupled disks and the example of the heavy sphere falling in the V–groove. Section 4 is devoted to open problems, remarks and conclusions.

1 Preliminaries

Example 1.

An homogeneous disk of mass m and radius R moves in the vertical plane being in contact with a rough guide inclined with slope angle α. Considering

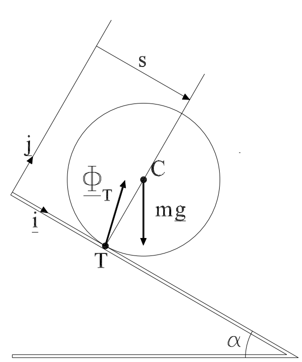

Figure 1: Rolling disk on an inclined plane

the system subject to the Coulomb's law of friction, with obvious notation clarified by Fig. 1, the feasibility of pure rolling condition of the disk can be determined with the following procedure:

0) we determine the relative velocity $\underline{\mathbf{v}}_T(t_0)$ of the contact point T of the disk at the instant t_0 with respect to the inclined plane as function of the initial data of the motion. The pure rolling condition requires of course that $\underline{\mathbf{v}}_T(t_0) = 0$. If so

1) we assume that the disk rolls without sliding on the inclined plane. Then the system has a single degree of freedom (for example the coordinate s of T along the inclined plane) and we can determine the equation of motion
$$mg \sin\alpha = \frac{3}{2} m\ddot{s}\,;$$

2) we determine the corresponding reaction $\underline{\mathbf{\Phi}}_T$ as a (in this case constant) function of time
$$\underline{\mathbf{\Phi}}_T = m\underline{\mathbf{a}}_C - m\underline{\mathbf{g}} = \left(-\frac{1}{3} mg\sin\alpha\right)\underline{\mathbf{i}} + (mg\cos\alpha)\,\underline{\mathbf{j}}\,;$$

3) we test the Coulomb's law of friction condition
$$\|\underline{\mathbf{\Phi}}_T^{\parallel}\| \leq \mu\,\|\underline{\mathbf{\Phi}}_T^{\perp}\| \quad \Leftrightarrow \quad \mu \geq \frac{1}{3}\tan\alpha$$
where $\underline{\mathbf{\Phi}}_T^{\parallel}$, $\underline{\mathbf{\Phi}}_T^{\perp}$ are the parallel and orthogonal component of $\underline{\mathbf{\Phi}}_T$ with respect to the inclined plane;

4) we assume that, if and until the Coulomb's condition is verified, the disk moves rolling on the plane and that if and when the Coulomb's condition is not verified, the disk changes its dynamic evolution beginning to slide on the plane (until the first time $t_1 > t_0$ such that $\underline{\mathbf{v}}_T(t_1) = 0$).

Some remarks are in order to focus the possibility to generalize the procedure to more complicated systems. Step 2 consists in the determination of the reaction acting on the disk as function of time. The utmost simplicity of the specific problem can hide the fact that in a more general situation the information about the reaction sufficient to analyze the rolling condition could require an explicit determination of the motion of the system as function of time.

Step 3 tests the compatibility of the reaction evaluated in Step 2 with the Coulomb's law of friction assumed as the constitutive characterization of the rough surface in contact with the disk. Of course the feasibility of the rolling condition can be tested with any other significative constitutive law.

In Step 4 we assume that, roughly speaking, if the disk can roll then it does. This is of course an arbitrary assumption, but the hypothesis is well confirmed by experimental results. In the next section, we will confirm this assumption in the more general situation of generic system.

To conclude the section, let us note that, in this very simple case, both the behaviors of the disk when the Coulomb's friction condition is or is not verified are determinable. In the general case, when the constitutive law is not verified, the behavior of the system turns out to be not so straight to determine, although some reasonable assumptions can be done. We will go back on this arguments in Section 4.

2 The general case

In this section, following a line of though similar to the one applied in the previous section, we discuss the possibility that a mechanical system $\mathcal{S}$ having two points T_1, T_2 in contact with a fixed surface Σ moves such that in both the contact points the rolling conditions can subsist respecting the Coulomb's law of friction. The arguments of the discussion can be easily extended to cases with more (but a finite number) than two contact points and possibly to different friction constitutive laws.

The discussion is based on the fact that, along the motion, the reactive forces acting on the system must validate the linear and angular momenta equations

$$\left\{ \begin{array}{rcl} \underline{\mathbf{R}}^{act} + \underline{\mathbf{R}}^{react} & = & M\underline{\mathbf{a}}_G \\ \\ \underline{\mathbf{M}}_G^{act} + \underline{\mathbf{M}}_G^{react} & = & \dfrac{d\underline{\Gamma}_G}{dt} \end{array} \right. \tag{1}$$

where M is the total mass of the system, G is the center of mass of the system and $\underline{\mathbf{R}}^{act}, \underline{\mathbf{R}}^{react}, \underline{\mathbf{M}}_G^{act}, \underline{\mathbf{M}}_G^{react}$ are respectively the sum of the active and reactive forces and active and reactive momenta acting on the whole system. In this specific situation we have that:

$$\left\{ \begin{array}{rcl} \underline{\mathbf{R}}^{react} & = & \underline{\mathbf{\Phi}}_{T_1} + \underline{\mathbf{\Phi}}_{T_2} \\ \\ \underline{\mathbf{M}}_G^{react} & = & \overrightarrow{GT_1} \times \underline{\mathbf{\Phi}}_{T_1} + \overrightarrow{GT_1} \times \underline{\mathbf{\Phi}}_{T_2} \end{array} \right. . \tag{2}$$

It is however well known [1] that Eqs. (1) are not sufficient to determine the motion of the mechanical system and the single reactions $\underline{\mathbf{\Phi}}_{T_1}, \underline{\mathbf{\Phi}}_{T_2}$ along the motion, since the system

$$\left\{ \begin{array}{rcl} \underline{\mathbf{R}}^{act} + \underline{\mathbf{\Phi}}_{T_1} + \underline{\mathbf{\Phi}}_{T_2} & = & M\underline{\mathbf{a}}_G \\ \\ \underline{\mathbf{M}}_G^{act} + \overrightarrow{GT_1} \times \underline{\mathbf{\Phi}}_{T_1} + \overrightarrow{GT_2} \times \underline{\mathbf{\Phi}}_{T_2} & = & \dfrac{d\underline{\Gamma}_G}{dt} \end{array} \right. \tag{3}$$

is by its very nature under–determined. In fact the projection of the angular momenta equation of (3) in the direction of $\overrightarrow{T_1T_2}$ is a pure equation of motion of the system where no reactions appear. Then (3) can give no more than 5 relations on the components of $\underline{\mathbf{\Phi}}_{T_1}$ and $\underline{\mathbf{\Phi}}_{T_2}$. Unfortunately, due to the roughness of the contacts, no preliminary conditions can be imposed on the components of the reactions, so that, even when the motion of the mechanical system is known, (3) is a linear system with 6 unknowns that is not of maximum rank.

Nevertheless the parametric solution of the system (2), and an assumption parallelizing the one of Step 4 of the case of Section 1, give us the possibility of determining if the rolling conditions in T_1 and T_2 are or not verified.

The procedure to test the feasibility of pure rolling condition of the disk can be then based on the following steps:

0) we test if the initial relative velocities of the contact points T_1, T_2 with respect to the surface are null or not. If they are null

1) we suppose that the system rolls without sliding in both the contact points. This assumption fixes the dynamics (for example the number of degrees of freedom...) of the system and consequently allows the determination of the motion of the system;

2) we write the linear and angular momenta equations for the whole system, for example in the form:

$$\begin{cases} \underline{\mathbf{\Phi}}_{T_1} + \underline{\mathbf{\Phi}}_{T_2} = M\underline{\mathbf{a}}_G - \underline{\mathbf{R}}^{act} \\ \overrightarrow{GT_1} \times \underline{\mathbf{\Phi}}_{T_1} + \overrightarrow{GT_2} \times \underline{\mathbf{\Phi}}_{T_2} = \dfrac{d\,\underline{\Gamma}_G}{dt} - \underline{\mathbf{M}}_G^{act} \end{cases} . \tag{4}$$

Since the motion of the system is known, both the right hand sides of the equations, together with the position vectors $\overrightarrow{GT_1}$ and $\overrightarrow{GT_2}$, are known as function of time. Therefore Eqs. (4) turn out to be a time–dependent under–determined linear system in the six scalar unknowns given by the components of the vectors $\underline{\mathbf{\Phi}}_{T_1}, \underline{\mathbf{\Phi}}_{T_2}$;

3) we solve the linear under–determined system (4), obtaining the expression of the reaction $\underline{\mathbf{\Phi}}_{T_1}$ and $\underline{\mathbf{\Phi}}_{T_2}$ as function of time and parameters $\lambda_1, \ldots, \lambda_r$, where of course the integer r is related to the rank of (4). Then we can determine the tangent and orthogonal components $\underline{\mathbf{\Phi}}^{\parallel}_{T_1}, \underline{\mathbf{\Phi}}^{\perp}_{T_1}, \underline{\mathbf{\Phi}}^{\parallel}_{T_2}, \underline{\mathbf{\Phi}}^{\perp}_{T_2}$ of the reactions with respect to the surface Σ as functions of $(t, \lambda_1, \ldots, \lambda_r)$. The pure rolling conditions then can subsist in both the contact points only in the time interval $[t_0, t_1]$ such that for every $t \in [t_0, t_1]$ there exists at least one admissible r–uple

$(\overline{\lambda}_1, \dots, \overline{\lambda}_r)$ such that the system

$$\begin{cases} \|\underline{\mathbf{\Phi}}^{\|}_{T_1}(t, \overline{\lambda}_1, \dots, \overline{\lambda}_r)\| \leq \mu_1 \, \|\underline{\mathbf{\Phi}}^{\perp}_{T_1}(t, \overline{\lambda}_1, \dots, \overline{\lambda}_r)\| \\ \\ \|\underline{\mathbf{\Phi}}^{\|}_{T_2}(t, \overline{\lambda}_1, \dots, \overline{\lambda}_r)\| \leq \mu_2 \, \|\underline{\mathbf{\Phi}}^{\perp}_{T_2}(t, \overline{\lambda}_1, \dots, \overline{\lambda}_r)\| \end{cases} \tag{5}$$

holds;

4) we assume that, if for every $t \in [t_0, t_1]$ there exists at least one admissible r–uple $(\overline{\lambda}_1, \dots, \overline{\lambda}_r)$ such that (5) are verified, the system moves rolling without sliding in both points T_1 and T_2 during the time interval $[t_0, t_1]$.

It is clear that the general procedure described above parallelizes as possible and generalizes the one of the disk on the inclined plane. The most significant differences consist in the explicit determination of the motion of the system (since otherwise Eqs. (4) could not admit a simple parametric solution for the reactions $\underline{\mathbf{\Phi}}_{T_1}$ and $\underline{\mathbf{\Phi}}_{T_2}$) and in the fact that, when the Coulomb conditions (5) are NOT verified, being understood that the system does not roll in both contact points, the determination of the behavior of the system could require a more subtle analysis. We will go back on these arguments in Section 4. We also remark that not all the r–uple $\lambda_1, \dots, \lambda_r$ could be admissible in the discussion of the inequalities (5). For example, if the system is leaned on the surface, we have to restrict our attention to the r–uple such that

$$\begin{cases} \underline{\mathbf{\Phi}}^{\perp}_{T_1}(t, \lambda_1, \dots, \lambda_r) \cdot \underline{\nu}_1 \geq 0 \\ \\ \underline{\mathbf{\Phi}}^{\perp}_{T_2}(t, \lambda_1, \dots, \lambda_r) \cdot \underline{\nu}_2 \geq 0 \end{cases} \tag{6}$$

(where $\underline{\nu}_i$ is the unit normal vector to the surface Σ in the point T_i and orientated toward the side of the system) since otherwise the system detaches from the surface.

3 Examples

A mechanical system is formed by two equal disks of mass m and radius R and a rod, of mass M and length L. The rod is constrained to remain orthogonal to the two planes of the disks with its endpoints coinciding with the two centers of the disks (see Fig. 3) so that the disks remain vertical. The whole system is leaned on a rough horizontal plane. The system has then 5 degrees of freedom: the coordinates x, y of the center of mass G of the rod, the angle ϑ formed by the plane of the disks with the xz plane and the two rotation angles φ_1, φ_2 of the disks. The rolling conditions in the

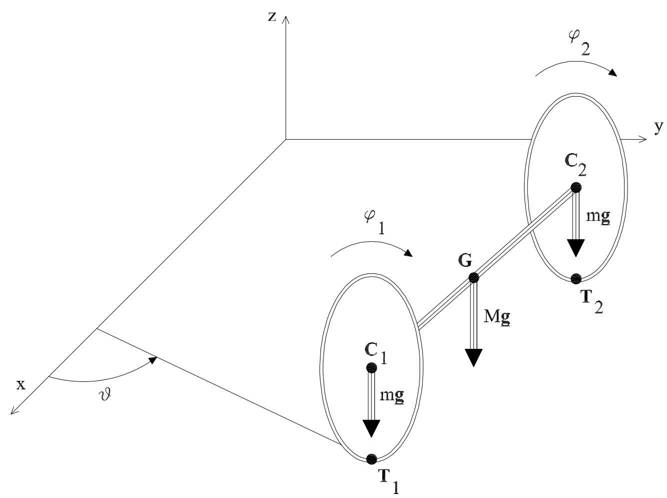

Figure 2: Rolling system on an horizontal plane

contact points T_1 and T_2 are equivalently expressed by:

$$\begin{cases} \dot{x} + \dfrac{1}{2}L\dot{\vartheta}\cos\vartheta - R\dot{\varphi}_1\cos\varphi_1 \;=\; 0 \\[2ex] \dot{y} + \dfrac{1}{2}L\dot{\vartheta}\sin\vartheta - R\dot{\varphi}_1\sin\varphi_1 \;=\; 0 \\[2ex] \dot{\vartheta} - \dfrac{R}{L}\left(\dot{\varphi}_1 - \dot{\varphi}_2\right) \;=\; 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} - \dfrac{1}{2}L\dot{\vartheta}\cos\vartheta - R\dot{\varphi}_2\cos\varphi_2 \;=\; 0 \\[2ex] \dot{y} - \dfrac{1}{2}L\dot{\vartheta}\sin\vartheta - R\dot{\varphi}_2\sin\varphi_2 \;=\; 0 \\[2ex] \dot{\vartheta} - \dfrac{R}{L}\left(\dot{\varphi}_1 - \dot{\varphi}_2\right) \;=\; 0 \end{cases} \tag{7}$$

Tedious but straightforward computations (see [5, 6]) give the equations of motion of the system

$$\begin{cases} \ddot{x} &=& \dfrac{1}{2}L\dot{\vartheta}^2\sin\vartheta - R\dot{\vartheta}\dot{\varphi}_1\sin\vartheta \\[2ex] \ddot{y} &=& -\dfrac{1}{2}L\dot{\vartheta}^2\cos\vartheta + R\dot{\vartheta}\dot{\varphi}_1\cos\vartheta \\ \ddot{\vartheta} &=& 0 \\ \ddot{\varphi}_1 &=& 0 \\ \ddot{\varphi}_2 &=& 0 \end{cases} \tag{8}$$

If we suppose assigned the almost generic initial data

$$\left\{\begin{array}{lcl} x(0) & = & x_0 \\ y(0) & = & y_0 \\ \vartheta(0) & = & \vartheta_0 \\ \varphi_1(0) & = & 0 \\ \varphi_2(0) & = & 0 \end{array}\right. \quad \left\{\begin{array}{lcl} \dot{x}(0) & = & \frac{1}{2}R\cos\vartheta_0(\dot{\varphi}_{10}+\dot{\varphi}_{20}) \\ \\ \dot{y}(0) & = & \frac{1}{2}R\sin\vartheta_0(\dot{\varphi}_{10}+\dot{\varphi}_{20}) \\ \\ \dot{\vartheta}(0) & = & \frac{R}{L}\left(\dot{\varphi}_{10}-\dot{\varphi}_{20}\right) \\ \\ \dot{\varphi}_1(0) & = & \dot{\varphi}_{10} \\ \dot{\varphi}_2(0) & = & \dot{\varphi}_{20} \end{array}\right. \tag{9}$$

with the only condition $\dot{\varphi}_{10} \neq \dot{\varphi}_{20}$, the motion of the system is given by

$$\left\{\begin{array}{lcl} x(t) & = & \frac{1}{2}L\frac{\dot{\varphi}_{10}+\dot{\varphi}_{20}}{\dot{\varphi}_{10}-\dot{\varphi}_{20}}\left[\sin\left(\frac{R}{L}\left(\dot{\varphi}_{10}-\dot{\varphi}_{20}\right)t+\vartheta_0\right)-\sin\vartheta_0\right]+x_0 \\ \\ y(t) & = & -\frac{1}{2}L\frac{\dot{\varphi}_{10}+\dot{\varphi}_{20}}{\dot{\varphi}_{10}-\dot{\varphi}_{20}}\left[\cos\left(\frac{R}{L}\left(\dot{\varphi}_{10}-\dot{\varphi}_{20}\right)t+\vartheta_0\right)-\cos\vartheta_0\right]+y_0 \\ \\ \vartheta(t) & = & \frac{R}{L}(\dot{\varphi}_{10}-\dot{\varphi}_{20})t \\ \\ \varphi_1(t) & = & \dot{\varphi}_{10}t \\ \\ \varphi_2(t) & = & \dot{\varphi}_{20}t \end{array}\right. \tag{10}$$

The linear and angular momenta equations for the system can be written as

$$\left\{\begin{array}{l} (2m+M)\underline{g}+\underline{\mathbf{\Phi}}_{T_1}+\underline{\mathbf{\Phi}}_{T_2} = (2m+M)\underline{\mathbf{a}}_G \\ \\ \overrightarrow{GC_1}\times m\underline{g}+\overrightarrow{GC_2}\times m\underline{g}+\overrightarrow{GT_1}\times\underline{\mathbf{\Phi}}_{T_1}+\overrightarrow{GT_2}\times\underline{\mathbf{\Phi}}_{T_2} \\ \\ \qquad = \mathbf{I}_{C_1}(\dot{\underline{\omega}}_1)+\underline{\omega}_1\times\mathbf{I}_{C_1}(\underline{\omega}_1)+m\overrightarrow{GC_1}\times\underline{\mathbf{a}}_{C_1} \\ \\ \qquad\quad +\mathbf{I}_G(\dot{\underline{\omega}}_{rod})+\underline{\omega}_{rod}\times\mathbf{I}_G(\underline{\omega}_{rod}) \\ \\ \qquad\qquad +\mathbf{I}_{C_2}(\dot{\underline{\omega}}_2)+\underline{\omega}_2\times\mathbf{I}_{C_2}(\underline{\omega}_2)+m\overrightarrow{GC_2}\times\underline{\mathbf{a}}_{C_2} \end{array}\right. \tag{11}$$

Taking into account the motion of the system (10) and introducing the orthonormal base $\{\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\mathbf{z}}\}$ with $\underline{\mathbf{u}} = \dfrac{\overrightarrow{C_2C_1}}{L}, \underline{\mathbf{z}} = \dfrac{\overrightarrow{T_1C_1}}{R}, \underline{\mathbf{v}} = \underline{\mathbf{z}}\times\underline{\mathbf{u}}$, with obvious notation we obtain

$$
\left\{
\begin{array}{rcl}
\Phi_{1_z} & = & \frac{1}{2}\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_2}_0^2-{\dot\varphi_1}_0^2)\right] \\
\Phi_{2_z} & = & \frac{1}{2}\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\right] \\
\Phi_{1_v} & = & 0 \\
\Phi_{2_v} & = & 0 \\
\Phi_{1_u}+\Phi_{2_u} & = & -\frac{1}{2}(2m+M)\frac{R^2}{L}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)
\end{array}
\right.
\tag{12}
$$

Note that, if the system leans on the horizontal plane, we must add the requirement

$$
|{\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2| \le \frac{(2m+M)}{(3m+M)}\frac{L^2}{R^2}\frac{g}{R}
\tag{13}
$$

since otherwise one between Φ_{1_z} and Φ_{2_z} becomes negative (and this is not acceptable, because the system lifts from the horizontal plane, and the initial assumptions of five degrees of freedom is violated).

If (13) is fulfilled, then we can chose for example $\Phi_{1_u}=\lambda$ and we find the reactions $\underline{\mathbf{\Phi}}_{T_1}, \underline{\mathbf{\Phi}}_{T_2}$ as functions of λ: Coulomb conditions (5) then takes the form:

$$
\left\{
\begin{array}{l}
|\lambda| \le \mu_1 \frac{1}{2}\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_2}_0^2-{\dot\varphi_1}_0^2)\right] \\
\left|\frac{1}{2}(2m+M)\frac{R^2}{L}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)+\lambda\right| \le \mu_2 \frac{1}{2}\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\right]
\end{array}
\right.
\tag{14}
$$

In conclusion, the pure rolling of the disks can subsist if and only if (13) holds and there is a λ such that

$$
\begin{aligned}
\max\Bigg\{&-\frac{1}{2}\mu_1\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_2}_0^2-{\dot\varphi_1}_0^2)\right],\\
&\qquad -\frac{1}{2}(2m+M)\frac{R^2}{L}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\\
&\qquad\qquad -\frac{1}{2}\mu_2\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\right]\Bigg\}
\end{aligned}
$$

$$\leq \lambda \leq$$

$$
\begin{aligned}
\min\Bigg\{&\frac{1}{2}\mu_1\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_2}_0^2-{\dot\varphi_1}_0^2)\right],\\
&\qquad \frac{1}{2}\mu_2\left[(2m+M)g-(3m+M)\frac{R^3}{L^2}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\right]\\
&\qquad\qquad -\frac{1}{2}(2m+M)\frac{R^2}{L}({\dot\varphi_1}_0^2-{\dot\varphi_2}_0^2)\Bigg\}.
\end{aligned}
$$

3.1 Example 3.

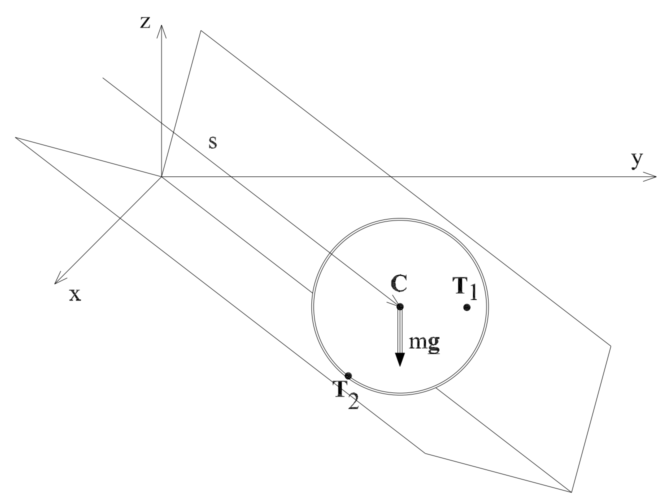

Figure 3: Sphere on a V–groove

A mechanical system is formed by a sphere of mass m and radius R leaned in an inclined V–groove whose walls are described by the equations

$$\pi_1 : 2x+y+z\;=\;0\,;\quad \pi_2 : -x+y+z\;=\;0$$

We introduce an orthonormal base $\{\underline{\mathbf{k}}_1^{\perp},\underline{\mathbf{k}}_2^{\perp},\underline{\mathbf{k}}^{\parallel}\}$ where $\underline{\mathbf{k}}_1^{\perp},\underline{\mathbf{k}}_2^{\perp}$ are orthogonal to π_1,π_2 respectively and $\underline{\mathbf{k}}^{\parallel}=\underline{\mathbf{k}}_1^{\perp}\times\underline{\mathbf{k}}_2^{\perp}$. The center C of the sphere is then determined by the vector $R\underline{\mathbf{k}}_1^\perp + R\underline{\mathbf{k}}_2^\perp - s\underline{\mathbf{k}}^\parallel$, where s is the distance of C from a fixed plane orthogonal to $\underline{\mathbf{k}}^\parallel$ (see Fig. 3.1). The rolling conditions in the contact points T_1, T_2 determines the angular velocity of the sphere in the form $\underline{\omega} = -\dfrac{\dot{s}}{R}(\underline{\mathbf{k}}_1^\perp - \underline{\mathbf{k}}_2^\perp)$ and the system has one degree of freedom: the coordinate s.

The linear and angular momenta equations for the system can be written as

$$\begin{cases} m\underline{g} + \underline{\mathbf{\Phi}}_{T_1} + \underline{\mathbf{\Phi}}_{T_2} = m\underline{\mathbf{a}}_C \\ \overrightarrow{T_1C} \times m\underline{g} + \overrightarrow{T_1T_2} \times \underline{\mathbf{\Phi}}_{T_2} = \mathbf{I}_C(\underline{\dot{\omega}}) + m\overrightarrow{T_1C} \times \underline{\mathbf{a}}_C \end{cases} \tag{15}$$

The projection of the angular momenta equation in the direction of $\overrightarrow{T_1T_2}$ gives the equation of motion of the sphere, that is $\ddot{s} = \dfrac{5\sqrt{2}}{18}g$. This relation suffices to obtain from (15) the under–determined system of the reactions: if we decompose the reactions along the basis introduced above

$$\underline{\mathbf{\Phi}}_{T_1} = \Phi_{1_N}\underline{\mathbf{k}}_1^\perp + \Phi_{1_u}\underline{\mathbf{k}}_2^\perp + \Phi_{1_v}\underline{\mathbf{k}}^\parallel$$

$$\underline{\mathbf{\Phi}}_{T_2} = \Phi_{2_u}\underline{\mathbf{k}}_1^\perp + \Phi_{2_N}\underline{\mathbf{k}}_2^\perp + \Phi_{2_v}\underline{\mathbf{k}}^\parallel$$

the system takes the form

$$\begin{cases} \Phi_{1_v} = \dfrac{\sqrt{2}}{9}mg \\ \Phi_{2_v} = \dfrac{\sqrt{2}}{9}mg \\ \Phi_{1_N} + \Phi_{2_u} = \dfrac{1}{\sqrt{6}}mg \\ \Phi_{1_u} + \Phi_{2_N} = \dfrac{1}{\sqrt{3}}mg \\ \Phi_{2_N} + \Phi_{2_u} = \dfrac{1}{\sqrt{3}}mg \end{cases} \tag{16}$$

To analyze the parametric solution of the system we chose $\Phi_{1_u} = \lambda mg$. In this case, and once again supposing the sphere leaned on the groove, we must require the condition $\lambda < \frac{1}{\sqrt{6}}$ since otherwise $\Phi_{1_N} < 0$ and the sphere comes off the groove. Conditions (5) take in this case the form

$$\begin{cases} \lambda^2 + \dfrac{2}{81} \le \mu_1^2 \left(\dfrac{1}{\sqrt{6}} - \lambda \right)^2 \\ \\ \lambda^2 + \dfrac{2}{81} \le \mu_2^2 \left(\dfrac{1}{\sqrt{3}} - \lambda \right)^2 \end{cases} \tag{17}$$

with $\lambda < \dfrac{1}{\sqrt{6}}$. A straightforward minimum computation for the functions on the left-hand side of (17) shows then that the system can roll on both the contact points if and only if

$$\begin{cases} \mu_1 \ge \dfrac{2}{\sqrt{31}} \\ \\ \mu_2 \ge \sqrt{\dfrac{2}{29}} \end{cases} \tag{18}$$

4 Conclusions

The procedure described in Sec. 2 in the case of two contact points can be generalized to (multibody) systems with three or more contact points (think for example of a "steering tricycle" formed by three vertical disks connected with three rods leaned on the horizontal plane). Of course, an increase of the number of contact points implies in general an increase of the technical difficulties in practical applications. This is principally due to the fact that Step 1 of the general procedure is not a straightforward passage. The effective knowledge of the motion of the system can be achieved only in some particular cases. Insurmountable technical difficulties can arise both for geometrical reasons (think of a convex rigid body moving in contact with a surface, both having generic shapes with the only requirement that the contact between rigid body and groove happens in two points. For a more detailed discussion on the argument, see, e.g. [9, 10]), and/or for computational reasons (even when the equations of motion of the system are explicitly obtained, it could be hard to integrate them to obtain the motion of the system). Nevertheless note that, as pointed out by the examples in Sec. 3, not for all the systems the explicit integration of the equations of motion is required.

A second remark is that the general procedure gives necessary conditions such that the pure rolling subsists in all contact points (conditions that become sufficient if we take into account Step 4 of the procedure) but it does not give any information on the behavior of the system if the pure rolling is not possible even in a single contact point. In fact, analogously to what happens in the simple case of Ex. 1, in the instant when (5) stops to hold, the dynamics of the system (for example, the number of degrees of freedom) changes abruptly.

To clarify this fact, suppose that, at the instant t_1 of the study of the system of Ex. 2, a sudden variation of the friction coefficient μ_2 in the point T_2 (an oil spot on the plane?) causes the invalidity of the second relation of (14), while the first relation still holds. Of course, even if we chose the assumption of Step 4 of the procedure as a fixed point of our argument, we cannot suppose that the system continues to roll in T_1 (and begins to slide in T_2) since the beginning of sliding in T_2 can affect the pure rolling behavior of the system in the point T_1. We must perform a new analysis of the behavior of the system, possibly supposing the system rolling in T_1 and sliding in T_2, we must determine (if possible) the new equations of motion of the system (with the additional difficulties of different friction laws in the point T_1 and T_2 and possibly increased number of degrees of freedom), the motion of the system, the new (parametric) system of reactions acting on the system and then we can test the Coulomb condition in the point T_1.

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