BOXICITY OF GRAPHS ON SURFACES

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ABSTRACT. The boxicity of a graph $G = (V, E)$ is the smallest integer k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leqslant i \leqslant k$, such that $E = E_1 \cap \cdots \cap E_k$. Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two and Thomassen proved in 1986 that planar graphs have boxicity at most three. In this note we prove that the boxicity of toroidal graphs is at most 7, and that the boxicity of graphs embeddable in a surface Σ of Euler genus g is at most $\frac{9}{2}g + 3$ if Σ is orientable, and at most $9g + 3$ otherwise. This result yields improved bounds on the dimension of the adjacency poset of graphs on surfaces.

1. Introduction

Given a collection $\mathcal{C}$ of subsets of a set Ω, the *intersection graph* of $\mathcal{C}$ is defined as the graph[1] with vertex set $\mathcal{C}$, in which two elements of $\mathcal{C}$ are adjacent if and only if their intersection is non-empty. A *d-box* is the Cartesian product $[x_1, y_1] \times \ldots \times [x_d, y_d]$ of d closed intervals of the real line. The *boxicity* $\text{box}(G)$ of a graph G is the smallest integer $d \geqslant 1$ such that G is the intersection graph of a collection of d-boxes[2]. An *interval graph* is a graph of boxicity one.

The *intersection* $G_1 \cap \cdots \cap G_k$ of k graphs $G_1, \ldots, G_k$ defined on the same vertex set V is the graph $(V, E_1 \cap \ldots \cap E_k)$, where E_i $(1 \leqslant i \leqslant k)$ denotes the edge set of G_i. Observe that the boxicity of a graph G can equivalently be defined as the smallest k such that G is the intersection of k interval graphs.

The concept of boxicity was introduced in 1969 by Roberts [17]. It is used as a measure of the complexity of ecological [18] and social [10] networks, and has applications in fleet maintenance [16]. Graphs with boxicity one (that is, interval graphs) can be recognized in linear time. On the other hand, Kratochvíl [13] proved that determining whether a graph has boxicity at most two is NP-complete.

Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two [19] and Thomassen proved in 1986 that planar graphs have boxicity at most three [23]. Other results on the boxicity of graphs can be found in [2, 6, 7] and the references therein.

Related to boxicity is the notion of adjacency posets of graphs, which was introduced by Felsner and Trotter [9]. The *adjacency poset* of a graph $G = (V, E)$ is the poset $\mathcal{P}_G = (W, \leqslant)$ with $W = V \cup V'$, where V' is a disjoint copy of V, and such that $u \leqslant v$ if and only if $u = v$, or $u \in V$ and $v \in V'$ and u, v correspond to two distinct vertices of G which are adjacent in G. Let us recall that the *dimension* $\dim(\mathcal{P})$ of a poset $\mathcal{P}$ is the minimum number of linear orders whose intersection is exactly $\mathcal{P}$.

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[1]Graphs in this paper are finite, simple, and undirected.

[2]It is sometimes considered that complete graphs have boxicity 0, but we find this confusing and hence do not take this convention. However we made sure that all results from papers following this convention that are quoted in this article are used safely in our proofs.

Felsner, Li, and Trotter [8] recently showed that $\dim(\mathcal{P}_G) \leqslant 5$ for every outerplanar graph G, and that $\dim(\mathcal{P}_G) \leqslant 8$ for every planar graph G. They also proved that $\dim(\mathcal{P}_G) \leqslant \frac{3}{2}\chi_a(G)(\chi_a(G)-1)$ for every graph G with $\chi_a(G) \geqslant 2$, where $\chi_a(G)$ denotes the *acyclic chromatic number* of G (the least integer k so that G can be properly colored with k colors, in such way that every two color classes induce a forest). Using a result of Alon, Mohar, and Sanders [5], this implies that the dimension of $\mathcal{P}_G$ is $O(g^{8/7})$ when G is embeddable in a surface of Euler genus g. (We recall that the *Euler genus* of a surface Σ is defined as twice its genus if Σ is orientable, as its non-orientable genus otherwise.) At the end of their paper, the authors of [8] write that it is likely that the $O(g^{8/7})$ upper bound on $\dim(\mathcal{P}_G)$ could be improved to $O(g)$.

In this note, we first observe that the boxicity can also be bounded from above by a function of the acyclic chromatic number, namely $\mathrm{box}(G) \leqslant \chi_a(G)(\chi_a(G)-1)$ for every graph G with $\chi_a(G) \geqslant 2$. Next, using a result of Adiga, Bhowmick, and Chandran [2], we relate $\dim(\mathcal{P}_G)$ to $\mathrm{box}(G)$ by observing that $\dim(\mathcal{P}_G) \leqslant 2\,\mathrm{box}(G) + \chi(G) + 4$ for every graph G (here $\chi(G)$ denotes the chromatic number of G). Then we prove that $\mathrm{box}(G) \leqslant 9g+3$ for every graph embeddable in a surface of Euler genus g (this upper bound can be reduced to $\frac{9}{2}g+3$ if the surface is orientable). This implies a $O(g)$ upper bound on $\dim(\mathcal{P}_G)$, thus confirming the suggestion of Felsner *et al.* [8] mentioned above. We also consider more closely the case of toroidal graphs and show that every such graph has boxicity at most 7, while there are toroidal graphs with boxicity 4. We conclude the paper with several remarks and open problems about the boxicity of graphs on surfaces.

2. Boxicity and acyclic coloring

It can be deduced from [2], or directly from [21], that the graph obtained from the complete graph K_n by subdividing each edge precisely once has boxicity $\Theta(\log \log n)$. This graph is 2-degenerate, hence the boxicity of a graph cannot be bounded from above by a function of its degeneracy[3] or chromatic number alone. However, the boxicity can be bounded by a function of the *acyclic* chromatic number, as we now show.

For a graph G and a subset X of vertices of G, we let $G[X]$ denote the subgraph of G induced by X, and let $G \setminus X$ denote the graph obtained from G by removing all vertices in X.

Consider a graph G and a subset X of vertices of G together with a mapping I from X to subintervals of some interval $[l,r]$ of $\mathbb{R}$. We call *the canonical extension of I to G* the interval graph I' defined by mapping the vertices of X to their corresponding intervals in I, and all other vertices of G to the interval $[l,r]$. Observe that if the interval graph defined by I is a supergraph of $G[X]$, then the canonical extension of I to G is a supergraph of G.

Lemma 1. $\mathrm{box}(G) \leqslant \chi_a(G)(\chi_a(G)-1)$ *for every graph G with* $\chi_a(G) \geqslant 2$.

Proof. Consider an acyclic coloring c of G with $k \geqslant 2$ colors. For any two distinct colors $i < j$, we consider the graph $G_{i,j}$ obtained from G by adding an edge between every pair of non-adjacent vertices u, v such that at most one of u, v is colored i or j.

We first show that $G = \bigcap_{i<j} G_{i,j}$. Since all $G_{i,j}$'s are supergraphs of G, we only need to show that for every pair u, v of non-adjacent vertices in G, there exist $i < j$ so that u and v are non-adjacent in $G_{i,j}$. Exchanging u and v if necessary, we may assume that $c(u) \leqslant c(v)$. If $c(u) < c(v)$ then $G_{c(u),c(v)}$ does not contain the edge uv. If $c(u) = c(v)$ then $G_{c(u),k}$ (if $c(u) < k$) or $G_{1,c(u)}$ (if $c(u) = k$) does not contain the edge uv.

[3] A graph G is *k-degenerate* if every subgraph of G has a vertex with degree at most k. The *degeneracy* of G is the smallest k such that G is k-degenerate.

We now prove that for every $i < j$, $\mathrm{box}(G_{i,j}) \leqslant 2$. This implies that $\mathrm{box}(G) \leqslant 2\binom{k}{2} = k(k-1)$. Observe that since c is an acyclic coloring of G, the subgraph $H_{i,j}$ of G induced by the vertices colored i or j is a forest, and thus has boxicity at most two (this follows from [19] but can also be proven independently fairly easily). Let $I_{i,j}$ and $J_{i,j}$ be two interval graphs on the vertex set $V(H_{i,j})$ such that $H_{i,j} = I_{i,j} \cap J_{i,j}$. Then $G_{i,j}$ is precisely the intersection of the canonical extensions of $I_{i,j}$ and $J_{i,j}$ to G, and thus has boxicity at most two. □

It follows from [15] and [12, 22] that graphs with no K_t-minor have acyclic chromatic number $O(t^2 \log t)$. Lemma 1 then implies that their boxicity is $O(t^4 \log^2 t)$.

Alon, Mohar and Sanders [5] proved that graphs embeddable in a surface of Euler genus g have acyclic chromatic number $O(g^{4/7})$. Using Lemma 1, this implies that such graphs have boxicity $O(g^{8/7})$. In the next section we show that the boxicity is bounded by a linear function of g.

3. Graphs on surfaces

We will prove that the boxicity of a graph embeddable in a surface of Euler genus g is $O(g)$, by induction on g. Before we do so, we need four simple lemmas that will be useful throughout the induction.

Lemma 2. *Let $G = (V, E)$ be a graph and let $X \subseteq V$ be such that $G[X]$ contains k pairwise disjoint pairs of non-adjacent vertices. Then* $\mathrm{box}(G) \leqslant \mathrm{box}(G \setminus X) + |X| - k$.

Proof. Let v_{2i-1}, v_{2i} $(1 \leqslant i \leqslant k)$ be k pairwise disjoint pairs of non-adjacent vertices in $G[X]$, and let $v_{2i+1}, \ldots, v_\ell$ be the remaining vertices of X. Consider t interval graphs $I_1, \ldots, I_t$ on the vertex set $V \setminus X$ such that $G \setminus X = \bigcap_{i=1}^t I_i$. We will prove that $\mathrm{box}(G) \leqslant t + \ell - k$.

For every pair v_{2i-1}, v_{2i} $(1 \leqslant i \leqslant k)$, we consider the interval graph J_i defined as follows: v_{2i-1} is mapped to $\{0\}$; v_{2i} is mapped to $\{2\}$; the common neighbors of v_{2i-1} and v_{2i} in G are mapped to $[0, 2]$; the remaining neighbors of v_{2i-1} are mapped to $[0, 1]$; the remaining neighbors of v_{2i} are mapped to $[1, 2]$; and the remaining vertices are mapped to $\{1\}$. The graph J_i is clearly a supergraph of G, and every non-neighbor of v_{2i-1} or v_{2i} in G is a non-neighbor of v_{2i-1} or v_{2i} (respectively) in J_i.

Next, for every $i \in \{2k+1, 2k+2, \ldots, \ell\}$, we define an interval graph J_i as follows: v_i is mapped to $\{0\}$; its neighbors in G are mapped to $[0, 1]$, and the remaining vertices are mapped to $\{1\}$. This is a supergraph of G, and every non-edge incident to v_i in G is a non-edge in J_i.

Let $I'_1, \ldots I'_t$ denote the canonical extensions of $I_1, \ldots, I_t$ to G. We claim that G is precisely the intersection of the I'_i's $(1 \leqslant i \leqslant t)$, and the J_i's $(i \in \{1, \ldots, k\} \cup \{2k+1, \ldots, \ell\})$. These graphs are clearly supergraphs of G. Moreover, every non-edge e of G is in one of these graphs, since e is either a non-edge in $G \setminus X$, or is incident to some vertex v_i with $i \in \{1, \ldots, \ell\}$. □

In all subsequent applications of Lemma 2, X will induce a cycle in G. In this case we obtain $\mathrm{box}(G) \leqslant \mathrm{box}(G \setminus X) + 3$ if $|X| = 3$, and $\mathrm{box}(G) \leqslant \mathrm{box}(G \setminus X) + \lceil |X|/2 \rceil$ if $|X| \geqslant 4$.

Lemma 3. *Let $G = (V, E)$ be a graph and let V_1, V_2, X be a partition of V such that no edge of G has an endpoint in V_1 and the other in V_2. Let G_1 be a graph obtained from $G[V_1 \cup X]$ by adding a (possibly empty) set of edges between pairs of vertices from X. Then* $\mathrm{box}(G) \leqslant \mathrm{box}(G_1) + \mathrm{box}(G[V_2 \cup X]) + 1$. *In particular* $\mathrm{box}(G) \leqslant \mathrm{box}(G[V_1 \cup X]) + \mathrm{box}(G[V_2 \cup X]) + 1$.

Proof. Consider k interval graphs $I_1, \ldots, I_k$ on the vertex set $V_1 \cup X$ such that $G_1 = \bigcap_{i=1}^k I_i$, and ℓ interval graphs $J_1, \ldots, J_\ell$ on the vertex set $V_2 \cup X$ such that $G[V_2 \cup X] = \bigcap_{i=1}^\ell J_i$.

Let $I'_1, \dots, I'_k$ be the canonical extensions of $I_1, \dots, I_k$ to G, and let $J'_1, \dots, J'_\ell$ be the canonical extensions of $J_1, \dots, J_\ell$ to G. Finally, let K be the interval graph defined by mapping all vertices of V_1 to $\{0\}$, all vertices of V_2 to $\{1\}$, and all vertices of X to $[0, 1]$.

It is clear that all the I_i's, J_i's and K are supergraphs of G, and that every non-edge of G appears in one of these graphs. Hence, $\mathrm{box}(G) \leqslant k + \ell + 1$. □

We now turn to the boxicity of graphs on surfaces. In this paper, a *surface* is a non-null compact connected 2-manifold without boundary. We refer the reader to the book by Mohar and Thomassen [14] for background on graphs on surfaces.

Consider a graph G embedded in a surface Σ. For simplicity, we use G both for the corresponding abstract graph and for the subset of Σ corresponding to the drawing of G. A cycle C of G is said to be *noncontractible* if C is noncontractible (as a closed curve) in Σ. Also, C is called *surface separating* if C separates Σ in two connected pieces. The *facewidth* $\mathrm{fw}(G)$ of G is the largest integer k such that every noncontractible simple closed curved in Σ meeting G only in vertices intersects at least k vertices of G. If G triangulates Σ then its facewidth is equal to the length of a shortest noncontractible cycle in G. Two cycles of G are *(freely) homotopic* in Σ if there is a continuous deformation mapping one to the other.

The following well-know fact (often called the *3-Path Property*) will be used: If P_1, P_2, P_3 are three internally disjoint paths with the same endpoints in an embedded graph, and P_1, P_2 are such that $P_1 \cup P_2$ is a noncontractible cycle, then at least one of the two cycles $P_1 \cup P_3$, $P_2 \cup P_3$ is also noncontractible (see for instance [14, Proposition 4.3.1]). This implies the following lemma.

Lemma 4. *Suppose that C is a noncontractible cycle of a graph G embedded in a surface. Then there exists a noncontractible induced cycle C' of G with $V(C') \subseteq V(C)$.*

The next two lemmas are standard facts about noncontractible cycles in embedded graphs, see [14, Chapter 4.2].

Lemma 5. *Suppose that C is a noncontractible cycle of a graph G embedded in an orientable surface of Euler genus $g \geqslant 2$. Then each component of $G \setminus V(C)$ is embeddable in an orientable surface of Euler genus $g - 2$.*

Lemma 6. *Suppose that C is a noncontractible cycle of a graph G embedded in a non-orientable surface of Euler genus $g \geqslant 1$. Then each component of $G \setminus V(C)$ is embeddable in a surface of Euler genus $g - 1$.*

Recall that Thomassen [23] proved that $\mathrm{box}(G) \leqslant 3$ for every planar graph G. We are now ready to state and prove the main result of this note, extending Thomassen's bound to general surfaces.

Theorem 7. *Let G be a graph embedded in a surface Σ of Euler genus g. Then* $\mathrm{box}(G) \leqslant \frac{9}{2}\, g + 3$ *if Σ is orientable, and* $\mathrm{box}(G) \leqslant 9g + 3$ *otherwise.*

Proof. We prove the result by induction on g. If $g = 0$ the bound follows from [23], so we can assume that $g \geqslant 1$. We can also assume that G triangulates Σ, since G is an induced subgraph of a triangulation of Σ and the boxicity is monotone by taking induced subgraphs.

First suppose that $\mathrm{fw}(G) \leqslant 5$. Since G is a triangulation, there exists a noncontractible cycle C of length at most 5. Using Lemma 4, we can further assume that C is an induced cycle of G. The boxicity of a graph is clearly the maximum boxicity of its components. Thus, if Σ is orientable, by Lemma 5 and the induction hypothesis, $\mathrm{box}(G \setminus V(C)) \leqslant \frac{9}{2}\,(g - 2) + 3$, and by Lemma 2, we deduce that $\mathrm{box}(G) \leqslant \frac{9}{2}\,(g - 2) + 3 + 3 \leqslant \frac{9}{2}\, g + 3$. If Σ is non-orientable, using Lemma 6 we obtain by induction that $\mathrm{box}(G \setminus V(C)) \leqslant \max\{\frac{9}{2}\,(g-1)+3, 9(g-1)+3\} = 9(g-1)+3$, and by Lemma 2 that $\mathrm{box}(G) \leqslant 9(g - 1) + 3 + 3 \leqslant 9g + 3$.

From now on, we assume that $\mathrm{fw}(G) \geqslant 6$, and we consider a shortest noncontractible cycle C in G. It follows from Lemma 4 that C is an induced cycle (otherwise, we could shorten it). Let V' be the set of vertices from $V(G) \setminus V(C)$ having at least one neighbor in C. Let H be the graph obtained from $G[V' \cup V(C)]$ by adding all possible edges between pairs of vertices from V'. By Lemma 3, we have $\mathrm{box}(G) \leqslant \mathrm{box}(H) + \mathrm{box}(G \setminus V(C)) + 1$. We will prove that $\mathrm{box}(H) \leqslant 8$, which gives $\mathrm{box}(G) \leqslant \mathrm{box}(G \setminus V(C)) + 9$. This in turn implies the theorem since, if Σ is orientable, Lemma 5 and the induction hypothesis imply then that $\mathrm{box}(G \setminus V(C)) \leqslant \frac{9}{2}(g-2)+3$, and hence $\mathrm{box}(G) \leqslant \frac{9}{2}g+3$, while if Σ is non-orientable, Lemma 6 and the induction hypothesis give that $\mathrm{box}(G \setminus V(C)) \leqslant 9(g-1)+3$, implying $\mathrm{box}(G) \leqslant 9g+3$. Therefore, in order to complete the proof, we only need to show that $\mathrm{box}(H) \leqslant 8$.

We remark that every vertex from V' has at most three neighbors in C. More precisely, if some vertex of V' does not belong to one of these four disjoint sets:

- S_1 : the vertices of V' with exactly one neighbor in C;
- S_2 : the vertices of V' with exactly two neighbors in C and such that these vertices are consecutive in C;
- S_3 : the vertices of V' with exactly two neighbors in C and such that these vertices are at distance two in C;
- S_4 : the vertices of V' with exactly three neighbors in C and such that these vertices are consecutive in C;

then, since C has length at least 6, the 3-Path Property implies that G contains a noncontractible cycle that is shorter than C, which is a contradiction.

Let H_1 be the graph obtained from H by adding all possible edges between $S_1 \cup S_3$ and $V(C)$, and let H_2 be the graph obtained from H by adding all possible edges between $S_2 \cup S_4$ and $V(C)$. We clearly have $\mathrm{box}(H) \leqslant \mathrm{box}(H_1) + \mathrm{box}(H_2)$. Now we prove that H_1 and H_2 have boxicity at most 2 and at most 6, respectively, thus completing the proof.

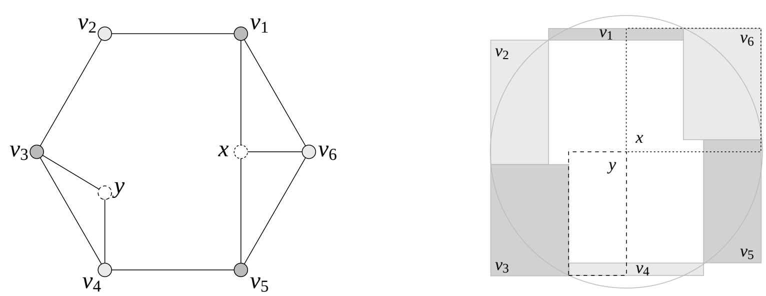

FIGURE 1. An example of the construction for H_1 with $k = 6$. The sets S_1 and S_3 are not depicted to avoid overloading the figure.

Enumerate the vertices of C as $v_1, \ldots, v_k$, in order. First we prove that $\mathrm{box}(H_1) \leqslant 2$ by showing that H_1 can be viewed as the intersection graph of some axis-parallel rectangles in the plane. Fix some small $\epsilon > 0$. For every $j \in \{1, \ldots, k\}$, we define the point $p_j = e^{(2j/k+\epsilon)i\pi}$. These k points are equally distributed on the unit circle, and taking ϵ sufficiently small ensures that no p_j is one of $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$. For every $i \in \{1, \ldots, k\}$, the vertex v_i is mapped to the rectangle with corners p_i, p_{i+1}, where indices are taken modulo k. The vertices of S_2 adjacent to v_{i-1} and v_i are mapped to the rectangle with corners $(0,0)$ and p_i, and the vertices of S_4 adjacent to v_{i-1}, v_i, v_{i+1} are mapped to the smallest rectangle containing $(0,0)$, p_i and p_{i+1}. All the other vertices are mapped to the rectangle with corners $(-1,-1)$ and $(1,1)$. An example of this construction with $k=6$ and each of S_2, S_4 reduced to a singleton is depicted in Figure 1. By construction, rectangles corresponding to vertices in V' all contain the point $(0,0)$, hence V' is a clique in the intersection graph. Since S_1 and S_3 are mapped to the rectangle with corners $(-1,-1)$ and $(1,1)$, the set $S_1 \cup S_3$ is complete to $V(C)$ in that graph. Moreover, for every $v \in S_2 \cup S_4$, the rectangle associated to v intersects precisely the rectangles corresponding to its neighbors in $V(C)$. Therefore, the intersection graph of rectangles is isomorphic to H_1, as desired.

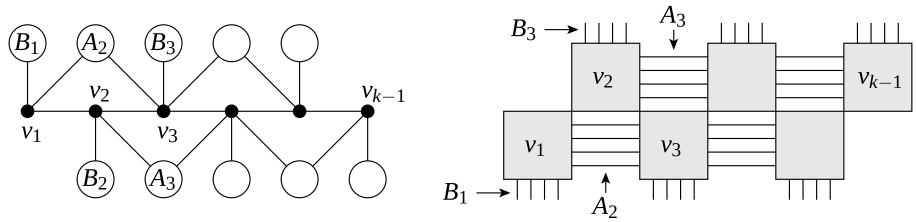

FIGURE 2. The triangle-free planar graph H_2' (left), and a representation of H_2' as intersection graph of axis-parallel rectangles (right).

Now we prove that $\text{box}(H_2) \leqslant 6$. Let H_2' be the graph obtained from $J = H_2[S_1 \cup S_3 \cup V(C)]$ by removing all edges between pairs of vertices of $S_1 \cup S_3$. It follows from [3, Lemma 7] that $\text{box}(J) \leqslant 2\text{box}(H_2')$. Since $\text{box}(H_2) = \text{box}(J)$, it follows that $\text{box}(H_2) \leqslant 2\text{box}(H_2')$. Observe that $H_2' \setminus v_k$ can be obtained from the path $v_1, \ldots, v_{k-1}$ by adding, for every $i \in \{2, \ldots, k-2\}$, some set A_i of vertices of degree two adjacent to v_{i-1} and v_{i+1}, and for every $i \in \{1, \ldots, k-1\}$, some set B_i of vertices of degree one adjacent to v_i. This shows that $H_2' \setminus v_k$ is triangle-free and planar (see Figure 2, left), which implies that it has boxicity at most two by [23] (this can also be proved independently quite easily in this specific case, see Figure 2, right). By Lemma 2, we then have $\text{box}(H_2') \leqslant 3$ and it follows that $\text{box}(H_2) \leqslant 6$, thus completing the proof. □

Theorem 7 implies that toroidal graphs have boxicity at most 12. We improve on this bound by using the following remarkable result of Schrijver [20]: Every graph embedded in the torus with facewidth k contains $\lfloor 3k/4 \rfloor$ vertex-disjoint noncontractible cycles. (Note that on the torus, such cycles are necessarily homotopic.)

Theorem 8. $\text{box}(G) \leqslant 7$ *for every toroidal graph* G.

Proof. Again, we may assume that G triangulates the torus.

Assume first that $\text{fw}(G) \leqslant 5$. Since G is a triangulation, there exists a noncontractible cycle C of length at most 5 such that $G \setminus V(C)$ is planar. Using Lemma 4, we can further assume that C is an induced cycle of G. Then, using Lemma 2 and the result of Thomassen about the boxicity of planar graphs, we deduce that $\text{box}(G) \leqslant 3 + 3 = 6$.

Assume now that $\text{fw}(G) \geqslant 6$. The aforementioned result of Schrijver implies that G contains 4 pairwise vertex-disjoint noncontractible cycles, say C_1, C_2, C_3, C_4 in this order. Because of C_2 and C_4 there are no edges between C_1 and C_3 in G. Further, we may assume by Lemma 4 that C_1 and C_3 are induced cycles in G. (Observe that every noncontractible cycle in $G[V(C_1)]$ or in $G[V(C_3)]$ is again homotopic to the four cycles C_1, C_2, C_3, C_4, because such a cycle is vertex-disjoint from C_2 and C_4.) The removal of C_1 and C_3 cuts the torus into two connected pieces Σ_1 and Σ_2. Let V_i $(i = 1, 2)$ be the set of vertices lying on Σ_i, and set $X = V(C_1) \cup V(C_3)$. Since $G[V_1 \cup X]$ and $G[V_2 \cup X]$ are planar, it follows from Lemma 3 that $\text{box}(G) \leqslant 3 + 3 + 1 = 7$. □

It was proved in [17] that for every $n \geqslant 1$, the graph G_{2n} obtained from K_{2n} by removing a perfect matching has boxicity exactly n. Since G_8 can be embedded on the torus (see Figure 3), there exist toroidal graphs with boxicity four.

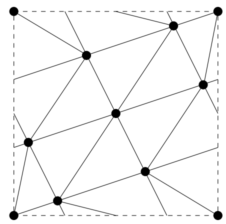

FIGURE 3. A toroidal embedding of the graph obtained from K_8 by removing a perfect matching (the four corners correspond to the same vertex).

Recall that, for a graph $G = (V, E)$, the adjacency poset $\mathcal{P}_G$ of G is defined as the poset $\mathcal{P}_G = (W, \leqslant)$ with $W = V \cup V'$, where V' is a disjoint copy of V, and $u \leqslant v$ if and only if $u = v$, or $u \in V$ and $v \in V'$ and u, v correspond to two distinct vertices of G which are adjacent in G. Let $\mathcal{P}^*_G$ denote the poset obtained from $\mathcal{P}_G$ by adding that $u \leqslant v$ for every $(u, v) \in V \times V'$ such that u and v correspond to the same vertex of G. Adiga, Bhowmick, and Chandran [2] recently proved that $\dim(\mathcal{P}^*_G)/2 - 2 \leqslant \mathrm{box}(G) \leqslant 2\dim(\mathcal{P}^*_G)$ for every graph G. Using this result, we may bound the dimension of $\mathcal{P}_G$ as follows.

Theorem 9. $\dim(\mathcal{P}_G) \leqslant 2\,\mathrm{box}(G) + \chi(G) + 4$ *for every graph* $G = (V, E)$.

Proof. We have that $\dim(\mathcal{P}^*_G) \leqslant 2\,\mathrm{box}(G) + 4$ by the aforementioned result of Adiga *et al.* [2], thus it is enough to show that $\dim(\mathcal{P}_G) \leqslant \dim(\mathcal{P}^*_G) + \chi(G)$. Consider a (proper) coloring $V_1, V_2, \ldots, V_k$ of G with $k = \chi(G)$ colors, and let $V'_1, V'_2, \ldots, V'_k$ denote the corresponding partition of V'. For $i \in \{1, \ldots, k\}$, let $\mathcal{L}_i = (W, \leqslant_i)$ be an arbitrary linear order satisfying that

$$V_1 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_k \leqslant_i V'_i \leqslant_i V_i \leqslant_i V'_1 \cup \cdots \cup V'_{i-1} \cup V'_{i+1} \cup \cdots \cup V'_k.$$

(Here $A \leqslant_i B$ means that $u \leqslant_i v$ for every $u \in A$ and $v \in B$.) Then it is easily checked that each $\mathcal{L}_i$ is a linear extension of $\mathcal{P}_G$, and that the intersection of these k linear orders with $\mathcal{P}^*_G$ is exactly $\mathcal{P}_G$. It follows that $\dim(\mathcal{P}_G) \leqslant \dim(\mathcal{P}^*_G) + k$, as desired. □

Corollary 10. *Let G be a graph embeddable in a surface Σ of Euler genus g. Then* $\dim(\mathcal{P}) \leqslant 9g + \frac{1}{2}(27 + \sqrt{1 + 24g})$ *if Σ is orientable, and* $\dim(\mathcal{P}) \leqslant 18g + \frac{1}{2}(27 + \sqrt{1 + 24g})$ *otherwise.*

Proof. For $g > 0$, this follows from Theorems 7 and 9, and Heawood's upper bound on the chromatic number of G, namely $\chi(G) \leqslant \frac{1}{2}(7 + \sqrt{1 + 24g})$. (For $g = 0$, the bound is of course implied by Thomassen's result for planar graphs.) □

This confirms what Felsner, Li, and Trotter [8] suggested as an improvement of their result.

4. Open questions

The first question is whether the bounds obtained in Section 3 are best possible. We believe that the boxicity of graphs embeddable in a surface of Euler genus g should rather be $O(\sqrt{g})$. Since the complete graph K_{2n} with a perfect matching removed has boxicity n, this would be optimal. This example also shows that the boxicity of graphs with no K_t-minor can be linear in t, while we only know a $O(t^4 \log^2 t)$ upper bound (see the remark after Lemma 1).

Kawarabayashi and Mohar [11] proved that for every fixed surface Σ, graphs embeddable in Σ with sufficiently large edgewidth are acyclically 7-colorable. It then follows from Lemma 1 that these graphs have boxicity at most 42. We believe that the following stronger statement is true:

Conjecture 11. *For every fixed surface Σ there exists an integer e_Σ so that every graph G embeddable on Σ with edgewidth at least e_Σ has boxicity at most three.*

It follows from a theorem of Thomassen [23] that triangle-free planar graphs have boxicity at most two. Since there exist trees that are not interval graphs, a natural question is whether, for every surface Σ, graphs embeddable in Σ and having sufficiently large girth (length of a shortest cycle) have boxicity at most two. We prove that the following slightly weaker statement holds:

Theorem 12. *For every fixed surface Σ there exists some integer g_Σ such that every graph with girth at least g_Σ embeddable in Σ has boxicity at most 4.*

Proof. It is well-known (see [4]) that there exists an integer g_Σ such that the vertex set of every graph G embeddable on Σ and having girth at least g_Σ can be partitioned into a forest F and a stable set S, in such way that every two vertices of S are at distance at least three in G.

Consider the graph G_1 obtained from G by adding an edge between every pair of non-adjacent vertices u, v, such that at least one of u, v is in S. As remarked in the proof of Lemma 1, $\mathrm{box}(G_1) \leqslant 2$. Observe now that every vertex of F has at most one neighbor in the stable set S. Using this property, it can be deduced from [7, Proof of Theorem 1] that the graph G_2 obtained from G by adding all possible edges between pairs of vertices of F has boxicity at most two. Since $G = G_1 \cap G_2$, it follows that $\mathrm{box}(G) \leqslant 4$. □

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