CUSPIDAL PLANE CUVES, SYZYGIES AND A BOUND ON THE MW-RANK

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Abstract. Let $C = Z(f)$ be a reduced plane curve of degree $6k$, with only nodes and ordinary cusps as singularities. Let I be the ideal of the points where C has a cusp. Let $\oplus S(-b_i) \to \oplus S(-a_i) \to S \to S/I$ be a minimal resolution of I. We show that $b_i \leq 5k$. From this we obtain that the Mordell-Weil rank of the elliptic threefold $W : y^2 = x^3 + f$ equals $2\#\{i \mid b_i = 5k\}$. Using this we find an upper bound for the Mordell-Weil rank of W, which is $\frac{1}{2}(15-\sqrt{15})k+l.o.t.$ and we find an upper bound for the exponent of (t^2-t+1) in the Alexander polynomial of C, which is $\frac{1}{4}(15-\sqrt{15})k+l.o.t.$. This improves a recent bound of Cogolludo and Libgober almost by a factor 2.

1. Introduction

In this paper we study reduced plane curves C of degree $d = 6k$ having only nodes and ordinary cusps as singularities. We allow C to be reducible. Let z_0, z_1, z_2 be coordinates on $\mathbf{P}^2$, and let $S = \mathbf{C}[z_0, z_1, z_2]$. Let $f \in S_{6k}$ be an equation for C. Let Σ be the set of cusps of C (we will ignore the nodes).

Consider now the elliptic threefold defined by

$$Z(-y^2 + x^3 + f) \subset \mathbf{P}(2k, 3k, 1, 1, 1).$$

Let $\mathrm{MW}(\pi)$ be the Mordell-Weil group, i.e., the group of rational sections of the elliptic fibration. It is known that the rank of $\mathrm{MW}(\pi)$ can be expressed in terms of the geometry of C (see Lemma 3.4), namely

$$\operatorname{rank} \mathrm{MW}(\pi) = 2 \dim \operatorname{coker} \left(S_{5k-3} \stackrel{\oplus ev_p}{\longrightarrow} \oplus_{p\in\Sigma} \mathbf{C} \right).$$

We can also consider the fundamental group $\pi_1(\mathbf{P}^2 \setminus C)$. With this group we can associate the so-called Alexander polynomial of C. It turns out that the exponent of $(t^2 - t + 1)$ in the Alexander polynomial equals

$$\dim \left(\operatorname{coker} S_{5k-3} \stackrel{\oplus ev_p}{\longrightarrow} \oplus_{p\in\Sigma} \mathbf{C} \right).$$

Hence both invariants coincide. Cogolludo and Libgober [2] noticed this and proved for a much larger class of singular plane curves that the degree of the Alexander polynomial is related with the Mordell-Weil group of an associated elliptic fibration.

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In this paper we give a non-trivial upper bound $g(k)$ for the Mordell-Weil rank, which also yields an upperbound for the exponent of t^2-t+1 in the Alexander polynomial. Asymptotically we have that

$$\lim_{k\to\infty}\frac{g(k)}{k}=\frac{15-\sqrt{15}}{2}\approx 5.56 \tag{1}$$

The best known previous upper bound for the exponent of (t^2-t+1) in the Alexander polynomial of a cuspidal curve seems to be due to Cogolludo and Libgober [2], and equals $5k-1$ (this implies that the MW-rank is at most $10k-2$). This bound is an immediate consequence of the Shioda–Tate formula. The divisibility theorem for the Alexander polynomial of LIbgober yields an upperbound of $6k-2$ for the exponent in the Alexanderpolynomial.

Our bound is deduced from two other bounds. Suppose we fix r,k and look for $C_{2r,k}$ the minimal number of cusps on a degree $6k$ curve such that the corresponding elliptic fibration has Mordell-Weil rank at least $2r$. We show that

$$C_{2,k}=6k^2 \text{ and } C_{2r,k}\geq 6k^2+3(r-1)k-\frac{3}{4}r(r-1)+O\left(\frac{1}{k}\right) \text{ for } k\to\infty. \tag{2}$$

The number of cusps on a degree d curve can be bounded by $\frac{5}{16}d^2-\frac{3}{8}d$ (see [8]). Combining both bounds yields the upper bound (1). This bound is very unlikely to be sharp. We expect that $C_{2r,k}$ can be bounded from below by a function of the form $h(r)k^2$, where h is increasing in r, rather than constant. However, the bound for $C_{2,k}$ is sharp. If we take a general polynomials $f_1\in S_{2k}$ and $f_2\in S_{3k}$, and set $f=f_1^3+f_2^2$, then f has $6k^2$ cusps and the Mordell-Weil rank is at least 2. The fact that $C_{2,k}\geq 6k^2$ holds, can also be obtained by different methods, namely if C has less then $6k^2$ cusps then $\pi_1(\mathbf{P}^2\setminus C)$ is abelian and therefore C has constant Alexander polynomial. In particular, the Mordell-Weil rank is zero in this case. ([12])

The main idea of the proofs is to consider the resolution of the ideal I of Σ:

$$0\to\oplus_{i=1}^{t}S(-b_i)\to\oplus_{i=1}^{t+1}S(-a_i)\to S\to S/I\to 0.$$

There are several restrictions on a_i,b_i coming from the fact that I is the ideal of a finite set of points in $\mathbf{P}^2$. These restrictions are classically known, see Proposition 2.1. We find further restrictions on the a_i and b_i by a combination of Bezout's theorem and an upper bound for the number of cusps on a degree d plane curve. See Proposition 2.5.

Using specializations to elliptic surfaces we show that $b_i\leq 5k$ for all i. Using an expression for the difference between the Hilbert polynomial of I and the Hilbert function of I we obtain that $\operatorname{rank}\operatorname{MW}(\pi)=2\#\{i\mid b_i=5k\}$. This fact is proved in Proposition 3.3. A statement that is equivalent with the fact that $b_i\leq 5k$ was already known to Zariski [12] in the case that C is irreducible. In the case of reducible curves we could not find an explicit place in the literature where this was proven, but the techniques to extend this result to reducible curves have been around since the beginning of the 1980s ([5],[9]). However, our proof is different from the existing proofs in the literature.

There are other examples where the highest degree syzygies of the ideal of the singular locus has a geometric interpretation. E.g., if we consider a minimal resolution of the ideals of the nodes, then a syzygy has degree at most the degree of the curve, and the number of highest degree syzygies is one less than the number of irreducible components of the curve. (Proposition 3.6)

In Section 4 we prove the bound (2) under an extra technical assumption on the a_i and b_i: After permuting the a_i and b_i we may assume that the a_i and b_i both form a descending sequence. A priory, we know that $b_i > a_i$. In Section 4 we assume that $a_i \leq b_{i+1}$ for all i.

In Section 5 we study the case when there is some i with $a_i > b_{i+1}$. We consider the ideal generated by all generators of I of degree less than a_i. This ideal defines a subscheme of $\mathbf{P}^2$ which is the union of a (possibly non-reduced) curve and a zero-dimensional scheme. We can analyze this situation in similar way as above and obtain further restrictions on the a_i and the b_i. These further obstructions allow us to construct a new sequence a'_i, b'_i that corresponds with an ideal I' such that $\#Z(I') \leq \#Z(I)$ and the a'_i, b'_i satisfy the extra condition used in Section 4. Hence also in this case the lower bound (2) holds.

2. Resolution of the locus of cusps

We cite first a result on the resolution of the ideal of finitely many points in $\mathbf{P}^2$.

Proposition 2.1. *Let I be the ideal of finitely many distinct points in $\mathbf{P}^2$. Then I has a free resolution*

$$0 \to \oplus_{i=1}^{t} S(-b_i) \to \oplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0, \tag{3}$$

such that

1. *for all i we have that $a_i, b_i \in \mathbf{Z}$ and $a_i > 0, b_i > 0$;*
2. $\sum_{i=1}^{t+1} a_i = \sum_{i=1}^{t} b_i$*;*
3. *for $i = 1, \dots t$ we have $b_i > a_i \geq a_{i+1}$ and for $j = 1, \dots, t-1$ we have $b_j \geq b_{j+1}$.*

Furthermore, let $B(s) = \frac{1}{2}(s+1)(s+2)$ then

$$\#Z(I) = B(s) - \sum_{i=1}^{t+1} B(s - a_i) + \sum_{i=1}^{t} B(s - b_i) =: c(\mathbf{a}, \mathbf{b})$$

Proof. This follows almost immediately from the fact that I has a free resolution of length 1 and that the Hilbert polynomial of I is constant. See [4, Section 3.1]. □

Remark 2.2. For each pair $\mathbf{a}, \mathbf{b}$ satisfying the three numerical conditions there is an ideal with a resolution of the form (3), see [4, Section 3.C].

For general $\mathbf{a}, \mathbf{b}$ the expression

$$B(k) - \sum_{i=1}^{t+1} B(k - a_i) + \sum_{i=1}^{t} B(k - b_i)$$

is a polynomial of degree 2 in k. However, from the three conditions from Proposition 2.1 it follows easily that the degree is zero, in fact,

$$c(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \left(\sum b_i^2 - \sum a_i^2 \right).$$

Definition 2.3. For d a positive integer, define $M(d)$ to be the maximal number of ordinary cusps on a (possibly reducible) degree d curve.

Remark 2.4. The best known upper bound for $M(d)$ we are aware of is a bound obtained in similar way as the Miyaoka bound (see [8]). This bound is

$$M(d) \leq \frac{5}{16}d^2 - \frac{3}{8}d.$$

In [2] a degree 12 curve with 39 cusps is constructed. From this it follows directly that

$$\limsup_{d\to\infty} \frac{M(d)}{d^2} \geq \frac{39}{144} = \frac{13}{48}.$$

Proposition 2.5. *Suppose I is the ideal of the locus of cusps of a plane curve C of degree d. Then*

$$c(\mathbf{a},\mathbf{b}) \leq \min\left(\frac{a_{t+1}d}{2}, M(d)\right) \leq \min\left(a_{t+1}, \frac{5}{8}d - \frac{3}{4}\right)\frac{d}{2}$$

Proof. The upper bound $c(\mathbf{a},\mathbf{b}) \leq M(d)$ is obvious. We prove now that $c(\mathbf{a},\mathbf{b}) \leq \frac{a_{t+1}d}{2}$.

Since the ideal I contains an element of degree a_{t+1}, there exists a curve C' of degree a_{t+1} such that all cusps of C are points of C'. By Bezout's theorem we have that if $2c(\mathbf{a},\mathbf{b}) > a_{t+1}d$ then C and C' have a common component C'' of degree $d'' \leq a_{t+1}$. Since cusps are irreducible singularities it follows that all the cusps of C that are also points of C' are actually cusps of C''. Hence we obtain that if $\frac{1}{2}a_{t+1}d < c(\mathbf{a},\mathbf{b})$ then

$$c(\mathbf{a},\mathbf{b}) \leq M(d'')+\frac{1}{2}(d-d'')(a_{t+1}-d'') \leq \frac{5}{16}(d'')^2+\frac{1}{2}a_{t+1}d+\frac{1}{2}(d'')^2-\frac{1}{2}d''(a_{t+1}+d)$$

Combining this with $c(\mathbf{a},\mathbf{b}) \geq \frac{1}{2}da_{t+1}$ yields

$$0 \leq \frac{13}{16}(d'')^2 - \frac{1}{2}d''(a_{t+1}+d).$$

Dividing this inequality by $d''/2$ and using $2d'' \leq a_{t+1}+d$ yields

$$0 \leq \frac{13}{8}d'' - (a_{t+1}+d) \leq \frac{-3}{8}d''.$$

Hence the degree of C'' is 0. Equivalently, the component C'' does not exists and $c(\mathbf{a},\mathbf{b}) \leq \frac{1}{2}a_{t+1}d$. □

3. Syzygies and MW-rank

Let $S := \mathbf{C}[z_0, z_1, z_2]$ be the polynomial ring in three variables. Let S_d be the subspace of homogeneous polynomials of degree d.

Fix an integer k and a square-free polynomial $f \in S_{6k}$, such that the plane curve $C = Z(f)$ has only nodes and ordinary cusps as singularities. Let Σ denote the set of cusps of C. Let $I \subset S$ be the ideal of Σ.

Let $W_f \subset \mathbf{P}(2k, 3k, 1, 1, 1)$ be the hypersurface given by the vanishing of

$$-y^2 + x^3 + f$$

The threefold W_f is birational to an elliptic threefold $\pi : X \to S$, where S is a rational surface and the elliptic fibration π is birational to the projection $\psi : W_f \setminus \{(1:1:0:0:0\} \to \mathbf{P}^2$ from $(1:1:0:0:0)$ onto the plane $\{x = y = 0\}$. The explicit construction of π is slightly complicated, see [10]. For $p \in \mathbf{P}^2$ the Zariski closure of $\psi^{-1}(p)$ is either an elliptic curve with j-invariant 0 or a cuspidal cubic, depending on whether $p \in C$ or not.

The Mordell-Weil group $\mathrm{MW}(\pi)$ of π is the group of rational sections of π. This is a finitely generated group, and if the singularities of C are "mild" then one has an algorithm to compute the rank of $\mathrm{MW}(\pi)$, see [6]. In our case (C has only A_1 and A_2 singularities) one gets

Proposition 3.1. *Suppose C is a degree $6k$ curve with only A_1 and A_2 singularities then*

$$\operatorname{rank}\mathrm{MW}(\pi) = 2\dim\left(\operatorname{coker} S_{5k-3} \stackrel{ev_P}{\longrightarrow} \oplus_{p\in\Sigma}\mathbf{C}\right). \tag{4}$$

Proof. For the case $k = 1$ see [7, Section 9]. The general case follows along the same lines:

An A_1 singularity of C yields an A_2 singularity of W_f, whereas an A_2 singularity of C yields a D_4 singularity on W_f.

Let Σ be the singular locus. We can now compute $H^4(W_f)_{\mathrm{prim}}$ as the cokernel of

$$H^4(\mathbf{P}(2k,3k,1,1,1)\setminus W_f) \cong H^3(W_f\setminus\Sigma) \to H^4_\Sigma(W_f).$$

An A_2 singularity of W_f does not contribute to H^4_Σ, whereas a D_4 singularities does [3, Example 1.9]. Actually, using the ideas from [3, Section 1] it follows that $H^4_p(W_f) = \mathbf{C}(-2)^2$ if p is a D_4 singularity of W_f, hence $H^4_\Sigma(W_f)$ is of pure Hodge type $(2,2)$. Then by the main results of [6] we have $\operatorname{rank}\mathrm{MW}(\pi) = h^4(W_f) - 1$.

Let ω be a third root of unity. The map $\omega : [x : y : z_0 : z_1 : z_1] \mapsto [\omega x : y : z_0 : z_1 : z_2]$ is an automorphism of W_f and fixes every point of Σ. The map $H^4(\mathbf{P}(2k,3k,1,1,1)\setminus W_f) \to H^4_\Sigma(W_f)$ is ω^*-equivariant, so we may decompose it in a ω and ω^2 eigenspace, which have both the same dimension, and a 1-eigenspace, which is trivial.

The argument used in [7, Section 9] shows in this case that the ω-eigenspace of the co-kernel of $H^4(\mathbf{P}(2k,3k,1,1,1)\setminus W_f) \to H^4_\Sigma(W_f)$ has dimension

$$\dim\left(\operatorname{coker} S_{5k-3} \stackrel{ev_P}{\longrightarrow} \oplus_{p\in\Sigma}\mathbf{C}\right).$$

□

Remark 3.2. With the fundamental group of $\mathbf{P}^2\setminus C$ one can associate the so-called Alexander polynomial. Cogolludo and Libgober [2] showed that the exponent of $t^2 - t + 1$ in this polynomial equals half the rank of $\mathrm{MW}(\pi)$.

Hence each statement which we make on the rank of $\mathrm{MW}(\pi)$ is also a statement on the exponent of $t^2 - t + 1$ in the Alexander polynomial of a cuspidal curve.

We will calculate $\operatorname{rank}\mathrm{MW}(\pi)$ using a projective resolution of I and use properties of the resolution to bound the Mordell-Weil rank in terms of k. Two more or less obvious restrictions on the resolution come from Bezout's theorem (Proposition 2.5) and from a Miyaoka-type bound for the maximal number of cusps on a degree d plane curve (Remark 2.4). The third restriction comes from considerations on the Mordell-Weil rank of elliptic surfaces:

Proposition 3.3. *Let*

$$0 \to \oplus_{i=1}^t S(-b_i) \to \oplus_{i=1}^{t+1} S(-a_i) \to S \to S/I \to 0$$

be a resolution of I. Then for all i we have that $b_i \leq 5k$, $a_i < 5k$ and

$$\operatorname{rank}\mathrm{MW}(\pi) = 2\#\{i \mid b_i = 5k\}.$$

Proof. Let $B(s)$ be as in the previous section. The Hilbert polynomial of I is given by

$$B(s)+\sum_{i=1}^{t} B(s-b_i)-\sum_{i=1}^{t+1} B(s-a_i).$$

The Hilbert function evaluated at s equals

$$B(s)+\sum_{i|b_i\leq s} B(s-b_i)-\sum_{i|a_i\leq s} B(s-a_i).$$

Since $B(-1)=B(-2)=0$ we may replace the condition "$\leq s$" by "$\leq s+2$" in the above formula. Hence the superabundance in degree s, i.e., the difference between the Hilbert polynomial and the Hilbert function evaluated at s, equals

$$\sum_{i|b_i\geq s+3} B(s-b_i)-\sum_{i|a_i\geq s+3} B(s-a_i).$$

From Proposition 3.1 it follows that the rank of $\mathrm{MW}(\pi)$ is twice the dimension of the cokernel of the evaluation map $S_{5k-3}\stackrel{ev_P}{\to}\oplus_{p\in\Sigma}\mathbf{C}$. The dimension of this cokernel is precisely the difference between the Hilbert function of I in degree $5k-3$, and the Hilbert polynomial of I (which is constant). Hence we proved the following lemma:

Lemma 3.4. *Let $b_i':=b_i-5k$ and $a_i':=a_i-5k$. We have*

$$\operatorname{rank}\mathrm{MW}(\pi)=\left(\sum_{i|b_i'\geq 0}(b_i'+1)(b_i'+2)-\sum_{i|a_i'\geq 0}(a_i'+1)(a_i'+2)\right).$$

Take a general line $\ell\subset\mathbf{P}^2$. The (projective) surface $\overline{\pi^{-1}(\ell)}\subset(\mathbf{P}(2k,3k,1,1,1)$ might be singular. Denote with $\widetilde{\pi^{-1}(\ell)}$ a resolution of singularities of this surface. This surface admits a natural elliptic fibration $\pi_\ell:\widetilde{\pi^{-1}(\ell)}\to\ell$. From the theory of elliptic surface we obtain the following well-known inequalities:

$$\operatorname{rank}\mathrm{MW}(\pi)\leq\operatorname{rank}\mathrm{MW}(\pi_\ell)\leq h^{1,1}(\widetilde{\pi^{-1}(\ell)})-2=10k-2. \tag{5}$$

The first inequality is a standard result on specializations. The second inequality follows from the Shioda-Tate formula. The final equality is a well-known fact for elliptic surfaces, see e.g., [11].

Define a_i' and b_i' as in the previous lemma. Suppose that for some i we have $b_i'>0$. Fix a positive integer w such that for some i we have

$$\begin{array}{ll}(b_i'w+1)(b_i'w+2)>10kw & \text{if } a_i'<0 \text{ or}\\ (b_i'w+1)(b_i'w+2)-(a_i'w+1)(a_i'w+2)>10kw & \text{if } a_i'\geq 0.\end{array}$$

Now take three general polynomials g_0,g_1,g_2 of degree w. Let $\varphi:\mathbf{P}^2\to\mathbf{P}^2$ be the map defined by $\varphi(z_0:z_1:z_2)=(g_0:g_1:g_2)$. Let $\tilde{f}=\varphi^*(f)\in S_{6kw}$ and let $\pi_w:X_w\to S'$ be the pull-back of the elliptic fibration $\pi:X\to S$.

For general g_i the curve defined by $\tilde{f}$ has only nodes and ordinary cusps as singularities and the locus $\tilde{\Sigma}$ consisting of the cusps of $Z(\tilde{f})$ equals $\varphi^{-1}(\Sigma)$. In particular, the corresponding ideal $\tilde{I}$ has the following free resolution

$$0\to\oplus_{i=1}^{t}S(-b_iw)\to\oplus_{i=1}^{t+1}S(-a_iw)\to S\to S/I\to 0.$$

From this and Lemma 3.4 it follows that

$$\operatorname{rank} \mathrm{MW}(\pi_w) = \left(\sum_{i \mid b'_i \geq 0} (b'_i w + 1)(b'_i w + 2) - \sum_{i \mid a'_i \geq 0} (a'_i w + 1)(a'_i w + 2) \right) \geq 10kw.$$

This contradicts the bound $\operatorname{rank} \mathrm{MW}(\pi_w) \leq 10kw - 2$ from (5). Hence $b'_i \leq 0$ and $b_i \leq 5k$. □

Remark 3.5. After distributing a preliminary version of this paper we learned the following: Zariski [12] proved that the Castelnuovo-Mumford regularity of the cuspidal locus of an *irreducible* plane curve is at most $5k - 1$. This is done by studying the cyclic degree $6k$ corver of $\mathbf{P}^2$ ramified along the curve C. The statement on the regularity implies that $b_i \leq 5k$ holds in the case of irreducible curves. In the case of reducible curves Zariski proved that the regularity of the cuspidal locus is at most $6k - 2$.

The statement $b_i \leq 5k$ in the case of reducible curves seems to be known to the experts, although we could not identify a proof for this statement in the literature. The techniques to extend Zariski's proof to the reducible case have been around since the beginning of the 1980s ([5],[9]). However, our proof is different from the existing proofs in the literature.

If C' is an *irreducible* curve of degree d and I' the ideal of the points of C' where C' has a node then each syzygy of I' has degree at most $d - 1$ (this is implied by the exercises 24 and 31 of [1, Appendix A]). Actually, a statement analogous to Proposition 3.3 holds for the locus of nodes of a plane curve, but we could not find this particular result in the literature.

Proposition 3.6. *Let $C' \subset \mathbf{P}^2$ be a reduced plane curve of degree d with only nodes and ordinary cusps as singularities. Let c be the number of irreducible components of C'. Define $\mathcal{N}$ to be the locus of nodes of C'. Let I' be the ideal of $\mathcal{N}$ and*

$$0 \to \oplus S(-b_i) \to \oplus S(-a_i) \to S \to S/I' \to 0$$

be a minimal resolution of I'. Then $b_i \leq d$ and

$$\#\{i \mid b_i = d\} = c - 1.$$

Proof. From the Mayer–Vietoris sequence it follows easily that $h^2(C') = c$.

We would like to use Dimca's method [3] to calculate $h^2(C')$ in terms of the defect of a linear system. However, [3] considers only hypersurfaces in weighted projective spaces of dimension at least 3. Large part of the proof below consists of showing that the ideas from [3] also work in the case of plane curves.

Let $\Sigma = C'_{\text{sing}}$. If Σ is empty then there is nothing to prove, so assume that Σ is non-empty. Let $C^* = C' \setminus \Sigma$, let $\mathbf{P}^* = \mathbf{P}^2 \setminus \Sigma$ and $U = \mathbf{P}^* \setminus C^* = \mathbf{P}^2 \setminus C'$. The linear systems studied in Dimca's paper are kernels of maps of the form $H^2(U) \to H^2_p(\Sigma)$. We are going to mimic his strategy to obtain such a map.

The first ingredient is the following part of the exact sequence for the pair (C', C^*):

$$H^1(C^*) \to H^2_\Sigma(C') \to H^2(C') \to H^2(C^*) = 0 \tag{6}$$

Since Σ contains at least one point of each irreducible component of C' we have that $H^2(C^*) = 0$.

We can relate $H^1(C^*)$ with $H^2(U)$ by using the Thom isomorphism $H^1(C^*) \cong H^3(\mathbf{P}^*, U)(1)$ (cf. [3, Section 2]). Consider the following part of the exact sequence for the pair $(\mathbf{P}^*, U)$:

$$H^2(U) \to H^3(\mathbf{P}^*, U) \to H^3(\mathbf{P}^*) \to H^3(U)$$

Since U is a affine and of dimension 2 we have that $H^3(U) = 0$. By Poincaré duality we have an isomorphism $H^3(\mathbf{P}^*) \cong H^1_c(\mathbf{P}^*)\check{}(-1)$. From the Gysin sequence

$$0 = H^0_c(\mathbf{P}^*) \to H^0_c(\mathbf{P}^2) \to H^0_c(\Sigma) \to H^1_c(\mathbf{P}^*) \to H^1_c(\mathbf{P}^2) = 0$$

it follows that

$$H^2(U)(1) \to H^1(C^*) \to H^0_c(\Sigma)\check{}(-1) \to H^0_c(\mathbf{P}^2)\check{}(-1) \to 0 \tag{7}$$

is exact. From [3, Section 2] it follows that $H^2(U)(1) \to H^1(C^*)$ is just the Poincaré residue map.

There is also a local version of the above paragraph. I.e, let $p \in \Sigma$. Let $V \subset \mathbf{P}^2$ be a small neighborhood of p, let $D = C' \cap V$ and $D^* = D \setminus \{p\}$. Then both

$$H^2(V \setminus D)(1) \to H^1(D^*) \to H^0(p)\check{}(-1) \to 0$$

and

$$H^1(D) \to H^1(D^*) \to H^2_p(D) \to H^2(D) = 0$$

are exact.

If p is a cusp then we may assume that $V = \mathbf{A}^2$ and $D = \{y^2 - x^3 = 0\}$. If p is a node then we may assume that $V = \mathbf{A}^2$ and $D = \{xy = 0\}$. In both cases we have $H^1(D) = 0$, hence $H^2_p(D) \cong H^1(D^*)$ holds. In the cuspidal case we have that D^* is isomorphic to $\mathbf{A}^1$ with one point deleted. From this if follows that $h^1(D^*) = 1$ and hence

$$H^2_p(D) \cong H^0(p)\check{}(-1). \tag{8}$$

In the nodal case we have that D^* is isomorphic to disjoint union of two copies of $\mathbf{A}^1$ with a point deleted. In this case $H^1(D^*)$ is two-dimensional and so is $H^2_p(D)$. Following the discussion after [3, Formula (1.7)] we obtain that $H^2(V \setminus D)$ is one-dimensional and that this vector space is spanned by

$$\frac{1}{xy} dx \wedge dy.$$

I.e., we have an isomorphism $H^2(V \setminus D) \cong \mathbf{C}$ mapping $\frac{g}{xy} dx \wedge dy \in \Omega^2(V \setminus D)$ to $g(p)$. In particular, if p is a node then we have an exact sequence

$$0 \to \mathbf{C}\frac{1}{xy} dx \wedge dy \to H^2_p(D) \to H^0(p)^*(-1) \to 0. \tag{9}$$

Collecting everything we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \vdots & & \\
 & & H^2(U)(1) & \dashrightarrow & \oplus_{p\in\mathcal{N}}\mathbf{C}\frac{1}{xy}(dx\wedge dy) & \dashrightarrow & K_1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
 & & H^1(C^*) & \longrightarrow & H^2_\Sigma(C) & \longrightarrow & H^2(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & K & \longrightarrow & H^0_c(\Sigma)\check{}(-1) & \longrightarrow & H^0_c(\mathbf{P})\check{}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The dotted arrows will be constructed below. We start by explaining where the plain arrows come from. The first column together with the third row is the exact sequence (7). The second column is the direct sum over all points in Σ of the local exact sequences (8) and (9). The second horizontal sequence is the exact sequence (6).

We will now construct the dotted arrows.

An easy diagram chase shows the existence of a unique map $H^2(U)(1) \to \oplus_{p\in\mathcal{N}}\mathbf{C}\frac{1}{xy}(dx\wedge dy)$. This map can be given explicitly as follows. Let $f \in S_d$ be a polynomial defining C'. Set

$$\Omega := z_0z_1z_2\left(\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}-\frac{dz_0}{z_0}\wedge\frac{dz_2}{z_2}+\frac{dz_0}{z_0}\wedge\frac{dz_1}{z_1}\right)$$

Following [3, Section 1] we have $F^3H^2(U)=0$,

$$F^2H^2(U)=\left\{\frac{g}{f}\Omega \mid g\in S_{d-3}\right\}$$

and $H^2(U)=F^0H^2(U)=F^1H^2(U)$. Moreover, the latter space is spanned by

$$\left\{\frac{g}{f}\Omega \mid g\in S_{d-3}\right\}\cup\left\{\frac{h}{f^2}\Omega \mid h\in S_{2d-3}\right\}.$$

Since $H^2(U)(1)\to H^2_\Sigma(C)$ is a morphism of Hodge structures and the image has only classes of type $(1,1)$, it follows that $Gr_1^F H^2(U)$ is mapped to zero in $H^4_\Sigma(W_f)$. Hence to determine the co-kernel of $H^2(U)(1)\to\oplus\mathbf{C}\frac{1}{xy}(dx\wedge dy)$ we can restrict the map to $F^2H^2(U)$. From the local construction of this map it follows directly that $\frac{g}{f}\Omega$ is mapped to $g(p)\frac{1}{xy}(dx\wedge dy)$. Let K_1 be the co-kernel of this map. Then

$$K_1\cong\operatorname{coker}\left(S_{d-3}\to\oplus_{p\in\mathcal{N}}\mathbf{C}\right).$$

This yields the desired linear system.

An easy diagram chase yields the existence of maps $K_1\to H^2(C')$ and $H^2(C')\to H^0_c(\mathbf{P}^2)\check{}(-1)$. The five lemma implies that the latter map is surjective. One further easy diagram chase shows that the kernel of the second map is precisely the image of the first map. Finally, the snake lemma shows that the map $K_1\to H^2(C')$ is injective.

From the exactness of the third column it follows that $\dim K_1 = c-1$, hence

$$\dim \operatorname{coker}(S_{d-3} \to \oplus_{p\in\mathcal{N}} \mathbf{C}) = c-1. \tag{10}$$

Consider next a minimal resolution of I'. We can now proceed as in Proposition 3.3. Let w be a positive integer, let f_0, f_1, f_2 be three general polynomials of degree w and $\varphi : \mathbf{P}^2 \to \mathbf{P}^2$ be the morphism $(z_0 : z_1 : z_2) \mapsto (f_0 : f_1 : f_2)$. Let c_w be the number of irreducible components of the pull back of C'. Using (10) and the argument used in Lemma 3.4 we obtain that

$$2c_w - 2 = \left(\sum_{i|b_i\geq d} (b_i w - dw + 1)(b_i w - dw + 2) - \sum_{i|a_i \geq d} (a_i w - dw + 1)(a_i w - dw + 2) \right).$$

If at least one of the b_i were bigger than d then the coefficient of w^2 in c_w would be positive. This contradicts the fact that c_w is at most the degree of the curve, which equals wd. Hence $b_i \leq d$. The same formula shows that

$$c_w - 1 = \#\{i \mid b_i = d\}$$

for all w, in particular $c - 1 = \#\{i \mid b_i = d\}$. □

4. Upper bound for rank MW(π) (strongly admissible case)

As discussed in the proof of Proposition 3.3 we have a naive upper bound for $\operatorname{rank} \mathrm{MW}(\pi)$, namely

$$\operatorname{rank} \mathrm{MW}(\pi) \leq 10k - 2.$$

Using the connection between the Mordell-Weil rank and the Alexander polynomial of C this gives an upper bound for the degree of the Alexander polynomial of C. The upper bound obtained by Cogolludo and Libgober [2] is precisely this bound.

In this and the next section we will give an upper bound $g(k)$ for $\operatorname{rank} \mathrm{MW}(\pi)$ such that

$$g(k) \sim \frac{15-\sqrt{15}}{2} k \approx 5.56k \text{ for } k \to \infty.$$

Definition 4.1. Fix a positive integer t. Let $a_1, \dots, a_{t+1}, b_1, \dots, b_t$ be a sequence of positive integers. If no confusion arises we write $\mathbf{a}, \mathbf{b}$ for this sequence. We call $\mathbf{a}, \mathbf{b}$ *k-admissible for rank $2r$ and lowest degree D_0* if

(1) $\sum a_i = \sum b_i$.
(2) $a_i < b_i$ for $i = 1, \dots, t$.
(3) $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$.
(4) $a_{t+1} = D_0$.
(5) $\#\{i \mid b_i = 5k\} \geq r$.
(6) $c(\mathbf{a}, \mathbf{b}) \leq \min(M(6k), 3ka_{t+1})$.

We call a sequence *strongly k-admissible* if $a_i \leq b_{i+1}$ for all $i = 1, \dots t-1$.

We call a (strongly) k-admissible sequence *reduced* if $\{a_i\} \cap \{b_i\} = \emptyset$.

Remark 4.2. Given a (strongly) k-admissible sequence $\mathbf{a}, \mathbf{b}$ we can construct a reduced k-admissible sequence by repeatedly throwing out b_i and a_j in case they are equal. The new sequence is easily to be seen k-admissible, since this operation does not change $c(\mathbf{a}, \mathbf{b})$. Moreover, this reduction transform an k-admissible sequence transforms into an k-admissible sequence and it transforms a strongly k-admissible sequence into a strongly k-admissible sequence.

The results of Sections 2 and 3 show that

Lemma 4.3. *Suppose that $y^2 = x^3 + f$ has Mordell-Weil rank $2r$. Let D_0 be the degree of a generator of minimal degree of I. Then there exists an k-admissible sequence of rank $2r$ and lowest degree D_0.*

Remark 4.4. In this section we will study strongly k-admissible sequences. In the next section we will show that if $y^2 = x^3 + f$ has rank $2r$ then there exists a strongly k-admissible sequence for rank $2r$ and lowest degree $D_0' \leq D_0$. Hence to prove the desired bounds we only have to consider strongly k-admissible sequences.

For technical reasons, we have to discuss the following type of sequences seperatly:

Lemma 4.5. *Suppose that $a_1 = \cdots = a_r = 5k-1, a_{r+1} = r$ and $b_1 = \cdots = b_r = 5k$. Then*

$$c(\mathbf{a}, \mathbf{b}) \geq \min\left(\frac{45}{4}k^2 - \frac{9}{4}k^2, 3kD_0\right) \geq \min\left(M(6k), 3kD_0\right)$$

unless $k = 1$ and $r = 3$.

Proof. In this case we have

$$c(\mathbf{a}, \mathbf{b}) = 5rk - \frac{1}{2}r(r+1)$$

and $D_0 = r$.

If $r < 4k - 1$ then

$$c(\mathbf{a}, \mathbf{b}) > 3kr = 3kD_0$$

hence we can exclude this case.

If $r \geq 4k$ then we can write $r = (4+\epsilon)k$, with $\epsilon \in [0, 1]$. We show now that than $c(\mathbf{a}, \mathbf{b}) \geq \frac{1}{4}(45k^2 - 9k)$ holds. Consider

$$\frac{1}{4}(45k^2 - 9k) - c(\mathbf{a}, \mathbf{b}) = \frac{k^2}{2}\left(\epsilon^2 - \left(2 - \frac{1}{k}\right)\epsilon - \frac{3}{2} - \frac{1}{2k}\right).$$

The right hand side is zero for

$$\epsilon = 1 - \frac{1}{2k} \pm \frac{\sqrt{10k^2 - 2k + 1}}{2k}.$$

One of the zeroes is negative, the other one is at least 1. Hence for all $\epsilon \in [0, 1]$ we have

$$c(\mathbf{a}, \mathbf{b}) > M(6k).$$

It remains to check the case $r = 4k - 1$. Then $c(\mathbf{a}, \mathbf{b}) = 12k^2 - 3k$. Now $c(\mathbf{a}, \mathbf{b}) \leq \frac{45}{4}k^2 - \frac{9}{4}k$ is equivalent with $k \in [0, 1]$. Hence the only case that might occur is $k = 1, r = 3$. □

Remark 4.6. This exceptional case $k = 1, t = r = 3$ does occur: Let C be the dual of a smooth cubic curve. This is a sextic curve with 9 cusps and no further singularities. Since $c(\mathbf{a}, \mathbf{b}) \leq 3kD_0$ it follows that $D_0 \geq 3$. Hence $b_i \in \{4, 5\}$ and the $a_i \in \{3, 4\}$.

Let r be the number of b_i that equals 5, let A_4 be the difference between the number of i such that $b_i = 4$ and the number of i such that $a_i = 4$, let A_3 be minus the number of i such that $a_i = 3$. Then we have the following three equalities

$$r + A_4 + A_3 + 1 = 0,\ 5r + 4A_4 + 3A_3 = 0,\ 25r + 16A_4 + 9A_3 = 18.$$

These equalities come from the following facts: there is one more a_i than b_i; we have $\sum a_i = \sum b_i$ and we have $\sum b_i^2 - \sum a_i^2 = 2c(\mathbf{a}, \mathbf{b})$.

The only solution to this system of equations is $r = 3, A_4 = -3, A_3 = -1$. This implies that $b_1 = b_2 = b_3 = 5$, $a_1 = a_2 = a_3 = 4$ and $a_{t+1} = 3$. If t were bigger than 3 then $b_4 = a_4 = 4$ which contradicts the fact that $\mathbf{a}, \mathbf{b}$ come from a minimal resolution. Hence $t = 3$. Hence this exceptional case does actually occur.

Proposition 4.7. *Suppose* $(\mathbf{a}', \mathbf{b}')$ *is strongly* k*-admissible for rank* $2r$ *and degree* D_0*. Then there exists a strongly* k*-admissible sequence* $(\mathbf{a}, \mathbf{b})$ *for rank* $2r$ *and degree* D_0 *such that* $c(\mathbf{a}, \mathbf{b}) \leq c(\mathbf{a}', \mathbf{b}')$ *and*

1. $k = 1, r = t = 3, a_1 = a_2 = a_3 = 4, a_4 = 3$;
2. $r = t$, *there exists an integer* w *between* 0 *and* $r-1$ *such that* $a_1 = \dots a_w = 5k - 1$; $a_w > a_{w+1} \geq a_{w+2}$, $a_{w+2} = a_{w+3} = a_{r+1} = D_0$ *and* $b_1 = \dots = b_r = 5k$ *or*
3. *there exists an integer* w *between* 0 *and* $r-1$ *such that* $a_1 = \dots a_w = 5k-1$; $a_{w+1} = \dots = a_{r+1} = D_0$; $b_1 = \dots = b_r = 5k$ *and* $b_i = a_i + 1$ *for* $r < i \leq t$.

Proof. We start by setting $\mathbf{a} := \mathbf{a}'$ and $\mathbf{b} := \mathbf{b}'$. We apply a series of modifications to $\mathbf{a}, \mathbf{b}$ in order to end up in one of the three above mentioned forms.

First of all we may reduce $\mathbf{a}, \mathbf{b}$, i.e. there are no pairs i, j such that $a_i = b_j$. Moreover, from Lemma 4.5 it follows that $D_0 < 5k - 1$.

Recall that $c(\mathbf{a}, \mathbf{b}) = \frac{1}{2}\left(\sum b_i^2 - \sum a_i^2\right)$. We apply several operations on $\mathbf{a}, \mathbf{b}$ that fix r and D_0, keep the sequence strongly k-admissible and reduced and lower the function c:

1. Let i be the smallest index such that $a_i < 5k - 1$, let $j > i$ such that $a_j > D_0$. Assume that $i < r$, hence $b_{i+1} = 5k > a_i + 1$. Replace in $\mathbf{a}$, a_i by $a_i + 1$ and a_j by $a_j - 1$. The new sequence is clearly strongly k-admissible (here one uses $i < r$) and has a lower value of $c(\mathbf{a}, \mathbf{b})$ (here one uses $a_i > a_j$).
2. If for some $r < i < t$ we have that $b_i - b_t \geq 2$ and $b_i - a_{i-1} \geq 2$ then we can decrease b_i by one and increase b_t by one.
3. If for some $r \leq i < t$ we have that $a_i > D_0$ then we can decrease both a_i and b_{i+1}.

It might be that one has to reorder the a_i and b_i after applying one of the above operations or that one has to reduce to sequence.

Applying the first operation several times brings us in the situation that at most one of the a_i is different from $5k - 1, D_0$, or that $a_{r-1} = 5k - 1$. If we are in the latter case and at least two of the $a_i \neq 5k - 1, D_0$ then $t > r$. Apply the third operation (combined with reducing and sorting if necessary) until either $t = r$ or $a_r = D_0$. Hence in we are now in the situation that at most one a_i is different from $5k - 1, D_0$.

If $t = r$ then all the b_i equal $5k$. From Lemma 4.5 it follows that at least two of the a_i are different from $5k - 1$, or $k = 1, t = r = 3$. Hence we are either in the first of in the second case of the Proposition.

Suppose now that $t > r$. Applying the third operation several times brings us in the situation where all the a_i are either $5k - 1$ or D_0.

Suppose now that $a_r = 5k - 1$. Let i be the largest index such that $a_i = 5k - 1$, let j be the largest such that $b_j = 5k$. Since $\mathbf{a}, \mathbf{b}$ is strongly k-admissible we have that $j > r$. Replace in $\mathbf{a}, \mathbf{b}$ a_i by D_0 and b_j by $D_0 + 1$, and sort $\mathbf{b}$. Then the new sequence has a lower value of c. Iterating this allows us to assume that $b_{r+1} < 5k$ and hence that $a_r = D_0$.

Let i be largest index such that $b_i \neq D_0 + 1$. If $i = r$ then our sequence is of the third from. Suppose now that $i > r$. From $\mathbf{a}, \mathbf{b}$ we obtain a new strongly k-admissible sequence of length $t+1$, by decreasing b_i by one and by setting $b_{t+1} = D_0 + 1$, $a_{t+2} = D_0$. The new sequence has a lower value of c. Iterating this yields a sequence $\mathbf{a}, \mathbf{b}$ such that a_i is either $5k-1$ or D_0 and b_i is either $5k$ or $D_0 + 1$, and such that $a_r = D_0$. □

Remark 4.8. If $(\mathbf{a}, \mathbf{b})$ is k-admissible then $c(\mathbf{a}, \mathbf{b}) \leq \min(M(6k), 3ka_{t+1})$ holds. Let $m(d)$ be the smallest integer bigger or equal than $2M(d)/d$. Then the above mentioned condition can be rephrased as $c(\mathbf{a}, \mathbf{b}) \leq 3ka_{t+1}$ if $a_{t+1} \leq m(6k)$ and $c(\mathbf{a}, \mathbf{b}) \leq M(6k)$ if $a_{t+1} \geq m(6k)$.

Proposition 4.9. *Suppose that* $\mathbf{a}, \mathbf{b}$ *is a strongly* k*-admissible sequence of rank* $2r$*. Then* $r < m(6k)$ *if* $k > 1$ *and* $r \leq m(6) = 3$ *if* $k = 1$*.*

Proof. Suppose we have a strongly k-admissible sequence $\mathbf{a}, \mathbf{b}$ with $r \geq m(6k)$. Without loss of generality we may assume that $\mathbf{a}, \mathbf{b}$ is reduced.

If $D_0 = r$ then by Lemma 4.5 if follows that $k = 1$ and $r = 3$. If $(k, r) \neq (1, 3)$ then we have that $D_0 > r$, hence at least two of the a_i are different from $5k - 1$.

By decreasing a_{t+1} by one and increasing a_t by one, we obtain a new sequence, that is again strongly k-admissible: the value of c decreases by this operation, and since the new D_0 is still larger or equal than $m(6k)$ we have that $\min(M(6k), 3ka_{t+1}) = M(6k)$. Since the value of c for the old sequence was already smaller than this quantity the value of c for this new sequence is that again. If necessary replace $\mathbf{a}, \mathbf{b}$ by its reduction. By iterating this we end up in the case that $a_t = 5k - 1$ and $D_0 = r$. This is impossible by Lemma 4.5. □

Proposition 4.10. *Suppose* $\mathbf{a}, \mathbf{b}$ *is a strongly* k*-admissible sequence of rank* $2r$*. Then*

$$c(\mathbf{a}, \mathbf{b}) \geq \frac{3k}{2}\left(r - 1 + 2k + \sqrt{-r^2 + 4kr + 1 - 4k + 4k^2}\right).$$

Proof. For fixed r, D_0 then minimum value of $c(\mathbf{a}, \mathbf{b})$ is attained by a sequence of the from described in Proposition 4.7.

The first case of Proposition 4.7 occurs only for $(k, r) = (1, 3)$. In this case $c(\mathbf{a}, \mathbf{b}) = 9$ holds and we have

$$c(\mathbf{a}, \mathbf{b}) = 9 = \frac{3k}{2}\left(r - 1 + 2k + \sqrt{-r^2 + 4kr + 1 - 4k + 4k^2}\right).$$

In the second and third case we will vary D_0 and the additional parameter w to determine the minimum value of $c(\mathbf{a}, \mathbf{b})$ for fixed r.

Consider now sequences of the form (2). Suppose first that $w < r - 1$. For this we need that $D_0 \geq \frac{1}{2}(5k + r + 1)$.

Set $w' = (5rk - rD_0 - D_0)/(5k - 1 - D_0)$. From

$$B(s - a_{w+1}) \leq (w - w')B(s - 5k + 1) + (1 - w + w')B(s - D_0)$$

it follows that

$$\begin{aligned}
c(\mathbf{a}, \mathbf{b}) &= rB(s - 5k) + B(s) - wB(s - 5k + 1) - B(s - a_{w+1}) - (r - w)B(s - D_0) \\
&\geq rB(s - 5k) + B(s) - w'B(s - 5k + 1) - (r + 1 - w')B(s - D_0) \\
&= \frac{1}{2}(5kD_0 + 5rk - rD_0 - D_0).
\end{aligned}$$

The function

$$b : D_0 \mapsto 3kD_0 - \frac{1}{2}(5kD_0 + 5rk - rD_0 - D_0)$$

is increasing in D_0. Since $c(\mathbf{a},\mathbf{b})$ is also increasing in D_0, we decrease D_0 until either $w = r-1$ (and hence $D_0 = \frac{1}{2}(5k+r+1)$) holds or that $b(D_0) = 0$ holds. This latter case cannot happen: For this it suffices to show that $b(1/2(5k+r+1)) \geq 0$. Consider

$$b\left(\frac{1}{2}(5k+r+1)\right) = \frac{1}{4}(k^2 - 4rk + 6k + 2r + 1 + r^2)$$

This quantity is positive for $r, k \geq 0$. Hence we may decrease D_0 until $w = r-1$ holds.

Suppose now that $w = r-1$ and hence $D_0 \leq \frac{1}{2}(5k+r+1)$. Then

$$c(\mathbf{a},\mathbf{b}) = \frac{1}{2}r(1-r) - D_0 + 5kD_0 - D_0^2 + D_0 r.$$

We look now for the smallest D_0 such that $c(\mathbf{a},\mathbf{b}) \leq 3kD_0$. Call this value $D_{0,\min}$. (For $D_0 < D_{0,\min}$ our sequence $\mathbf{a},\mathbf{b}$ is not k-admissible, for $D_0 > D_{0,\min}$ we find a higher value of $c(\mathbf{a},\mathbf{b})$.) Solving

$$3kD_0 = \frac{1}{2}r(1-r) - D_0 + 5kD_0 - D_0^2 + D_0 r$$

yields

$$D_{0,\min} = \frac{1}{2}\left(2k - 1 + r + \sqrt{4k^2 + 4kr - r^2 - 4k + 1}\right).$$

The minimal value of $c(\mathbf{a},\mathbf{b})$ is then $3kD_{0,\min}$ which is precisely the bound mentioned in the statement.

Consider now case (3) of Proposition 4.7. We have $b_1 = \cdots = b_r = 5k$, $b_{r+1} = \cdots = b_t = D_0 + 1$, $a_1 = \cdots = a_w = 5k-1$ and $a_{w+1} = \cdots = a_{t+1} = D_0$, $w \leq r$. For the same reason as above we have that $D_0 > r$. In order to have $\sum a_i = \sum b_i$ we need

$$t = (5k - 1 - D_0)(w - r) + D_0.$$

Note that t is increasing in w and the function $2c(\mathbf{a},\mathbf{b})$ equals

$$(D_0 + 1 - 5k)(5k - 2 - D_0)w + D_0 - 5rk + D_0^2 + 25rk^2 - 10rkD_0 + D_0 r + D_0^2 r.$$

This function is either constant or is decreasing in w, hence we may take w as large as possible, namely $w = r-1$, and therefore

$$t = 1 - 5k + 2D_0.$$

Note that $D_0 \geq \frac{1}{2}(r + 5k - 1)$. Differentiating c with respect to D_0 yields that c increases as function of D_0 for $D_0 \geq \frac{1}{2}(r+5k-1)$. Hence the minimal value for c is attained at $D_0 = \frac{1}{2}(r+5k-1)$. In that case we have

$$c(\mathbf{a},\mathbf{b}) = \frac{1}{4}(25k^2 + 2rk - 10k - r^2 + 1).$$

A straightforward calculation shows that $3kD_{0,\min} - \frac{1}{4}(25k^2 + 2rk - 10k - r^2 + 1)$ has a maximum in $r = 2k$, and that for $r = 2k$ this function is negative. Hence for each D_0 we have that $c(\mathbf{a},\mathbf{b}) \geq 3kD_{0,\min}$ which finishes the proof. $\square$

Remark 4.11. For $r = 1$ we find that C has at least $6k^2$ cusps. If we expand the right hand side of the inequality as a function for $k \to \infty$ we obtain

$$\frac{3k}{2}\left(r-1+2k+\sqrt{-r^2+4kr+1-4k+4k^2}\right) = 6k^2+3(r-1)k+\frac{3}{4}r(1-r)+O\left(\frac{1}{k}\right).$$

Remark 4.12. We can specialize to the case $k = 1$. Then we find that to have $r = 1$ one needs at least 6 cusps, to have $r = 2$ one needs at least 8 cusps, to have $r = 3$ one needs at least 9 cusps and for $r > 3$ one would need at least 10 cusps. Since a sextic has at most 9 cusps, this is not possible. The above mentioned bounds for the minimal number of cusps are sharp, see [7, Theorem 9.2].

Corollary 4.13. *There is a function $g : \mathbf{Z}_{>0} \to \mathbf{Z}_{>0}$ such that for each $r > g(k)$ there does not exist a strongly k-admissible sequence $\mathbf{a}, \mathbf{b}$ of rank $2r$. Moreover one has for $k \geq 2$ that*

$$g(k) \leq \frac{1}{4}(15k-1-\sqrt{15k^2-18k+7}) \sim \frac{1}{4}(15-\sqrt{15})k \approx 2.78k.$$

Proof. We know that for fixed r and k we have by the previous Proposition that

$$c(\mathbf{a},\mathbf{b}) \geq \frac{3k}{2}\left(r-1+2k+\sqrt{-r^2+4kr+1-4k+4k^2)}\right).$$

If $\mathbf{a}, \mathbf{b}$ is k-admissible than

$$c(\mathbf{a},\mathbf{b}) \leq \frac{45}{4}k^2 - \frac{9}{4}k.$$

Hence if for r there exists strongly k-admissible $\mathbf{a}, \mathbf{b}$ then

$$\frac{3k}{2}\left(r-1+2k+\sqrt{-r^2+4kr+1-4k+4k^2)}\right) \leq c(\mathbf{a},\mathbf{b}) \leq \frac{45}{4}k^2 - \frac{9}{4}k.$$

One can solve this for r and one obtain that

$$r \leq \frac{1}{4}(15k-1-\sqrt{15k^2-18k+7})$$

or

$$r \geq \frac{1}{4}(15k-1+\sqrt{15k^2-18k+7}).$$

In the latter case we have that $r \geq 4k-1 \geq m(6k)$. This case is excluded by Proposition 4.9. □

5. Admissible case

We consider now the case that where the resolution of I, the ideal of the cusps, yields a sequence $\mathbf{a}', \mathbf{b}'$ that is k-admissible but not strongly k-admissible. We show that in this case there is a curve of small degree containing many of the cusps. This yields additional numerical constraints on $\mathbf{a}', \mathbf{b}'$. We will use these extra constraints to construct a strongly k-admissible sequence $\mathbf{a}, \mathbf{b}$ for the same rank such that $c(\mathbf{a}, \mathbf{b}) \leq c(\mathbf{a}', \mathbf{b}')$:

Proposition 5.1. *Suppose $\mathbf{a}', \mathbf{b}'$ form a sequence coming from the resolution of the ideal of cusps of a cuspidal curve for rank $2r$. Suppose that $\mathbf{a}', \mathbf{b}'$ is not strongly k-admissible. Then there exists a* strongly k-admissible *sequence $\mathbf{a}, \mathbf{b}$ for rank $2r$ such that $c(\mathbf{a}, \mathbf{b}) \leq c(\mathbf{a}, \mathbf{b})$.*

Proof. Set $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$. We are going to modify $\mathbf{a}$ and $\mathbf{b}$ such that they become strongly admissible.

Step 1: Set-up and goal

To study non-strongly k-admissible sequence we need to introduce some further notation. Without loss of generality we may assume that $\mathbf{a}, \mathbf{b}$ is reduced, in particular $a_i \neq b_{i+1}$.

Since the a_i, b_i are not strongly k-admissible, there is an i such that $a_i > b_{i+1}$. Let i_0 be the smallest index such that $a_{i_0} > b_{i_0+1}$ holds. Let $A = a_{i_0}$ and $D_2 = \sum_{i=1}^{i_0} b_i - a_i$.

Define

$$h_1 := \sum_{i=1}^{i_0} B(s-b_i) - B(s-a_i)$$

and

$$h_2 := B(s) - B(s-a_{t+1}) + \sum_{i=i_0+1}^{t} B(s-b_i) - B(s-a_i).$$

Then $h_1 + h_2$ is the Hilbert Polynomial of I. In particular, $\#\Sigma = h_1 + h_2$.

Consider now the ideal I' generated by the elements of I of degree strictly less than A. Let f_1 be the greatest common divisor of the elements of I'. Let I'' be the ideal generated by the elements of I' divided by f_1. Let $D_1 := \deg(f_1)$.

Now I'' defines a zero-dimensional scheme, which is possibly non-reduced, moreover I'' might be not saturated. If I'' were saturated then its resolution has length 1, but otherwise the resolution might be of length 2. From this it follows that I' has a resolution of length at most 2. Hence there might exist a $s_0 \in \mathbf{Z}_{\geq 0}$ and $c_j, d_j \in \mathbf{Z}$ for $j = 1, \dots, s_0$ such that $A \leq c_j < d_j$ and

$$0 \to \oplus_{j=1}^{s_0} S(-d_j) \to \oplus_{j=1}^{s_0} S(-c_j) \bigoplus \oplus_{i>i_0} S(-b_i) \to \oplus_{i>i_0} S(-a_i) \to S \to S/I' \to 0$$

is a minimal resolution of I'. Let $h_3 = \sum_{j=1}^{s_0} (B(s-c_j) - B(s-d_j))$.

With this notation we have that the Hilbert polynomial of I equals $h_1 + h_2$, the Hilbert Polynomial of I' equals $h_2 + h_3$ and the Hilbert polynomial of I'' equals $h_2 + h_3 - h_C$, with $h_C(s) = D_1 s - \frac{1}{2} D_1(D_1+3)$, the Hilbert polynomial of the ideal (f_1).

We will find some additional restrictions.

Note that the Hilbert polynomial of I'' is constant. The coefficient of s in $h_2(s) + h_3(s)$ equals $D_2 + \sum d_j - c_j$. Hence $D_1 = D_2 + \sum (d_j - c_j)$. Since $d_j > c_j$ it follows that $D_2 \leq D_1$. Using $i_0 \geq r$ and $D_2 = \sum_{i=1}^{i_0} (b_i - a_i)$ we obtain that $D_2 \geq r$.

Suppose the curve $f_1 = 0$ contains more than $3kD_1$ cusps of C. Then C and $Z(f_1)$ have a common component. Using the same reasoning as in Proposition 2.5 one can show that the common component has non-positive degree. From this it follows that $Z(f_0)$ contains at most $3kD_1$ points of Σ. Therefore $Z(I'')$ contains at least $c(\mathbf{a}, \mathbf{b}) - 3kD_1$ points of Σ and

$$3kD_1 \geq \#Z(I) - \#Z(I'') \geq h_1 + h_2 - h_2 - h_3 + h_C = h_1 - h_3 + h_C.$$

Note that up to degree $A - 1$ the generators and syzygies of I and I' agree. Hence $h_I(A-1) = h_{I'}(A-1)$. Now $h_I(A-1) = h_2(A-1)$ and $h_{I'} = h_{I''} + h_C$. From this we get

$$0 \leq h_{I''}(A-1) = h_{I'}(A-1) - h_C(A-1) = h_2(A-1) - h_C(A-1).$$

Summarizing we found sequences $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and integers i_0, D_0, D_1, D_2, A such that (with h_1, h_2, h_3, h_C as above)

(1) $a_i \geq A$ for $i = 1, \dots, i_0$,
(2) $b_i < A$ for $i = i_0, \dots, t$.
(3) $D_0 \leq a_i < b_i \leq 5k$ for $i = 1, \dots t$.
(4) $b_i = 5k$ for $i = 1, \dots, r$.
(5) $D_0 = a_{t+1} = \sum_{i=1}^{t}(b_i - a_i)$.
(6) $A \leq c_j < d_j$ for $j = 1, \dots s_0$.
(7) $a_i < b_{i+1}$ for $i = 1, \dots, i_0 - 1$
(8) $D_2 = \sum_{i=1}^{i_0}(b_i - a_i)$.
(9) $r \leq D_2 \leq D_1 \leq D_0 \leq A \leq 5k - 1$.
(10) $h_1 + h_2 \leq 3kD_0$.
(11) $h_1 + h_2 \leq \frac{1}{4}(45k^2 - 9k)$.
(12) $h_1 - h_3 + h_C \leq 3kD_1$.
(13) $h_2(A-1) \geq h_C(A-1)$.

We want to show that for given r, a sequence $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and integers D_1, D_2, A satisfying the above conditions there exists a sequence $\mathbf{a}', \mathbf{b}'$ with same rank r, but that is strongly k-admissible and $c(\mathbf{a}, \mathbf{b}) \geq c(\mathbf{a}', \mathbf{b}')$. We do this by changing the above mentioned parameters in such a way that $c(\mathbf{a}, \mathbf{b})$ decreases and such that in the end we have $D_0 = A$ or $D_0 = D_1 = D_2$. In the former case we clearly have a strongly k-admissible sequence. In the latter case we use that $D_0 = D_2$ implies $i_0 = t$, hence $a_i < b_{i+1}$ for $i = 1, \dots i_0 - 1 = t - 1$, which in turn implies that the sequence is strongly k-admissible.

Step 2: Optimization of h_1, h_2, h_3 without changing D_0, D_1, D_2, A and r. We first optimize our $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ without changing D_0, D_1, D_2, A and r. Specifically, we aim at decreasing the constant coefficients of h_1 and h_2 and at increasing the constant coefficient of h_3. Hence at this stage we only have to consider the conditions (1)-(8).

The sequence $a_1, \dots, a_{i_0}, D_1; b_1, \dots, b_{i_0}$ is a strongly k-admissible sequence. This means that we can apply the same transformations as in the proof of Proposition 4.7, only that we need to impose $a_i \geq A$ for $i = 1, \dots, i_0$.

If for some $i < j \leq i_0$ we have $5k - 1 > a_i \geq a_j > A$, then we can increase a_i by one and decrease a_j by one then reduce and sort. The new sequence still satisfies the above mentioned conditions, but $c(\mathbf{a}, \mathbf{b})$ decreases. So $a_i \in \{5k - 1, A\}$ for all but at most one $i \leq i_0$.

Suppose $i_0 > r$ and for some $i < i_0$ we have that $A < a_i < 5k - 1$. Let j be such that $b_j \neq 5k$. If $b_j < a_i$ then we can increase both a_i and b_j by $5k - 1 - a_i$, and hence all the $a_i \in \{5k - 1, A\}$. If $b_j > a_i$ then we can lower them both with $a_i - A$ and sort the b_i if necessary. From this it follows that all the $a_i \in \{5k - 1, A\}$.

Suppose $i_0 > r$ and $b_{i_0+1} > A + 1$, then we lower b_{i_0} by one and increase the length of our original sequence $\mathbf{a}, \mathbf{b}$ by adding A to $\mathbf{a}$ and adding $A + 1$ to $\mathbf{b}$. Now sort and reduce.

The optimization of h_1 allows us to assume for $i \leq i_0$ that either

(1) $i_0 = r$, $b_i = 5k$, $i = 1, \dots r$ and there exists a $w \leq r - 1$ such that $a_1 = \dots a_w = 5k - 1 > a_{w+1} \geq a_{w+2} = \dots = a_r = A$. (If $w = r - 1$ then $a_{w+1} = A$. In this case we might disregard a_{w+2}, since $w + 2 > i_0$.)
(2) $i_0 > r$, $b_i = 5k$, $i = 1, \dots r$, $b_i = A + 1$ for $i = r + 1, \dots, i_0$ and there exists a $w \leq r - 1$ such that $a_1 = \dots a_w = 5k - 1$, $a_{w+1} = \dots a_{i_0} = A$.

This description of $\mathbf{a},\mathbf{b}$ implies $D_2\geq 5k-A+r-1$ or, equivalently, $A\geq 5k+r-1-D_2$.

Suppose we are in case (1). Then A and D_2 determine both w and a_{w+1} hence the function h_1 depends only on A and D_2. Denote this function by $h_{1,a}(A,D_2)$. If we are in case (2) then A,D_2 and w determine i_0. Hence we have a function $h_{1,b}(A,D_2,w)$. Now $h_{1,b}$ is decreasing in w, hence we may assume that $w=r-1$. Since

$$h_{1,b}(A,D_2,r-1)\leq h_{1,a}(A,D_2)+\frac{1}{2}(A+2-5k)(A-5k-r+d_1+1)$$

we have that the constant polynomial $h_{1,b}-h_{1,a}$ is negative. Hence we may assume that $h_1=h_{1,b}(A,D_2,r-1)$, i.e.,

$$h_1(s)=-D_2s-r+1+5kr-rA-D_2+D_2A-\frac{15}{2}k+\frac{3}{2}A+\frac{1}{2}A^2+\frac{25}{2}k^2-5kA.$$

The optimization of h_3 is relatively easy. First lowering c_j and d_j simultaneously increases the constant coefficient of h_3, so we may assume that $c_j=A$ for all j. Suppose that for some j we have $d_j\neq A+1$. We can lower d_j by one as follows: we increase the length of $\mathbf{c}$ by one by setting $c_{s_0+1}=A$. We increase the length of $\mathbf{d}$ by setting $d_{s_0+1}=A+1$, and decreasing d_j by one. Then the constant coefficient of h_3 increases under this operation. This allows us to assume that $d_j=A+1$.

$$h_3=(D_1-D_2)s+(D_1-D_2)(1-A).$$

We can optimize h_2 as follows. If for some $j>i_0$ we decrease b_j and a_j simultaneously by one then the constant coefficient of h_2 decreases. If we extend $\mathbf{a},\mathbf{b}$ by setting $a_{t+2}=D_0$, $b_{t+1}=D_0+1$, and lowering one of the b_j for some $j>i_0$ then $c(\mathbf{a},\mathbf{b})$ decreases. However, we have to stop as soon as $h_2(A-1)=h_C(A-1)$. I.e., this allows us to assume that $h_2=\max(h_{2,a},h_{2,b})$ with

- $h_{2,a}=B(s)-B(s-D_0)+(D_0-D_2)(B(s-D_0-1)-B(s-D_0))=D_2s+(\frac{1}{2}D_0^2+\frac{1}{2}D_0-D_2(D_0+1)$,
- $h_{2,b}=D_2s+\frac{1}{2}(D_1-D_1^2)+D_1A-D_2(A-1)$.

In the next step we are going to vary A,D_0,D_1,D_2 in such a way that if we start with an k-admissible sequence, not strongly k-admissible and satisfying the conditions (1)-(13) then the new sequence is still k-admissible, satisfies (1)-(13), but might have a lower value of $c(\mathbf{a},\mathbf{b})=h_1+\max(h_{2,a},h_{2,b})$. It turns out that we end up either in the case $A=D_0$ or $D_2=D_0$.

The only remaining variables are A,D_0,D_1,D_2, hence we may disregard the coniditions (1)-(8). Also we want to minimize h_1+h_2, hence condition (11) is a priori fullfilled. By replacing h_2 by $\max h_{2,a},h_{2,b}$ we forced condition (13) to hold. Hence we need only to consider the conditions (9), (10) and (12). We consider (9) as describing a domain in which the parameters A,D_0,D_1,D_2 may vary, and try to minimize h_1+h_2 in such a way that (10) and (12) hold.

Step 3: Elimination of A

Note that the main conditions we are considering are

$$h_1+h_{2,a}\leq 3kD_0, h_1+h_{2,b}\leq 3kD_0, h_1+h_C-h_3\leq 3kD_1.$$

Since $h_{2,b}+h_3-h_C$ is a constant polynomial, $h_3(A-1)=0$ and $h_{2,b}(A-1)=h_C(A-1)$ it follows that $h_{2,b}=h_C-h_3$. So the left hand side of the second and third inequality agree. Since $D_0>D_1$ we might ignore the second inequality.

Both $h_1 + h_{2,a}$ and $h_1 + h_{2,b}$ are increasing as a function of A, for $A \geq 5k + r - 1 - D_2$. Hence we might take A as small as possible, which means that either $A = D_0$ or $A = 5k + r - 1 - D_2$. In the first case we are done. So from now on we assume that

$$A = 5k + r - 1 - D_2.$$

Step 4: Elimination of D_1

Substituting $A = 5k+r-1-D_2$ yields new functions h_1, $h_{2,a}$, $h_{2,b}$ and h_3. The function $h_1 + h_{2,a}$ increases with D_0, whereas $h_1 + h_{2,b}$ is independent of D_0. So we might decrease D_0 until one of the following three case occurs:

$$D_0 = D_1, h_{2,b} = h_{2,a} \text{ or } h_1 + h_{2,a} = 3kD_0$$

We claim now that even in the second and in the third case we may also assume that $D_0 = D_1$ holds.

If D_0 is such that $h_{2,b} = h_{2,a}$ then the only interesting inequalities are

$$h_1 + h_{2,b} \leq \min\left(3kD_0, \frac{1}{4}(45k^2 - 9k)\right), h_1 + h_{2,b} \leq 3kD_1$$

Now h_1 and $h_{2,b}$ are independent of D_0. Since $D_1 < D_0$ we may simplify these bounds to $h_1 + h_{2,b} \leq \min(3kD_1, 45k^2 - 9k)$, which is completely independent of D_0. Hence we may decrease D_0 such that $D_1 = D_0$ since $c(\mathbf{a}, \mathbf{b})$ is increasing in D_0 and D_0 is not involved in any of the further bounds except $D_1 \leq D_0$.

Suppose now that D_0 is such that $h_1 + h_{2,a} = 3kD_0$ and $h_{2,a} > h_{2,b}$. Then $3kD_1 - h_1 - h_{2,b}$ is increasing in D_1 if $D_2 + D_1 \geq r + 2k$. Now $D_1 \geq D_2 \geq r$, hence this condition is automatically satisfied as soon as $r \geq 2k$.

If $D_1 + D_2 \geq r + 2k$ then this implies that we may increase D_1 until we reach $D_1 = D_0$ or $h_{2,a} = h_{2,b}$. The latter case we can apply the above argument to obtain $D_0 = D_1$. So if $D_1 + D_2 \geq r + 2k$ we may assume $D_0 = D_1$.

If $D_1 + D_2 \leq r + 2k$ and $h_{2,a} > h_{2,b}$ then $c(\mathbf{a}, \mathbf{b})$ is independent of D_1. Hence we might decrease D_1, since this decreases $h_{2,b} + h_1$ and increases $h_{2,a} - h_{2,b}$. Hence we get in the situation that $D_1 = D_2$. The new function $3kD_2 - h_{2,b} + h_1$ decreases with D_2, whereas $c(\mathbf{a}, \mathbf{b})$ increases with D_2. So we may decrease $D_2 = D_1$ as long as all lower bounds for D_2 are satisfied. However, the only bound for D_2 is $D_2 \geq r$. Hence we are in the situation that $D_1 = D_2 = r$. From this it follows that $A = 5k - 1$. Substituting this in $h_1 + h_{2,b} \leq 3kD_1$ yields

$$-2kn + (1/2)k + (1/2)k^2 \geq 0.$$

In particular $r \geq 4k - 1$. Since $D_2 + D_1 \geq r + 2k$ implies $r \leq 2k$ this case does not occur.

Step 5

Assume now $D_0 = D_1$. Consider the following bounds

$$h_1 + h_{2,a} \leq 3kD_0, h_1 + h_{2,b} \leq 3kD_0 \text{ and } h_1 + h_C - h_3 \leq 3kD_1$$

The second and third inequality coincide: as remarked before we have that $h_{2,b} = h_C - h_3$, moreover we assumed now that $D_0 = D_1$.

Suppose that $h_{2,b} \leq h_{2,a}$. Then we only need to consider the first inequality $h_1 + h_{2,a} \leq 3kD_0$. Now $h_1 + h_{2,a}$ increases with D_2 hence in this case we might decrease D_2 until either $D_2 = r$ holds, or $h_{2,a} = h_{2,b}$ holds.

Suppose that $h_{2,b} \geq h_{2,a}$. Then we only need to consider the first inequality $h_1 + h_{2,b} \leq 3kD_0$. Now $h_1 + h_{2,b}$ decreases with D_2 hence in this case we might increase D_2 until either $D_2 = D_1 = D_0$ holds or $h_{2,a} = h_{2,b}$ holds.

So we have either $h_{2,a} = h_{2,b}$, $D_2 = r$ or $D_0 = D_1 = D_2$. In the latter case we are done.

Subcase $D_2 = r$. If $D_2 = r$, then

$$h_1 + h_{2,b} = 5kD_0 - \frac{1}{2}D_0(D_0 + 1).$$

One of the conditions we need to check is that $h_1 + h_{2,b} \leq 3kD_0$. This inequality implies that $D_0 \geq 4k - 1$. Now, $h_1 + h_{2,b}$ is increasing in D_0 for $D_0 < 5k$. In particular

$$h_1 + h_{2,b} \geq 5k(4k - 1) - \frac{1}{2}(4k - 1)4k = 12k^2 - 3k$$

This contradicts $h_1 + h_{2,b} \leq \frac{45}{4}k^2 - \frac{9}{4}k$, unless $k = 1$ and $D_0 \geq 3$. Now if $k = 1$ and $D_0 \geq 3$ then $a_i \in \{3, 4\}$ and $b_i \in \{4, 5\}$. An easy computation shows that $4 - r$ of the a_i equal 3. In particular, we have $b_1 = 5, b_2 = 4, a_1 = a_2 = a_3 = 3$, $b_1 = b_2 = 5, a_1 = 4, a_2 = a_3 = 3$ or $b_1 = b_2 = b_3 = 5, a_1 = a_2 = a_3 = 4, a_4 = 3$. These are all strongly k-admissible.

Subcase $h_{2,a} = h_{2,b}$. Hence the remaining case is $h_{2,a} = h_{2,b}$. Using that $D_0 = D_1$, $A = 5k + r - 1 - D_2$ it follows that $h_{2,a} = h_{2,b}$ implies

$$0 = (D_0 - D_2)(D_0 + 1 - r - 5k + D_1).$$

Hence we have either $D_0 = D_2$ or $D_0 = r - 1 + 5k - D_2 = A$. In both case we are done. □

Theorem 5.2. *Suppose $C = Z(f)$ is a reduced degree $6k$ curve with only nodes and ordinary cusps as singularities. Then both twice the exponent of $t^2 - t_1$ in the Alexander polynomial of C and the Mordell-Weil rank of the elliptic 3-fold given by $y^2 = x^3 + f$ are at most*

$$\frac{1}{2}(15k - 1 - \sqrt{15k^2 - 18k + 7}).$$

Proof. Let $2r$ be the Mordell-Weil rank. By Proposition 5.1 there exists a strongly k-admissible sequence of rank $2r$. From Corollary 4.13 it follows that

$$r \leq \frac{1}{4}(15k - 1 - \sqrt{15k^2 - 18k + 7}).$$

□

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