Intertwining of simple characters in GL(n)

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Abstract. Let F be a non-Archimedean local field and let G be the general linear group $G = \mathrm{GL}_n(F)$. Let θ_1, θ_2 be simple characters in G. We show that θ_1 intertwines with θ_2 if and only if θ_1 is endo-equivalent to θ_2. We also show that any simple character in G is a G-type.

Let F be a non-Archimedean local field and let $G = \mathrm{GL}_n(F)$, for some $n \geqslant 1$. Following [1], the category $\mathrm{Rep}\, G$ of smooth complex representations of G decomposes as a direct sum of indecomposable blocks,

$$\mathrm{Rep}\, G = \coprod_{\mathfrak{s} \in \mathcal{B}(G)} \mathrm{Rep}_{\mathfrak{s}}\, G,$$

indexed by a certain set $\mathcal{B}(G)$. Let $\mathfrak{S}$ be a finite subset of $\mathcal{B}(G)$. As in [7], an $\mathfrak{S}$-type in G is an irreducible smooth representation ρ, of some compact open subgroup of G, with the property that an irreducible smooth representation of G contains ρ if and only if it lies in $\mathrm{Rep}_{\mathfrak{s}}\, G$, for some $\mathfrak{s} \in \mathfrak{S}$.

One knows [8] how to construct an $\{\mathfrak{s}\}$-type in G, for any $\mathfrak{s} \in \mathcal{B}(G)$. Those types are all built from *simple characters* in groups $\mathrm{GL}_m(F)$, in the sense of [6], for various integers $m \leqslant n$. Here, we return to the simple characters themselves and prove:

Type Theorem. *Let $G = \mathrm{GL}_n(F)$, for some $n \geqslant 1$, and let θ be a simple character in G. The character θ is then an $\mathfrak{S}_\theta$-type in G, for some finite subset $\mathfrak{S}_\theta$ of $\mathcal{B}(G)$.*

The proof, and a description of $\mathfrak{S}_\theta$, are given in §4 below.

We use the Type Theorem to prove a powerful result concerning the intertwining properties of simple characters. As part of the definition, a simple character in $G = \mathrm{GL}_n(F)$ is attached, in an invariant manner, to a hereditary order in the matrix algebra $A = \mathrm{M}_n(F)$. A cornerstone of the theory is the fact ((3.5.11) of [6]) that two simple characters in G, attached to the same order and which intertwine in G, are actually conjugate. This result is taken one further level in [4]. There is a canonical procedure for transferring simple characters between hereditary orders, in possibly different matrix algebras. Given two simple characters θ_i in $\mathrm{GL}_{n_i}(F)$, attached to hereditary orders $\mathfrak{a}_i$, one can find an integer n and a hereditary order $\mathfrak{a}$ in $\mathrm{M}_n(F)$ to which both characters may be transferred. If the transferred characters are conjugate in $\mathrm{GL}_n(F)$, one says they are *endo-equivalent.* One knows that endo-equivalence is an equivalence relation on

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the class of simple characters in all general linear groups. Moreover, endo-equivalent characters attached to the same order are necessarily conjugate. Here, we consider a general pair of simple characters in one group. We prove:

Intertwining Theorem. *Let θ_1, θ_2 be simple characters in $G = \mathrm{GL}_n(F)$. The characters θ_1, θ_2 intertwine in G if and only if they are endo-equivalent.*

To the specialist in the area, these results provide clear and satisfying conclusions to several lines of development, but the non-specialist may wish for more motivation. This is first provided by our examination [5] of the congruence properties of the local Langlands correspondence, where the Intertwining Theorem provides a crucial step in the argument: see [5] 4.3 Lemma.

The results also give a framework in which to investigate representations in more general settings. There is a fully functional theory of simple characters and endo-equivalence spanning all inner forms $\mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$, where D is a finite-dimensional central F-division algebra [2], [9], [10]. However, certain new structures come into play, and one is led to ask how these are reflected or clarified in analogues of our results. In a different direction, one may consider smooth representations of $\mathrm{GL}_n(F)$ over fields of positive characteristic ℓ. Provided ℓ is not the residual characteristic of F, one has an identical general theory of simple characters. One deduces readily that the Intertwining Theorem holds unchanged. However, as Vincent Sécherre reminds us, a simple character need not be a type in this situation.

1. A review of simple characters

Let $\mathfrak{o}_F$ be the discrete valuation ring in F and $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. We choose a smooth character ψ of F which is of level one, in the sense that $\mathrm{Ker}\,\psi$ contains $\mathfrak{p}_F$ but not $\mathfrak{o}_F$.

Let V be an F-vector space of finite dimension and set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Let $\mathfrak{a}$ be a hereditary $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathfrak{p}_\mathfrak{a}$. A *simple character in G,* attached to $\mathfrak{a}$, is one of the following objects. The *trivial* simple character attached to $\mathfrak{a}$ is the trivial character of the group $U^1_\mathfrak{a} = 1+\mathfrak{p}_\mathfrak{a}$: we denote this $1^1_\mathfrak{a}$.

To define a *non-trivial* simple character attached to $\mathfrak{a}$, we recall briefly the definition [6] (1.5.5) of a simple stratum $[\mathfrak{a}, l, 0, \beta]$ in A. First, $\beta \in G$ and the algebra $E = F[\beta]$ is a field. The hereditary order $\mathfrak{a}$ is *E-pure,* in that $x^{-1}\mathfrak{a}x = \mathfrak{a}$ for $x \in E^\times$. The integer l is positive and given by $\beta^{-1}\mathfrak{a} = \mathfrak{p}^l_\mathfrak{a}$. The quadruple $[\mathfrak{a}, l, 0, \beta]$ is then a simple stratum in A if β satisfies a technical condition "$k_0(\beta, \mathfrak{a}) < 0$" *loc. cit.* Since we will not use this directly, we say no more of it.

Following the recipes of [6] 3.1, the simple stratum $[\mathfrak{a}, \beta] = [\mathfrak{a}, l, 0, \beta]$ defines open subgroups $H^1(\beta, \mathfrak{a}) \subset J^1(\beta, \mathfrak{a}) \subset J^0(\beta, \mathfrak{a})$ of the unit group $U_\mathfrak{a} = \mathfrak{a}^\times$, such that $J^1(\beta, \mathfrak{a}) = J^0(\beta, \mathfrak{a}) \cap U^1_\mathfrak{a}$. The choice of ψ then gives rise to a finite set $\mathcal{C}(\mathfrak{a}, \beta, \psi)$ of smooth characters of $H^1(\beta, \mathfrak{a})$, called *simple characters:* see [6] 3.2 for the full definition. The choice of ψ is essentially irrelevant, so we treat it as fixed and henceforth omit it from the notation.

We recall a fundamental property of simple characters attached to a fixed hereditary order [6] (3.5.11).

Intertwining implies conjugacy. *For $i = 1, 2$, let $[\mathfrak{a}, \beta_i]$ be a simple stratum in A and let $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$. If the characters θ_1, θ_2 intertwine in G then they are conjugate by an element of $U_\mathfrak{a}$.*

We shall also need systems of *transfer maps.* Let $[\mathfrak{a}, l, 0, \beta]$ be a simple stratum in A, as before. Suppose we have another F-vector space V' of finite dimension, an F-embedding $\boldsymbol{\iota}$: $F[\beta] \to A' = \mathrm{End}_F(V')$, and an $F[\boldsymbol{\iota}\beta]$-pure hereditary order $\mathfrak{a}'$ in A': any two such embeddings

$\boldsymbol{\iota}$ are $U_{\mathfrak{a}'}$-conjugate, so we are justified in omitting $\boldsymbol{\iota}$ from the notation. There is a unique integer l' such that $[\mathfrak{a}', l', 0, \beta]$ is a simple stratum in A'. There is a canonical bijection

$$\tau^{\beta}_{\mathfrak{a},\mathfrak{a}'} : \mathcal{C}(\mathfrak{a}, \beta) \xrightarrow{\approx} \mathcal{C}(\mathfrak{a}', \beta). \tag{1.1}$$

This family of maps is transitive with respect to the orders: in the obvious notation, we have

$$\tau^{\beta}_{\mathfrak{a},\mathfrak{a}''} = \tau^{\beta}_{\mathfrak{a}',\mathfrak{a}''} \circ \tau^{\beta}_{\mathfrak{a},\mathfrak{a}'}.$$

Full details may be found in [6] section 3.6 and [4] section 8.

Lemma 1. *For $j = 1, 2$, let $[\mathfrak{a}_j, l_j, 0, \beta_j]$ be a simple stratum in $A_j = \mathrm{End}_F(V_j)$.*

(1) *There exists a finite-dimensional F-vector space V, a hereditary order $\mathfrak{a}$ in $A = \mathrm{End}_F(V)$ and a pair of F-embeddings $\boldsymbol{\iota}_j : F[\beta_j] \to A$, such that $\mathfrak{a}$ is $F[\boldsymbol{\iota}_j\beta_j]$-pure, for $j = 1, 2$.*

(2) *Let $\theta_j \in \mathcal{C}(\mathfrak{a}_j, \beta_j)$. The following are equivalent:*

  (a) *There exists a system $(V, \mathfrak{a}, \boldsymbol{\iota}_j)$, as in (1), such that $\tau^{\beta_1}_{\mathfrak{a}_1,\mathfrak{a}}\theta_1$ intertwines with $\tau^{\beta_2}_{\mathfrak{a}_2,\mathfrak{a}}\theta_2$ in $G = \mathrm{Aut}_F(V)$.*

  (b) *For any system $(V, \mathfrak{a}, \boldsymbol{\iota}_j)$, as in (1), the character $\tau^{\beta_1}_{\mathfrak{a}_1,\mathfrak{a}}\theta_1$ intertwines with $\tau^{\beta_2}_{\mathfrak{a}_2,\mathfrak{a}}\theta_2$ in $G = \mathrm{Aut}_F(V)$.*

Proof. Part (1) is elementary. If (2)(a) holds, then $[F[\beta_1]{:}F] = [F[\beta_2]{:}F]$ by [6] (3.5.1) and the intertwining implies conjugacy property. Part (2) is then given by Theorem 8.7 of [4]. □

Developing this theme, if the (non-trivial) simple characters θ_j of Lemma 1(2) satisfy condition (a), we say they are *endo-equivalent.* In particular, in the context of (1.1), θ is endo-equivalent to $\tau^{\beta}_{\mathfrak{a},\mathfrak{a}'}\theta$. Further, two endo-equivalent simple characters attached to the same order intertwine, and so are conjugate. It follows that endo-equivalence is indeed an equivalence relation on the class of non-trivial simple characters *cf.* [4] 8.10.

It is convenient to extend this framework to include the trivial simple characters. We set $\tau_{\mathfrak{a},\mathfrak{a}'}1^1_{\mathfrak{a}} = 1^1_{\mathfrak{a}'}$ and deem that any two trivial simple characters are endo-equivalent. Any two such characters in the same group intertwine, so the approach is consistent with the main case. Moreover, a trivial simple character can never intertwine with a non-trivial one: this follows from [3] Theorem 1 and [6] (2.6.2).

2. Heisenberg extensions

Let θ be a simple character in $G = \mathrm{Aut}_F(V)$, attached to a hereditary order $\mathfrak{a}$ in $A = \mathrm{End}_F(V)$. Thus θ is a character of an open subgroup H^1_θ of $U^1_{\mathfrak{a}}$. Let J^0_θ denote the $U_{\mathfrak{a}}$-normalizer of θ and put $J^1_\theta = J^0_\theta \cap U^1_{\mathfrak{a}}$. If θ is non-trivial, we choose a simple stratum $[\mathfrak{a}, \beta]$ in A such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. We then get $J^k_\theta = J^k(\beta, \mathfrak{a})$ and $H^1_\theta = H^1(\beta, \mathfrak{a})$, in the notation of §1. If θ is the trivial simple character $1^1_{\mathfrak{a}}$ attached to $\mathfrak{a}$, we have $H^1_\theta = J^1_\theta = U^1_{\mathfrak{a}}$ and $J^0_\theta = U_{\mathfrak{a}}$.

With $E = F[\beta]$ (if θ is non-trivial) or F (otherwise), let $B = \mathrm{End}_E(V)$ be the A-centralizer of E and set $\mathfrak{b} = \mathfrak{a} \cap B$. Thus $\mathfrak{b}$ is a hereditary $\mathfrak{o}_E$-order in B with radical $\mathfrak{q} = \mathfrak{p}_{\mathfrak{a}} \cap B$. We then have $J^0_\theta = J^1_\theta U_{\mathfrak{b}}$ and $J^1_\theta \cap U_{\mathfrak{b}} = U^1_{\mathfrak{b}}$.

Let $\eta = \eta_\theta$ be the unique irreducible representation of J^1_θ which contains θ [6] (5.1.1). Thus $\eta|_{H^1_\theta}$ is a multiple of θ. Let $\mathcal{R}^0(\theta)$ be the set of equivalence classes of irreducible representations of J^0_θ which contain θ. Let $\mathcal{H}^0(\theta)$ be the set of $\kappa \in \mathcal{R}^0(\theta)$ with the following two properties. First, $\kappa|_{J^1_\theta} \cong \eta_\theta$. Second, κ is intertwined by every element of G which intertwines θ. In the language of [6], $\mathcal{H}^0(\theta)$ consists of the "β-extensions" of η_θ and is non-empty [6] (5.2.2).

In particular, $\mathcal{R}^0(1^1_{\mathfrak{b}})$ is the set of equivalence classes of irreducible representations of $U_{\mathfrak{b}}$ trivial on $U^1_{\mathfrak{b}}$. For $\sigma \in \mathcal{R}^0(1^1_{\mathfrak{b}})$, there is a unique irreducible representation σ_θ of J^0_θ which agrees with σ on $U_{\mathfrak{b}}$ and is trivial on J^1_θ.

Lemma 2. *Let $\kappa \in \mathcal{H}^0(\theta)$, $\sigma \in \mathcal{R}^0(1^1_{\mathfrak{b}})$. The representation $\kappa \otimes \sigma_\theta$ of J^0_θ is irreducible, and lies in $\mathcal{R}^0(\theta)$. For any $\kappa \in \mathcal{H}^0(\theta)$, the map*

$$\begin{aligned} \mathcal{R}^0(1^1_{\mathfrak{b}}) &\longrightarrow \mathcal{R}^0(\theta), \\ \sigma &\longmapsto \kappa \otimes \sigma_\theta, \end{aligned}$$

is a bijection.

Proof. The restriction of $\kappa \otimes \sigma_\theta$ to H^1_θ is surely a multiple of θ. The other assertions are given by [5] 1.5 Proposition. □

3. Residually cuspidal representations

We continue in the same situation. Let $\Bbbk_E$ denote the residue field of E. The group J^0_θ/J^1_θ takes the form

$$J^0_\theta/J^1_\theta \cong U_{\mathfrak{b}}/U^1_{\mathfrak{b}} \cong \prod_{i=1}^r \mathrm{GL}_{m_i}(\Bbbk_E),$$

for integers $r, m_i \geqslant 1$ such that $\sum_{1\leqslant i\leqslant r} m_i = n/[E{:}F]$. In particular, $U_{\mathfrak{b}}/U^1_{\mathfrak{b}}$ is the group of rational points of a connected reductive $\Bbbk_E$-group. We fix $\kappa \in \mathcal{H}^0(\theta)$. If $\lambda \in \mathcal{R}^0(\theta)$ then, by Lemma 2, $\lambda \cong \kappa\otimes\sigma_\theta$ where σ is the inflation of a uniquely determined irreducible representation $\tilde\sigma$ of $U_{\mathfrak{b}}/U^1_{\mathfrak{b}}$. We say that λ is *residually cuspidal* if the representation $\tilde\sigma$ is cuspidal. The representation κ is uniquely determined, up to tensoring with a character of the form $(\phi\circ\det_B)_\theta$, where ϕ is a character of U_E trivial on U^1_E [6] (5.2.2), so this property of λ does not depend on the choice of κ. We denote by $\mathcal{R}^0_c(\theta)$ the subset of residually cuspidal elements of $\mathcal{R}^0(\theta)$.

Proposition 1. *Let θ be a simple character in G, and let $[\mathfrak{a}, \beta]$ be a simple stratum in A such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. Let E denote the field $F[\beta]$.*

1. *Let $\mathfrak{a}'$ be an E-pure hereditary $\mathfrak{o}_F$-order in A, containing $\mathfrak{a}$. Let $\theta' = \tau^\beta_{\mathfrak{a},\mathfrak{a}'}\theta$, and let $\lambda \in \mathcal{R}^0_c(\theta)$. An irreducible representation π of G containing λ then contains some element of $\mathcal{R}^0(\theta')$.*
2. *Suppose that $\lambda \in \mathcal{R}^0(\theta)$ is not residually cuspidal. There exists an E-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{a}''$ in A, with $\mathfrak{a}'' \subsetneq \mathfrak{a}$, and an element $\lambda'' \in \mathcal{R}^0_c(\theta'')$, where $\theta'' = \tau^\beta_{\mathfrak{a},\mathfrak{a}''}\theta$, with the following property: any irreducible representation of G containing λ also contains λ''.*

Proof. All assertions follow from (8.3.5) Proposition of [6]. □

Remark 1. Proposition 1 applies equally when θ is a trivial simple character, as noted in [6], following (8.3.5).

The simple characters θ', θ'' of Proposition 1 are both endo-equivalent to θ.

We say that a simple stratum $[\mathfrak{a}, \beta]$ in A is *m-simple* if $\mathfrak{a}$ is maximal among $F[\beta]$-pure hereditary $\mathfrak{o}_F$-orders in A. We say that a simple character θ is *m-simple* if $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, where $[\mathfrak{a}, \beta]$ is m-simple. (This depends on θ, not the choice of $[\mathfrak{a}, \beta]$.) Similarly for trivial characters.

Proposition 2. *Let $\lambda \in \mathcal{R}^0(\theta)$. The following are equivalent:*

1. *θ is m-simple and λ is residually cuspidal;*
2. *λ is contained in some irreducible cuspidal representation of G;*
3. *any irreducible representation of G containing λ is cuspidal.*

Proof. The equivalence of (2) and (3) is [6] (6.2.1, 6.2.2). The implication (1) ⇒ (2) is [5] (6.2.3). For the converse, suppose that either θ is not m-simple or that λ is not residually cuspidal. Let π be an irreducible representation of G containing λ. In either case, part (2) of Proposition 1 implies the existence of the following objects:

(1) a non-maximal E-pure hereditary order $\mathfrak{a}'$ in A,
(2) a simple character θ' attached to $\mathfrak{a}'$ and endo-equivalent to θ,
(3) a representation $\lambda' \in \mathcal{R}_c^0(\theta')$ occurring in π.

The representation π is then not cuspidal, by [6] (8.3.3 or 7.3.16). □

Corollary 1. *An irreducible cuspidal representation π of G contains exactly one conjugacy class of simple characters θ, and all of those characters are m-simple.*

Proof. This follows from Proposition 2 and [6] (6.2.4). □

So, if π is an irreducible cuspidal representation of G, all simple characters contained in π belong to the same endo-equivalence class, which we denote $\vartheta(\pi)$.

4. The Type Theorem

Let $\mathfrak{a}$ be a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$, with Jacobson radical $\mathfrak{p}_\mathfrak{a}$. Thus

$$\mathfrak{a}/\mathfrak{p}_\mathfrak{a} \cong \prod_{i=1}^{r} \mathrm{M}_{n_i}(\Bbbk_F),$$

for positive integers n_i with sum n. Let $M_\mathfrak{a}$ be an F-Levi subgroup of G such that $M_\mathfrak{a} \cong \prod_{1\leqslant i\leqslant r} \mathrm{GL}_{n_i}(F)$. The group $M_\mathfrak{a}$ is determined uniquely, up to conjugation in G. If M is an F-Levi subgroup of G, we say that M is *subordinate to* $\mathfrak{a}$ if M is G-conjugate to a Levi subgroup of $M_\mathfrak{a}$.

We recall some further definitions. A *cuspidal datum in* G is a pair (M,σ), where M is a Levi subgroup of G and σ is an irreducible cuspidal representation of M. The set of such data carries the equivalence relation "G-inertial equivalence", as in [7] §1. The set of equivalence classes for this relation will be denoted $\mathcal{B}(G)$.

If π is an irreducible smooth representation of G, there is a cuspidal datum (M,σ) in G and a parabolic subgroup P of G, with Levi component M, such that π is equivalent to a subquotient of the induced representation $\mathrm{Ind}_P^G\,\sigma$. The inertial equivalence class of (M,σ) is thereby uniquely determined: we call it the *inertial support* of π and denote it $\mathcal{I}(\pi)$. If $\mathfrak{S}$ is a finite subset of $\mathcal{B}(G)$, an $\mathfrak{S}$-type in G is a pair (K,ρ), where K is a compact open subgroup of G and ρ is an irreducible smooth representation of K such that, if π is an irreducible smooth representation of G, then π contains ρ if and only if $\mathcal{I}(\pi) \in \mathfrak{S}$ [7] 4.1, 4.2.

Let θ be a (possibly trivial) simple character in G, attached to the hereditary order $\mathfrak{a}$. Let $\varTheta$ denote the endo-equivalence class of θ.

Definition. *Let $\mathfrak{s} \in \mathcal{B}(G)$ be the G-inertial equivalence class of (M,σ), where*

$$M \cong \prod_{j=1}^{s} \mathrm{GL}_{m_j}(F), \qquad \sigma = \bigotimes_{j=1}^{s} \sigma_j,$$

and σ_j is an irreducible cuspidal representation of $\mathrm{GL}_{m_j}(F)$. *We say that $\mathfrak{s}$ is* subordinate to θ *if M is subordinate to $\mathfrak{a}$ and $\vartheta(\sigma_j) = \varTheta$, for all j.*

We prove the following version of the Type Theorem.

Theorem 3. *Let V be a finite-dimensional F-vector space, and let G denote the group $\mathrm{Aut}_F(V)$. Let θ be a simple character in G. Let $\mathfrak{S}_\theta$ be the set of $\mathfrak{s} \in \mathcal{B}(G)$ that are subordinate to θ. The character θ is then an $\mathfrak{S}_\theta$-type in G.*

Proof. We have to show that an irreducible representation π of G contains θ if and only if the inertial support of π is an element of $\mathfrak{S}_\theta$. We assume that θ is non-trivial: the proof for trivial simple characters is parallel but easier, so we omit it. We choose a simple stratum $[\mathfrak{a}, \beta]$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ and set $E = F[\beta]$.

We start in a slightly more general situation, with a cuspidal datum $\mathfrak{s}$ of the form (M, σ) such that

$$M \cong \prod_{k=1}^{s} \mathrm{GL}_{n_k}(F), \quad \sigma = \bigotimes_{k=1}^{s} \sigma_k, \tag{4.1}$$

for various integers $n_k \geqslant 1$, and $\vartheta(\sigma_k) = \Theta$ for all k. Replacing M by a G-conjugate and each σ_k by an equivalent representation, we can assume we are in the following situation. First, M is the G-stabilizer of a decomposition $V = \bigoplus_{1 \leqslant k \leqslant s} V_k$, in which the V_k are non-zero E-subspaces of V. Second, each σ_k contains a simple character $\theta_k \in \mathcal{C}(\mathfrak{a}_k, \beta)$, endo-equivalent to θ, for some simple stratum $[\mathfrak{a}_k, \beta]$ in $\mathrm{End}_F(V_k)$. Observe that, by Corollary 1, each θ_k is m-simple, so $J^0_{\theta_k}/J^1_{\theta_k} \cong \mathrm{M}_{n_k/[E:F]}(\Bbbk_E)$.

We may impose a further normalization. We suppose given an E-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A = \mathrm{End}_F(V)$ such that $\mathfrak{A}/\mathfrak{p}_\mathfrak{A} \cong \prod_k \mathrm{GL}_{n_k}(\Bbbk_F)$, the integers n_k being as in (4.1). There is then an integer $N > 0$ such that $[\mathfrak{A}, N, 0, \beta]$ is a simple stratum in A. Let $\theta_\mathfrak{A} = \tau^\beta_{\mathfrak{a},\mathfrak{A}}\theta \in \mathcal{C}(\mathfrak{A}, \beta)$. In particular, $\theta_\mathfrak{A}$ is endo-equivalent to θ. Theorem 7.2 of [8] gives an $\mathfrak{s}$-type in G of the form $(J^0(\beta, \mathfrak{A}), \lambda_\mathfrak{s})$, where $\lambda_\mathfrak{s} \in \mathcal{R}^0_c(\theta_\mathfrak{A})$.

Remark 2. To be more precise, the construction in [8] 7.2 yields an $\mathfrak{s}$-type (K, τ) where, in our notation, $H^1(\beta, \mathfrak{A}) \subset K \subset J^0(\beta, \mathfrak{A})$. The representation of $J^0(\beta, \mathfrak{A})$ induced by τ is our $\lambda_\mathfrak{s}$.

Let $\mathfrak{s} \in \mathfrak{S}_\theta$. Thus $\mathfrak{s}$ is subordinate to θ and we may therefore choose $\mathfrak{A} \subset \mathfrak{a}$. Let π be an irreducible representation of G of inertial support $\mathfrak{s}$. By definition, π contains $\lambda_\mathfrak{s}$. By Proposition 1, π contains a simple character $\theta' \in \mathcal{C}(\mathfrak{a}, \beta)$ which is endo-equivalent to $\theta_\mathfrak{A}$. It follows that θ' is endo-equivalent to θ, and hence G-conjugate to θ. In particular, π contains θ.

Conversely, let π be an irreducible representation of G which contains θ. Proposition 1(2) gives an E-pure hereditary order $\mathfrak{a}' \subset \mathfrak{a}$, a simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta)$ and a representation $\lambda \in \mathcal{R}^0_c(\theta')$ which occurs in π. Comparing with Theorem 7.2 of [8] again, we see that λ is an $\mathfrak{s}$-type in G, for some $\mathfrak{s} \in \mathfrak{S}_\theta$. Consequently, the inertial support of π is an element of $\mathfrak{S}_\theta$, as required. □

5. The Intertwining Theorem

We prove the Intertwining Theorem. Let $G = \mathrm{GL}_n(F)$, $A = \mathrm{M}_n(F)$. Let θ_1, θ_2 be simple characters in G, with endo-classes Θ_1, Θ_2 respectively.

Suppose first that $\Theta_1 = \Theta_2$. We have to show that θ_1 intertwines with θ_2 in G. If the θ_i are trivial, this is clear, so suppose otherwise. Choose simple strata $[\mathfrak{a}_i, \beta_i]$ in A such that $\theta_i \in \mathcal{C}(\mathfrak{a}_i, \beta_i)$ and put $E_i = F[\beta_i]$. The relation $\Theta_1 = \Theta_2$ implies that the field extensions E_i/F have the same ramification indices and the same residue class degrees [4] (8.11). So, there exists an E_2-pure hereditary order $\mathfrak{a}$ in A which is isomorphic to $\mathfrak{a}_1$. Set $\theta = \tau^{\beta_2}_{\mathfrak{a}_2,\mathfrak{a}}\theta_2$. According to Lemma 1, the simple characters θ_1, θ must intertwine in G and hence be G-conjugate, say

$\theta = \theta_1^g$, for some $g \in G$. By [6] (3.6.1), the characters θ_2, θ agree on $H^1(\beta_2, \mathfrak{a}_2) \cap H^1(\beta_2, \mathfrak{a})$. Therefore g intertwines θ_1 with θ_2, as required.

For the converse, suppose that θ_1 intertwines with θ_2. Abbreviating $H_i = H^1(\beta_i, \mathfrak{a}_i)$, this hypothesis implies the existence of a non-trivial G homomorphism

$$c\text{-Ind}_{H_1}^G \theta_1 \longrightarrow c\text{-Ind}_{H_2}^G \theta_2. \tag{5.1}$$

Frobenius Reciprocity, for compact induction from open subgroups, implies that the space $\Pi_i = c\text{-Ind}_{H_i}^G \theta_i$ is generated over G by its θ_i-vectors. Since θ_i is a type in G (Theorem 3), every irreducible G-subquotient of Π_i contains θ_i [7] 4.1. The existence of the non-trivial map (5.1) implies there exists an irreducible representation π of G containing both θ_i. If the inertial support of π is of the form $\left(\prod_k \mathrm{GL}_{n_k}(F), \bigotimes_k \sigma_k\right)$ then, by Theorem 3 again, $\vartheta(\sigma_k) = \Theta_1 = \Theta_2$, for all k. □

Since endo-equivalence is an equivalence relation, the Intertwining Theorem implies that simple characters, in a fixed group, exhibit the following surprising property.

Corollary 2. *Let θ_1, θ_2, θ_3 be simple characters in $G = \mathrm{GL}_n(F)$. If θ_1 intertwines with θ_2 and θ_2 intertwines with θ_3, then θ_1 intertwines with θ_3.*

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