AFFINE OPEN SUBSETS IN $\mathbb{A}^3$ WITHOUT THE CANCELLATION PROPERTY

ADRIEN DUBOULOZ

Abstract. We give families of examples of principal open subsets of the affine space $\mathbb{A}^3$ which do not have the cancellation property. We show as a by-product that the cylinders over Koras-Russell threefolds of the first kind have a trivial Makar-Limanov invariant.

Introduction

The generalized Cancellation Problem asks if two algebraic varieties X and Y with isomorphic cylinders $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are isomorphic themselves. Although the answer turns out to be affirmative for a large class of varieties including the case when one of the varieties is the affine plane $\mathbb{A}^2$ [10, 14], counter-examples exists for affine varieties in any dimension≥ 2, and the particular case when one of the two varieties is an affine space $\mathbb{A}^n$, $n \geq 3$, still remains a widely open problem.

The first counter-example for complex affine varieties has been constructed by Danielewski [1] in 1989: he exploited the fact that the non isomorphic affine surfaces $S_1 = \{xz = y^2 - 1\}$ and $S_2 = \{x^2 z = y^2 - 1\}$ in $\mathbb{A}^3_{\mathbb{C}}$ can be equipped with free actions of the additive group $\mathbb{G}_a$ admitting geometric quotients in the form of non trivial $\mathbb{G}_a$-bundles $\rho_i : S_i \to \tilde{\mathbb{A}}^1$, $i = 1, 2$ over the affine line with a double origin. It then follows that the fiber product $S_1 \times_{\tilde{\mathbb{A}}^1} S_2$ inherits simultaneously the structure of a $\mathbb{G}_a$-bundle over S_1 and S_2 via the first and the second projection respectively, but since S_1 and S_2 are both affine, the latter are both trivial, providing isomorphisms $S_1 \times \mathbb{A}^1 \simeq S_1 \times_{\tilde{\mathbb{A}}^1} S_2 \simeq S_2 \times \mathbb{A}^1$. Since then, Danielewski's fiber product trick has been the source of many new counter-examples in any dimension [7, 3, 8, 5], some of these being very close to affine spaces either from an algebraic or a topological point of view.

However, a counter-example over the field of real numbers was constructed earlier by Hochster [9] using the algebraic counterpart of the classical fact from differential geometry that the tangent bundle of the real sphere S^2 is non trivial but 1-stably trivial. His argument actually applies more generally to the situation when a finitely generated domain R over a field k admits a non trivial projective module M of rank $n - 1 \geq 1$ such that $M \oplus R \simeq R^{\oplus n} = R^{\oplus n-1} \oplus R$. Indeed, these hypotheses immediately imply that the varieties $X = \mathrm{Spec}_R(\mathrm{Sym}\,(M))$ and $Y = \mathrm{Spec}(R[x_1, \ldots, x_n])$ are not isomorphic as schemes over $Z = \mathrm{Spec}(R)$ while their cylinders are. Of course, there is no reason in general that X and Y are not isomorphic as k-varieties, but this holds for instance when Z does not admit any dominant morphism from an affine space $\mathbb{A}^n_k$ since then any isomorphism between X and Y necessarily descends to an automorphism of Z ([10, 2]). Recently, Jelonek [11] gave revival to Hochster idea by constructing families of examples of non uniruled affine open subsets of affine spaces of any dimension ≥ 8 with 1-stably trivial but non trivial vector bundles, which fail the cancellation property.

While affine affine open subsets of affine spaces of dimension ≤ 2 always have the cancellation property (see e.g. *loc.cit*), we derive in this note from a variant of Danielewski's fiber product trick that cancellation already fails for suitably chosen principal open subsets of $\mathbb{A}^3$.

As an application of our construction we also obtain that all cylinders over Koras-Russell threefolds $X_{d,k,l} = \{x^d z = y^l + x - t^k = 0\} \subset \mathbb{A}^4$, $d \geq 2$ and $2 \leq l < k$ relatively prime [12], have a trivial Makar-Limanov invariant [13].

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1. Principal open subsets in $\mathbb{A}^3$ without the cancellation property

For every $d \geq 1$ and $l \geq 2$, we denote by $B_{d,l}$ the surface in $\mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}\,[x,y,z])$ defined by the equation $f_{d,l} = y^l + x - x^d z = 0$ and by $U_{d,l} = \mathbb{A}^3 \setminus B_{d,l} \simeq \mathrm{Spec}(\mathbb{C}\,[x,y,z]_{f_{d,l}})$ its open complement. By construction, $U_{d,l}$ comes equipped with a flat isotrivial fibration $f_{d,l}\,|_{U_{d,l}}\colon U_{d,l} \to \mathbb{A}^1_* = \mathrm{Spec}(\mathbb{C}\,[t^{\pm 1}])$ with closed fibers isomorphic to the surface $S_{d,l} \subset \mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}\,[X,Y,Z])$ defined by the equation $X^d Z = Y^l + X - 1$. A surface $S_{d,l}$ having no non constant invertible function, an isomorphism $\varphi : U_{d,l} \xrightarrow{\sim} U_{d',l'}$ necessarily maps closed fibers of $f_{d,l}$ isomorphically onto that of $f_{d',l'}$. But since $S_{d,l}$ is isomorphic to $S_{d',l'}$ if and only if $(d',l') = (d,l)$ (see e.g. [6, Theorem 3.2 and Proposition 3.6]), it follows that the threefolds $U_{d,l}$, $d \geq 1$, $l \geq 2$, are pairwise non isomorphic. In contrast, we have the following result:

Theorem 1. *For every fixed $l \geq 2$, the fourfolds $U_{d,l} \times \mathbb{A}^1$, $d \geq 1$, are all isomorphic.*

Proof. We exploit the fact that every $U_{d,l}$ admits a free $\mathbb{G}_a$-action defined by the locally nilpotent derivation $x^d\partial_y + ly^{l-1}\partial_z$ of its coordinate ring $\mathbb{C}\,[x,y,z]_{f_{d,l}}$. A free $\mathbb{G}_a$-action being locally trivial in the étale topology, it follows that a geometric quotient $\nu_{d,l} : U_{d,l} \to \mathfrak{S}_{d,l} = U_{d,l}/\mathbb{G}_a$ exists in the form of an étale locally trivial $\mathbb{G}_a$-bundle over a certain algebraic space $\mathfrak{S}_{d,l}$. Then it is enough to show that for every fixed $l \geq 2$, the algebraic spaces $\mathfrak{S}_{d,l}$ are all isomorphic, say to a fixed algebraic space $\mathfrak{S}_l$. Indeed, if so, then for every $d, d' \geq 1$, the fiber product $U_{d,l} \times_{\mathfrak{S}_l} U_{d',l}$ will be simultaneously a $\mathbb{G}_a$-bundle over $U_{d,l}$ and $U_{d',l}$ via the first and second projection respectively whence will be simultaneously isomorphic to the trivial $\mathbb{G}_a$-bundles $U_{d,l} \times \mathbb{A}^1$ and $U_{d',l} \times \mathbb{A}^1$ as $U_{d,l}$ and $U_{d',l}$ are both affine.

The algebraic spaces $\mathfrak{S}_{d,l}$ can be described explicitly as follows: one checks that the isotrivial fibration $f_{d,l} : U_{d,l} \to \mathbb{A}^1_*$ becomes trivial on the Galois étale cover $\xi_l : \mathbb{A}^1_* = \mathrm{Spec}(\mathbb{C}\,[u^{\pm 1}]) \to \mathbb{A}^1_*$, $u \mapsto t = u^l$, with isomorphism $\Phi_{d,l} : S_{d,l} \times \mathbb{A}^1_* \xrightarrow{\sim} U_{d,l} \times_{\mathbb{A}^1_*} \mathbb{A}^1_*$ given by $(X,Y,Z,u) \mapsto (u^l X, uY, u^{(1-d)l}Z, u)$. The group μ_l of l-th roots of unity acts freely on $S_{d,l} \times \mathbb{A}^1_*$ by $\varepsilon \cdot (X,Y,Z,u) = \big(X, \varepsilon^{-1}Y, Z, \varepsilon u\big)$ and the μ_l-invariant morphism $\pi_{d,l} = \mathrm{pr}_1 \circ \Phi_{d,l} : S_{d,l} \times \mathbb{A}^1_* \to U_{d,l}$ descends to an isomorphism $(S_{d,l} \times \mathbb{A}^1_*)/\mu_l \simeq U_{d,l}$. The $\mathbb{G}_a$-action on $U_{d,l}$ lifts via the proper étale morphism $\pi_{d,l}$ to the free $\mathbb{G}_a$-action on $S_{d,l} \times \mathbb{A}^1_*$ commuting with that of μ_l defined by the locally nilpotent derivation $u^{ld-1}(X^d\partial_Y + lY^{l-1}\partial_Z)$ of its coordinate ring $\mathbb{C}\,[X,Y,Z] \,/\, \big(X^dZ - Y^l - X + 1\big)\,[u^{\pm 1}]$. The principal divisor $\{X = 0\}$ of $S_{d,l} \times \mathbb{A}^1_*$ is $\mathbb{G}_a$-invariant and it decomposes into the disjoint union of irreducible divisors $D_\eta = \{X = Y - \eta = 0\}_{\eta \in \mu_l} \simeq \mathrm{Spec}(\mathbb{C}\,[Z]\,[u^{\pm 1}])$ on which μ_l acts by $D_\eta \ni (Z,u) \mapsto (Z, \varepsilon u) \in D_{\varepsilon\eta}$. Now a similar argument as in [7, Lemma 1.2] implies that for every $\eta \in \mu_l$, the $\mathbb{G}_a$-invariant morphism $\mathrm{pr}_X \times \mathrm{id} : S_{d,l} \times \mathbb{A}^1_* \to \mathbb{A}^1 \times \mathbb{A}^1_*$ restricts on $(S_{d,l} \times \mathbb{A}^1_*) \setminus \bigcup_{\varepsilon \in \mu_l \setminus \{\eta\}} D_\varepsilon$ to a trivial $\mathbb{G}_a$-bundle over $\mathbb{A}^1 \times \mathbb{A}^1_*$. Letting $C\,(l)$ be the scheme over $\mathbb{A}^1 = \mathrm{Spec}\,(\mathbb{C}\,[X])$ obtained by gluing l copies C_η, $\eta \in \mu_l$, of $\mathbb{A}^1 = \mathrm{Spec}\,(\mathbb{C}\,[X])$ outside their respective origins, it follows that $\mathrm{pr}_X \times \mathrm{id}$ factors through a μ_l-equivariant $\mathbb{G}_a$-bundle $\rho_{d,l} \times \mathrm{id} : S_{d,l} \times \mathbb{A}^1_* \to C\,(l) \times \mathbb{A}^1_*$, where μ_l acts freely on $C\,(l) \times \mathbb{A}^1_*$ by $C_\eta \times \mathbb{A}^1_* \ni (X,u) \mapsto (X, \varepsilon u) \in C_{\varepsilon\eta} \times \mathbb{A}^1_*$.

A quotient $(C(l) \times \mathbb{A}^1_*)/\mu_l$ exist in the category of algebraic spaces in the form of a principal μ_l-bundle $\sigma_l : C(l) \times \mathbb{A}^1_* \to \mathfrak{S}_l$, and the above description implies that $\rho_{d,l} \times \mathrm{id}$ descends to an étale locally trivial $\mathbb{G}_a$-bundle $\tilde{\nu}_{d,l} : U_{d,l} \to \mathfrak{S}_l$ for which the diagram

$$\begin{array}{ccc} S_{d,l} \times \mathbb{A}^1_* & \xrightarrow{\pi_{d,l}} & U_{d,l} \simeq (S_{d,l} \times \mathbb{A}^1_*)/\mu_l \\ \downarrow {\scriptstyle \rho_{d,l} \times \mathrm{id}} & & \downarrow {\scriptstyle \tilde{\nu}_{d,l}} \\ C(l) \times \mathbb{A}^1_* & \xrightarrow{\sigma_l} & \mathfrak{S}_l, \end{array}$$

is cartesian. By virtue of the universal property of categorical quotients one has necessarily $\mathfrak{S}_{d,l} \simeq \mathfrak{S}_l$ for every $d \geq 1$. In particular, the isomorphy type of $\mathfrak{S}_{d,l}$ depends only on l, which completes the proof. □

Remark 2. The algebraic spaces $\mathfrak{S}_l = (C\,(l) \times \mathbb{A}^1_*)/\mu_l$, $l \geq 2$, considered in the proof above cannot be schemes: indeed, otherwise the image in $\mathfrak{S}_l$ of the point $(0,1) \in C_1 \times \mathbb{A}^1_* \subset C\,(l) \times \mathbb{A}^1_*$ would have a Zariski open affine neighborhood V. But then the inverse image of V by the finite étale cover $\sigma_l : C\,(l) \times \mathbb{A}^1_* \to \mathfrak{S}_l$ would be a μ_l-invariant affine open neighborhood of $(0,1)$ in $C\,(l) \times \mathbb{A}^1_*$, which is absurd since $(0,1)$ does not even have a separated μ_l-invariant open neighborhood in $C(l)\times\mathbb{A}^1_*$. This implies in turn that the free $\mathbb{G}_a$-action on $U_{d,l}$ defined by the locally nilpotent derivation $x^d\partial_y+ly^{l-1}\partial_z$ is not locally trivial in the Zariski topology. In contrast, the latter property holds for its lift to $S_{d,l}\times\mathbb{A}^1_*$ via the étale Galois cover $\pi_{d,l}:S_{d,l}\times\mathbb{A}^1_*\to U_{d,l}$.

Remark 3. In Danielewski's construction for the surfaces $S_i=\{x^iz=y^2-1\}\subset\mathbb{A}^3$, $i=1,2$, the geometric quotients $S_i/\mathbb{G}_a\simeq\tilde{\mathbb{A}}^1$, $i=1,2$, were obtained from the categorical quotients $S_i//\mathbb{G}_a=\mathrm{Spec}(\mathbb{C}[x])$ taken in the category of affine schemes by replacing the origin by two copies of itself, one for each orbit in the zero fiber of the quotient morphism $q=\mathrm{pr}_x:S_i\to\mathbb{A}^1$. For the threefolds $U_{d,l}$, the difference between the quotients $U_{d,l}//\mathbb{G}_a=\mathrm{Spec}(\mathbb{C}[x,y,z]_{f_{d,l}}^{\mathbb{G}_a})$ taken in the category of (affine) schemes and the geometric quotients $\mathfrak{S}_l=U_{d,l}/\mathbb{G}_a$ is very similar : indeed, we may identify $U_{d,l}$ with the closed subvariety of $\mathbb{A}^3\times\mathbb{A}^1_*=\mathrm{Spec}(\mathbb{C}[x,y,z][t^{\pm1}])$ defined by the equation $x^dz=y^l+x-t$ in such a way that $f_{d,l}:U_{d,l}\to\mathbb{A}^1_*$ coincides with the projection $\mathrm{pr}_t\mid_{U_{d,l}}$. Then, the kernel of the locally nilpotent derivation $x^d\partial_y+ly^{l-1}\partial_z$ of the coordinate ring of $U_{d,l}$ coincides with the subalgebra $\mathbb{C}[x,t^{\pm1}]$ and so, the $\mathbb{G}_a$-invariant morphism $q=\mathrm{pr}_{x,t}:U_{d,l}\to\mathbb{A}^1\times L=\mathrm{Spec}(\mathbb{C}[x][t^{\pm1}])$ is a categorical quotient in the category of affine schemes. One checks easily that q restricts to a trivial $\mathbb{G}_a$-bundle over the principal open subset $\{x\neq0\}$ of $\mathbb{A}^1\times\mathbb{A}^1_*$ whereas the inverse image of the punctured line $\{x=0\}\simeq L$ is isomorphic to $\tilde{L}\times\mathbb{A}^1=\mathrm{Spec}(\mathbb{C}[y,t^{\pm1}]/(y^l-t)[z])$ where $\mathbb{G}_a$ acts by translations on the second factor. So we may interpret the geometric quotient $\mathfrak{S}_l=U_{d,l}/\mathbb{G}_a$ as being obtained from $U_{d,l}//\mathbb{G}_a=\mathbb{A}^1\times L$ by replacing the punctured line $\{x=0\}\simeq L$ not by l disjoint copies of itself but, instead, by the total space $\tilde{L}$ of the nontrivial étale Galois cover $\mathrm{pr}_t:\tilde{L}\to L$.

The Koras-Russell threefolds $X_{d,k,l}$ are smooth complex affine varieties defined by equations of the form $x^dz=y^l+x-t^k$, where $d\geq2$ and $2\leq l<k$ are relatively prime.[1] While all diffeomorphic to the euclidean space $\mathbb{R}^6$, none of these threefold is algebraically isomorphic to the affine $\mathbb{A}^3$. Indeed, it was established by Kaliman and Makar-Limanov [13, 12] that they have fewer algebraic $\mathbb{G}_a$-actions than the affine space $\mathbb{A}^3$: the subring $\mathrm{ML}(X_{d,k,l})$ of their coordinate ring consisting of regular functions invariant under all algebraic $\mathbb{G}_a$-actions on $X_{d,k,l}$ is equal to the polynomial ring $\mathbb{C}[x]$, while $\mathrm{ML}\left(\mathbb{A}^3\right)$ is *trivial*, consisting of constants only. However, it was observed by the author in [4] that the Makar-Limanov invariant ML fails to distinguish the cylinder over the so-called Russell cubic $X_{2,2,3}$ from the affine space $\mathbb{A}^4$. This phenomenon holds more generally for cylinders over all Koras-Russell threefolds $X_{d,k,l}$:

Corollary 4. *All the cylinders $X_{d,k,l}\times\mathbb{A}^1$ have a trivial Makar-Limanov invariant.*

Proof. We consider $X_{d,k,l}\times\mathbb{A}^1$ as the subvariety of $\mathrm{Spec}(\mathbb{C}[x,y,z,t][v])$ defined by the equation $f_{d,l}-t^k=0$. Since $\mathrm{ML}(X_{d,k,l}\times\mathbb{A}^1)\subset\mathrm{ML}(X_{d,k,l})=\mathbb{C}[x]$, it is enough to construct a locally nilpotent derivation of $\mathbb{C}[x,y,z][v]/(f_{d,l}-t^k)$ which does not have x in its kernel. One checks easily that $\mathrm{ML}(U_{1,l})=\mathbb{C}[f_{1,l}^{\pm1}]$ is the intersection of the kernels of the locally nilpotent derivations $x\partial_y+ly^{l-1}\partial_z$ and $ly^{l-1}\partial_x+(z-1)\partial_y$ of $\mathbb{C}[x,y,z]_{f_{1,l}}$. Theorem 1 above implies in particular that $\mathrm{ML}(U_{d,l}\times\mathbb{A}^1)\simeq\mathrm{ML}(U_{1,l}\times\mathbb{A}^1)=\mathbb{C}[f_{1,l}^{\pm1}]$ and so, there exists a locally nilpotent derivation $\delta_{d,l}$ of $\Gamma(U_{d,l}\times\mathbb{A}^1,\mathcal{O}_{U_{d,l}\times\mathbb{A}^1})=\mathbb{C}[x,y,z]_{f_{d,l}}[v]$ which does not have x in its kernel. Up to multiplying it by a suitable power of $f_{d,l}\in\mathrm{Ker}(\delta_{d,l})$, we may further assume that $\delta_{d,l}$ is the extension to $\mathbb{C}[x,y,z]_{f_{d,l}}[v]$ of a locally nilpotent derivation of $\mathbb{C}[x,y,z][v]$ which has $f_{d,l}$ but not x in its kernel. This implies in particular that $B_{d,l}\times\mathbb{A}^1=\mathrm{Spec}(\mathbb{C}[x,y,z]/(f_{d,l})[v])$ is invariant under the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^4=\mathrm{Spec}(\mathbb{C}[x,y,z][v])$. The projection $p=\mathrm{pr}_{x,y,z,v}:X_{d,k,l}\times\mathbb{A}^1\to\mathbb{A}^4$ being a finite Galois cover with branch locus $B_{d,l}\times\mathbb{A}^1$, it follows that the $\mathbb{G}_a$-action on $\mathbb{A}^4$ lifts to a one on $X_{d,k,l}\times\mathbb{A}^1$ for which $p:X_{d,k,l}\times\mathbb{A}^1\to\mathbb{A}^4$ is $\mathbb{G}_a$-equivariant. By construction, the corresponding locally nilpotent derivation of $\mathbb{C}[x,y,z][v]/(f_{d,l}-t^k)$ does not have x in its kernel. □

In the proof above, we used the following classical fact that we include here because of a lack of an appropriate reference.

[1] These are called Koras-Russell threefolds of the first kind in [15].

Lemma 5. *Let X be a variety defined over a field of characteristic zero and equipped with a non trivial $\mathbb{G}_a$-action, let Z be a normal variety and let $p : Z \to X$ be a finite surjective morphism. Suppose that there exists a $\mathbb{G}_a$-invariant affine open subvariety U of X over which p restricts to an étale morphism. Then there exists a unique $\mathbb{G}_a$-action on Z for which $p : Z \to X$ is a $\mathbb{G}_a$-equivariant morphism.*

Proof. The induced $\mathbb{G}_a$-action on the invariant affine open subvariety U of X is determined by a locally nilpotent derivation ∂ of $\Gamma(U, \mathcal{O}_U)$. Since $p : p^{-1}(U) \to U$ is étale and proper, $p^{-1}(U)$ is an affine open subvariety of Z and ∂ lifts in a unique way to a derivation of $\Gamma(p^{-1}(U), \mathcal{O}_{p^{-1}(U)})$ which is again locally nilpotent by virtue of [17]. By construction, the latter defines a $\mathbb{G}_a$-action on $p^{-1}(U)$ for which the restriction of p to $p^{-1}(U)$ is equivariant. Now the assertion follows from [16, Lemma 6.1] which guarantees that the $\mathbb{G}_a$-action on $p^{-1}(U)$ can be uniquely extended to a one on Z with the desired property. □

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CNRS, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UNIVERSITÉ DE BOURGOGNE, 9 AVENUE ALAIN SAVARY - BP 47870, 21078 DIJON CEDEX, FRANCE
E-mail address: adrien.dubouloz@u-bourgogne.fr