

# Self-similar asymptotics of solutions to the Navier-Stokes system in two dimensional exterior domain

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## Abstract

We consider the 2D incompressible Navier-Stokes equations with Dirichlet boundary condition in the exterior of one obstacle. Assuming that the circulation at infinity of the velocity is sufficiently small, we prove that the large time behavior of the corresponding solution to the initial-boundary value problem is described by the Lamb-Oseen vortex. The later is the well-known explicit self-similar solution to the Navier-Stokes system in the whole space  $\mathbb{R}^2$ .

## 1 Introduction

It is well-known that the large time behavior of solutions of the initial-value problem for the Navier-Stokes equations considered either in the whole space  $\mathbb{R}^n$ ,  $n \geq 2$ , or in an exterior domain depends on integrability properties of initial conditions. In the finite energy case, that is when the velocity is square integrable, a solution tends to zero in  $L^2(\mathbb{R}^n)$  as time goes to infinity, see *e.g.* [2, 18, 19] and references therein. In this case, nonlinear effects are negligible for large values of time and asymptotics of solutions is determined by the corresponding Stokes semigroup.

On the other hand, when an initial velocity is not square integrable, a solution of the initial value problem for the Navier-Stokes in  $\mathbb{R}^n$  with  $n \geq 2$  is constructed in a so-called scaling invariant space (*e.g.* in a homogeneous Besov space or in a weak  $L^n$ -space) under suitable smallness assumption on initial conditions, see the review article [3] and the book [16]. Here, the large time behavior of solutions is described by self-similar solutions to the Navier-Stokes system.

In this work, we contribute to the theory on the asymptotic behavior of solutions of the Navier-Stokes system in a two dimensional exterior domain. First, however, we recall that the Navier-Stokes system in the whole space  $\mathbb{R}^2$  has an explicit self-similar solution called the Lamb-Oseen vortex

$$(1.1) \quad \Theta(t, x) = \frac{x^\perp}{2\pi|x|^2} \left( 1 - e^{-\frac{|x|^2}{4t}} \right), \quad \text{with } x^\perp = (x_2, -x_1),$$

which appears in the large time expansions of other infinite energy solutions of this system. Let us explain this result.

For every initial vorticity  $\omega_0 \in L^1(\mathbb{R}^2)$ , one obtains the corresponding divergence-free initial velocity field  $u_0$  via the Biot-Savart law. It is well-known that constructed-in-this-way initial condition belongs to the scaling invariant space  $L^{2,\infty}(\mathbb{R}^2)$  (the weak  $L^2$ -space) and the Navier-Stokes equations have a unique global-in-time solution corresponding to such an initial datum, see

[12]. Moreover, the large time behavior of solutions to the initial value problem for the 2D Navier-Stokes equations with an initial vorticity from  $L^1(\mathbb{R}^2)$  is given by the multiple of the Lamb-Oseen vortex  $\alpha\Theta$ , with the *circulation at infinity*  $\alpha \equiv \int_{\mathbb{R}^2} \omega_0(x) dx$ . This result was proved in [11] if  $\omega_0$  is small in  $L^1$ , in [5] in the case of small circulation, and in [9] in the general case. In fact, due to the regularizing effect of the Navier-Stokes equations, as far as large time behavior is concerned, an initial vorticity can be an arbitrary bounded Radon measure in  $\mathbb{R}^2$ , see [8].

The aim of this paper is to show an analogous result on the large time behavior of solutions of the 2D Navier-Stokes equations in an exterior domain with the Dirichlet boundary condition, when the initial velocity is not square integrable. Here, however, due to the fact that a vorticity does not verify any reasonable boundary conditions, we cannot use the vorticity equation. Hence, we formulate our hypothesis and results in terms of velocity rather than of vorticity. To see which hypothesis should be imposed on an initial velocity, we recall that, for every bounded compactly supported vorticity, one can construct the corresponding velocity field in an exterior domain, which behaves when  $|x| \rightarrow \infty$  as the vector field  $x^\perp/|x|^2$ , see [13, Sec. 2.2] and [14, Sec. 3] for more details. For this reason, we assume in this work that our initial velocity is a small multiple of the particular vector field  $x^\perp/|x|^2$  plus a large  $L^2$  part.

Let us now be more precise. Assume that  $\Omega \subset \mathbb{R}^2$  is an exterior domain, whose complement is a bounded, open, connected and simply connected set, with a smooth boundary  $\Gamma$ . Moreover, without loss of generality, we can assume that  $B(0, 1) \subset \mathbb{R}^2 \setminus \Omega$ . We consider the incompressible Navier-Stokes equations in  $\Omega$  with the Dirichlet boundary condition

$$(1.2) \quad \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{for } t > 0, \quad x \in \Omega,$$

$$(1.3) \quad u(t, x) = 0 \quad \text{for } t > 0, \quad x \in \Gamma,$$

$$(1.4) \quad u(0, x) = u_0(x) \quad \text{for } x \in \Omega.$$

Above,  $u_0$  must be divergence free and tangent to the boundary. In the following, we assume that the initial condition is of the following particular form

$$(1.5) \quad u_0 = \tilde{u}_0 + \alpha H_\Omega$$

where  $\tilde{u}_0 \in L^2_\sigma(\Omega)$  is an arbitrary square integrable, divergence free, and tangent to the boundary vector field, and  $H_\Omega$  the unique harmonic vector field in  $\Omega$  (*i.e.* the unique vector field on  $\Omega$  which is divergence free, curl free, vanishing at infinity, tangent to the boundary, and with circulation equal to 1 on the boundary  $\Gamma$ ). It was proved in [13, Sec. 2.3] that such a harmonic vector field  $H_\Omega$  exists and behaves at infinity like  $x^\perp/(2\pi|x|^2)$ . Moreover, it follows directly from [14, Lemma 6 with  $\varepsilon = 1$ ] that every velocity field with a compactly supported bounded vorticity can be written under the form (1.5). Notice, however, that in an exterior domain, the circulation at infinity  $\alpha$  is not the integral of the vorticity as it is in the full plane case. Namely, here, one has to subtract the circulation of the velocity on the boundary, so the integral of the vorticity is in fact the total circulation of the velocity, see [13, Sec. 3.1] for more details.

If the circulation at infinity is sufficiently small, we are able to prove a counterpart of the result from [11, 5, 9] on the large time behavior of the Navier-Stokes in the whole plane. The following theorem contains the main result of this work.

**Theorem 1.** *For every  $\tilde{u}_0 \in L^2_\sigma(\Omega)$  there exists a constant  $\alpha_0 = \alpha_0(\tilde{u}_0, \Omega) > 0$  such that for all  $|\alpha| \leq \alpha_0$  the solution of problem (1.2)-(1.5) satisfies*

$$(1.6) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|u(t) - \alpha\Theta(t)\|_{L^p(\Omega)} = 0$$

for each  $p \in (2, \infty)$ .

In other words, Theorem 1 says that the large time behavior of solutions to the Navier-Stokes system in an exterior domain, supplemented with the Dirichlet boundary condition and particular initial condition (1.5) is described by the explicit self-similar solution (1.1) of the Navier-Stokes system.

*Remark 2.* The global-in-time well-posedness for problem (1.2)-(1.5) was established by Kozono and Yamazaki [15, Thm.4]. The existence part of that result requires an initial velocity  $u_0$  to satisfy a smallness condition of the form  $\limsup_{R \rightarrow \infty} R |\{x \in \Omega : |u_0(x)| > R\}|^{1/2} \ll 1$ . This condition is satisfied for every  $\tilde{u}_0 \in L^2_\sigma(\Omega)$ . Since  $H_\Omega$  is bounded, the lim sup above is always zero in this case.

We apply the following strategy to prove Theorem 1. In the next section, we prove the limit relation (1.6) for the linear evolution, that is when the nonlinear term  $u \cdot \nabla u$  is skipped in equation (1.2). This is achieved by combining results in [14] with a rescaling technique used by Carpio in [5]. Next, in Section 3, we show that we can assume, without loss of generality, that  $u_0$  is small in the norm of the space  $L^{2,\infty}(\Omega)$ , by replacing the initial condition in (1.4) with  $u(t_0, x)$  with sufficiently large  $t_0$  and choosing sufficiently small  $|\alpha|$  (see Lemma 11 below). Finally, using the integral representation of solutions to problem (1.2)-(1.4), we apply a stabilization argument inspired from [1, 4] to show that, for small data in  $L^{2,\infty}(\Omega)$ , the asymptotic stability at the level of the Stokes equation implies the asymptotic stability at the level of the Navier-Stokes equations.

**Notation.** In the following, the space  $L^p_\sigma(\Omega)$  is the closure of the set of smooth, divergence-free, and compactly supported vector fields  $C^\infty_c(\Omega)$  with respect to the usual  $L^p$ -norm. We denote by  $\mathbb{P}_\Omega$  the Leray projection, *i.e.* the  $L^2$  orthogonal projection onto  $L^2_\sigma(\Omega)$ , which can be extended to a bounded operator on  $L^p(\Omega)$  for every  $p \in (1, \infty)$ . Thus, the space  $L^p_\sigma(\Omega)$  is the image of  $L^p(\Omega)$  by  $\mathbb{P}_\Omega$ . In a similar way, for every  $p \in (1, \infty)$ , we define  $L^{p,\infty}_\sigma(\Omega) = \mathbb{P}_\Omega(L^{p,\infty}(\Omega))$ , where  $L^{p,\infty}(\Omega)$  is the Marcinkiewicz weak  $L^p$ -space. Hence  $u \in L^{p,\infty}_\sigma(\Omega)$  if  $u \in L^{p,\infty}(\Omega) \times L^{p,\infty}(\Omega)$ ,  $\operatorname{div} u = 0$  in  $\Omega$  and  $u \cdot n = 0$  on  $\Gamma$ , where  $n$  is the normal vector to the boundary  $\Gamma$ . The ball  $B(0, R) \subset \mathbb{R}^2$  is centered at zero and of radius  $R > 0$ . By the letter  $E$ , we denote the extension operator of functions defined on  $\Omega$  to  $\mathbb{R}^2$  with zero values outside the domain of definition.

## 2 Asymptotics of solutions to the linear evolution

It is well-known that the Stokes operator associated with the following linear boundary value problem

$$(2.1) \quad \partial_t v - \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0 \quad \text{for } t > 0, \quad x \in \Omega,$$

$$(2.2) \quad v(t, x) = 0 \quad \text{for } t > 0, \quad x \in \Gamma,$$

$$(2.3) \quad v(0, x) = v_0(x) \quad \text{for } x \in \Omega,$$

where  $v_0$  is divergence free and tangent to the boundary, generates an analytic semigroup  $S(t)$  on  $L^p_\sigma(\Omega)$ , for each  $1 < p < \infty$ , see [10]. Moreover, this semigroup satisfies the following decay  $L^p$  estimates.

**Proposition 3.** *Assume that  $1 < q < \infty$ .*

*Let  $q \leq p \leq \infty$ . There exists  $K_1 = K_1(\Omega, p, q) > 0$  such that for every  $v_0 \in L^q_\sigma(\Omega)$*

$$(2.4) \quad \|S(t)v_0\|_{L^p(\Omega)} \leq K_1 t^{\frac{1}{p} - \frac{1}{q}} \|v_0\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

*If, in addition, we assume that  $q < p \leq \infty$ , then for every  $v_0 \in L^{q,\infty}_\sigma(\Omega)$  we also have*

$$(2.5) \quad \|S(t)v_0\|_{L^p(\Omega)} \leq K_1 t^{\frac{1}{p} - \frac{1}{q}} \|v_0\|_{L^{q,\infty}(\Omega)} \quad \text{for all } t > 0.$$

There exists  $K_2 = K_2(\Omega, q) > 0$  such that for every  $v_0 \in L^{q,\infty}_\sigma(\Omega)$  we have the inequality

$$(2.6) \quad \|S(t)v_0\|_{L^{q,\infty}(\Omega)} \leq K_2 \|v_0\|_{L^{q,\infty}(\Omega)} \quad \text{for all } t > 0.$$

Let  $q \leq p \leq 2$ . There exists  $K_3 = K_3(\Omega, p, q) > 0$  such that

$$(2.7) \quad \|\nabla S(t)v_0\|_{L^p(\Omega)} \leq K_3 t^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|v_0\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

Assume  $q \geq 2$  and let  $q \leq p < \infty$ . Then there exists  $K_4 = K_4(\Omega, p, q) > 0$  such that for every matrix  $F \in L^q(\Omega; M_{2 \times 2}(\mathbb{R}))$

$$(2.8) \quad \|S(t)\mathbb{P}_\Omega \operatorname{div} F\|_{L^p(\Omega)} \leq K_4 t^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|F\|_{L^q(\Omega)} \quad \text{for all } t > 0,$$

with the divergence  $\operatorname{div}$  computed along rows of the matrix  $F$ .

Estimates (2.4)–(2.7) were proved in [6, 7, 15, 17] and estimate (2.8) follows from (2.7) by a duality argument because the adjoint of  $\nabla S(t)$  on  $L^p_\sigma(\Omega)$  is  $S(t)\mathbb{P} \operatorname{div}$ .

*Remark 4.* Recall the following scale invariance of the Stokes equations: the vector  $(v(t, x), p(t, x))$  is a solution of system (2.1) on  $\Omega$  if and only if for every  $\lambda > 0$  the vector  $(\lambda v(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x))$  is a solution of the same system on  $\Omega/\lambda = \{x \in \mathbb{R}^2; \lambda x \in \Omega\}$ . It follows from this scale invariance that the constants  $K_1, \dots, K_4$  associated to  $\Omega/\lambda$  are independent of  $\lambda$ .

The following corollary contains a minor improvement of the decay estimate (2.4).

**Corollary 5.** *Assume that  $1 < q < \infty$  and let  $v_0 \in L^q_\sigma(\Omega)$ . Then for every  $p \in (q, \infty)$*

$$\lim_{t \rightarrow \infty} t^{\frac{1}{q} - \frac{1}{p}} \|S(t)v_0\|_{L^p(\Omega)} = 0.$$

*Proof.* This limit relation is clear when the initial datum is smooth and compactly supported. To show it for all  $v_0 \in L^q_\sigma(\Omega)$ , it suffices to use a standard density argument combined with estimate (2.4).  $\square$

Now, we consider the linear problem (2.1)–(2.3) with the initial datum  $v_0 = H_\Omega$ , where  $H_\Omega$  the unique harmonic vector field in  $\Omega$ . The main goal of this section is to show that the large time behavior of  $S(t)H_\Omega$  is described by the Lamb-Oseen vortex  $\Theta$ . More precisely, we will prove the following theorem.

**Theorem 6.** *For every  $p \in (2, \infty)$ , we have  $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|S(t)H_\Omega - \Theta(t)\|_{L^p(\Omega)} = 0$ .*

The remainder of this section is devoted to the proof of this theorem. Here, we use a scaling argument that was also applied in [5] to study large time asymptotics for the Navier-Stokes equations. Hence, for every  $\lambda \geq 1$ , we define

$$\Omega_\lambda \equiv \Omega/\lambda = \{x \in \mathbb{R}^2; \lambda x \in \Omega\}.$$

The vector field  $\lambda H_\Omega(\lambda x)$  is divergence free, curl free, tangent to the boundary of  $\Omega_\lambda$ , vanishes at infinity and has circulation equal to 1 on  $\partial\Omega_\lambda$ . Thus, by [13, Prop. 2.1], this rescaled vector field has to be equal to the unique harmonic vector field on  $\Omega_\lambda$ , namely, we have the identity

$$(2.9) \quad H_{\Omega_\lambda}(x) = \lambda H_\Omega(\lambda x).$$

Let us now denote by  $S_\lambda(t)$  the Stokes semi-group on the domain  $\Omega_\lambda$  and let us define

$$(2.10) \quad H_\lambda(t, x) \equiv S_\lambda(t)H_{\Omega_\lambda}.$$

By the scaling invariance of equations (2.1), by (2.9), and by the uniqueness of solutions to the Stokes problem, we infer that

$$H_\lambda(t, x) = \lambda H_1(\lambda^2 t, \lambda x),$$

where we put  $H_1(t, x) = S(t)H_\Omega$ . Recalling, moreover, the scaling property of the Lamb-Oseen vortex  $\lambda\Theta(\lambda^2 t, \lambda x) = \Theta(t, x)$ , we observe that the conclusion of Theorem 6 is equivalent to

$$\lim_{\lambda \rightarrow \infty} \|H_\lambda(1) - \Theta(1)\|_{L^p(\Omega_\lambda)} = 0 \quad \text{for every } p \in (2, \infty).$$

In the following, we denote by  $E$  the extension operator to  $\mathbb{R}^2$  with zero values outside the domain of definition. Since  $\Theta(1)$  is a bounded function, we immediately obtain that  $\lim_{\lambda \rightarrow \infty} \|\Theta(1)\|_{L^p(\mathbb{R}^2 \setminus \Omega_\lambda)} = 0$ . Hence, in order to prove Theorem 6, it suffices to show that

$$(2.11) \quad EH_\lambda(1, x) \xrightarrow{\lambda \rightarrow \infty} \Theta(1, x) \quad \text{strongly in } L^p(\mathbb{R}^2) \quad \text{for every } p \in (2, \infty).$$

First, we state a result on the weak convergence.

**Lemma 7.** *Let  $H_\lambda(t, x) = \lambda H_1(\lambda^2 t, \lambda x)$ . Then*

$$(2.12) \quad EH_\lambda(1, x) \xrightarrow{\lambda \rightarrow \infty} \Theta(1, x) \quad \text{weakly in } L^p(\mathbb{R}^2) \quad \text{for every } p \in (2, \infty).$$

*Proof.* Observe now that due to the identity  $\|H_{\Omega_\lambda}\|_{L^2_\sigma(\Omega_\lambda)} = \|H_\Omega\|_{L^2_\sigma(\Omega)}$  for every  $\lambda \geq 1$ , the scaling invariant estimate (2.5) implies that the family  $\{EH_\lambda(1)\}_{\lambda \geq 1}$  is bounded in  $L^p(\mathbb{R}^2)$  for every  $p \in (2, \infty)$ , hence weakly compact in these spaces.

From now on, we follow the reasoning from [14], where the authors considered the Navier-Stokes equations in  $\Omega_\lambda$  with a more general initial velocity. In our case, the initial vorticity vanishes while in [14] the vorticity is smooth, independent of  $\lambda$  and compactly supported in  $\mathbb{R}^2 \setminus \{0\}$ . The difference between the Stokes and the Navier-Stokes equations is the bilinear term  $u \cdot \nabla u$  which only complicates matters. Therefore, ignoring all additional difficulties caused by the bilinear term, the results proved in [14] go through to our case. Note that the smallness assumption required in [14] is irrelevant in this work since we deal with a linear equation.

Let us be more precise. It was proved in [14] (see Proposition 18 and the end of the proof of Theorem 22) that  $\mathbb{P}_{\mathbb{R}^2}[\eta^\lambda EH_\lambda]$  converges to the Lamb-Oseen vortex  $\Theta$  when  $\lambda \rightarrow \infty$ , up to a subsequence, uniformly in time with values in  $H_{loc}^{-3}(\mathbb{R}^2)$ . The precise definition of the cut-off function  $\eta^\lambda$  is not required here (the interested reader can find it in relation (4.1) of [14] with  $\varepsilon = 1/\lambda$ ). We only need to know that  $0 \leq \eta^\lambda \leq 1$ , that  $\eta^\lambda$  vanishes in the neighborhood of the boundary of  $\Omega_\lambda$  and that  $\eta^\lambda(x) \equiv 1$  for all  $|x| > C/\lambda$ .

In particular, we have that  $\mathbb{P}_{\mathbb{R}^2}[\eta^\lambda EH_\lambda(1)] \rightarrow \Theta(1)$  in  $H_{loc}^{-3}(\mathbb{R}^2)$  when  $\lambda \rightarrow \infty$ , up to a subsequence. On the other hand, the sequence  $\mathbb{P}_{\mathbb{R}^2}[\eta^\lambda EH_\lambda(1)]$  is bounded in  $L^p(\mathbb{R}^2)$  since  $H_\lambda(1)$  is bounded in  $L^p(\Omega_\lambda)$ . By uniqueness of limits, we infer that  $\mathbb{P}_{\mathbb{R}^2}[\eta^\lambda EH_\lambda(1)] \rightarrow \Theta(1)$  weakly in  $L^p(\mathbb{R}^2)$  as  $\lambda \rightarrow \infty$ .

Finally, we observe that

$$\begin{aligned} \|\mathbb{P}_{\mathbb{R}^2}[\eta^\lambda EH_\lambda(1)] - EH_\lambda(1)\|_{L^p(\mathbb{R}^2)} &= \|\mathbb{P}_{\mathbb{R}^2}[(\eta^\lambda - 1)EH_\lambda(1)]\|_{L^p(\mathbb{R}^2)} \\ &\leq C\|(\eta^\lambda - 1)EH_\lambda(1)\|_{L^p(\mathbb{R}^2)} \leq C\|H_\lambda(1)\|_{L^\infty(\Omega)} \text{mes}(B(0, C/\lambda))^{\frac{1}{p}} \leq C\lambda^{-\frac{2}{p}} \xrightarrow{\lambda \rightarrow \infty} 0. \end{aligned}$$

This completes the proof of the lemma.  $\square$

Consequently, to prove the strong convergence (2.11), in view of the weak convergence (2.12), it suffices to show that  $\{EH_\lambda(1)\}_{\lambda \geq 1}$  is relatively compact in  $L^p(\mathbb{R}^2)$  for every  $p \in (2, \infty)$ . Here, we proceed in two steps; we show that the family  $\{EH_\lambda(1)\}_{\lambda \geq 1}$  is:

i) relatively compact in  $L^p_{loc}(\mathbb{R}^2)$  for every  $p \in (2, \infty)$  (Lemma 8, below),

ii) small in the  $L^p$ -sense for large  $|x|$ , uniformly in  $\lambda \geq 1$  (Lemma 9).

Then, the relative compactness of the family  $\{EH_\lambda(1)\}_{\lambda \geq 1}$  in the space  $L^p(\mathbb{R}^2)$  is a consequence of a standard diagonal argument. Here, a set is called to be relatively compact in  $L^p_{loc}(\mathbb{R}^2)$  if it is relatively compact in  $L^p(B(0, R))$  for every  $R > 0$ .

In the following two lemmas,  $R > 1$  is a sufficiently large constant and

$$(2.13) \quad h_R(x) = h(x/R), \quad \text{where } h \in C^\infty(\mathbb{R}^2)$$

is such that  $h(x) = 0$  for  $|x| < 1$  and  $h(x) = 1$  for  $|x| > 2$ .

**Lemma 8.** *Let  $H_\lambda(t)$  be defined in (2.10). The set  $\{EH_\lambda(1)\}_{\lambda \geq 1}$  is relatively compact in  $L^p_{loc}(\mathbb{R}^2)$  for every  $p \in (2, \infty)$ .*

*Proof.* Here, as the usual practice, one could show  $L^p$ -estimates for  $\nabla EH_\lambda(1)$  which are uniform in  $\lambda \geq 1$ . Unfortunately, we do not know any scaling invariant gradient estimate for solutions of the Stokes equation with initial conditions from  $L^{2,\infty}(\Omega)$ . Thus, we have to proceed in a different manner.

Recall first that, by [13, Prop. 2.1], the vector field  $H_\Omega$  is smooth, bounded, and there is a constant  $C > 0$  such that  $|H_\Omega(x)| \leq C/|x|$  for all  $x \in \Omega$  (recall that  $\Omega \subset \mathbb{R}^2 \setminus B(0, 1)$ ). Since the rescaled harmonic vector field  $H_{\Omega_\lambda}$  is divergence free and tangent to the boundary, we can write the following decomposition

$$H_{\Omega_\lambda} = \mathbb{P}_{\Omega_\lambda} H_{\Omega_\lambda} = \mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda}) + \mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}],$$

with the cut-off function  $h_R$  defined in (2.13).

Obviously, the Leray projector  $\mathbb{P}_{\Omega_\lambda}$  is a bounded operator on the space  $L^q(\Omega_\lambda)$  for each  $1 < q < \infty$ , with norm independent of  $\lambda$ . Thus, for fixed  $q \in (1, 2)$ , using the identity (2.9) we estimate

$$(2.14) \quad \begin{aligned} \|\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]\|_{L^q(\Omega_\lambda)} &\leq C\|(1 - h_R)H_{\Omega_\lambda}\|_{L^q(\Omega_\lambda)} \leq C\lambda\|H_\Omega(\lambda \cdot)\|_{L^q(\Omega_\lambda \cap B(0, 2R))} \\ &= C\lambda^{1-\frac{2}{q}}\|H_\Omega\|_{L^q(\Omega \cap B(0, 2R\lambda))} \leq C\lambda^{1-\frac{2}{q}}\left\|\frac{1}{|\cdot|}\right\|_{L^q(1 < |x| < 2R\lambda)} \\ &\leq C(q, \Omega)R^{\frac{2}{q}-1}. \end{aligned}$$

Therefore, the quantity  $\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]$  is bounded in  $L^q(\Omega_\lambda)$  with  $q \in (1, 2)$ , uniformly with respect to  $\lambda$ . Now, we deduce from the scaling invariant decay estimates (2.4) and (2.7) that  $\{S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]\}_{\lambda \geq 1}$  is bounded in  $H^1(\Omega_\lambda)$ . Moreover, since  $S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]$  vanishes on the boundary of  $\Omega_\lambda$  we have the relation

$$E\nabla S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}] = \nabla E S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]$$

which implies that  $\{E S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]\}_{\lambda \geq 1}$  is bounded in  $H^1(\mathbb{R}^2)$ . By the compactness of the Sobolev imbedding  $H^1(\mathbb{R}^2) \subset L^p_{loc}(\mathbb{R}^2)$ , we infer that the set  $\{E S_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1 - h_R)H_{\Omega_\lambda}]\}_{\lambda \geq 1}$  is relatively compact in  $L^p_{loc}(\mathbb{R}^2)$  for all  $p \in (2, \infty)$ .

On the other hand, calculations similar to those in (2.14) with  $p \in (2, \infty)$  lead to the inequality

$$\|\mathbb{P}_{\Omega_\lambda} h_R H_{\Omega_\lambda}\|_{L^p(\Omega_\lambda)} \leq C\lambda^{1-\frac{2}{p}}\left\|\frac{1}{|\cdot|}\right\|_{L^p(|x| > R\lambda)} \leq C(p, \Omega)R^{\frac{2}{p}-1}.$$

Using the decay estimate (2.4) we infer that

$$(2.15) \quad \begin{aligned} \|ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda})\|_{L^p(\mathbb{R}^2)} &= \|S_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda})\|_{L^p(\Omega_\lambda)} \\ &\leq C\|\mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda})\|_{L^p(\Omega_\lambda)} \leq C(p, \Omega)R^{\frac{2}{p}-1}. \end{aligned}$$

Finally, since  $ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda})$  tends to zero in  $L^p(\mathbb{R}^2)$  as  $R \rightarrow \infty$  uniformly in  $\lambda$  and since the family  $\{ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1-h_R)H_{\Omega_\lambda}]\}_{\lambda \geq 1}$  is relatively compact in  $L^p_{loc}(\mathbb{R}^2)$  for every fixed  $R$ , we infer that

$$(2.16) \quad EH_\lambda(1) = ES_\lambda(1)H_{\Omega_\lambda} = ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_R H_{\Omega_\lambda}) + ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}[(1-h_R)H_{\Omega_\lambda}]$$

is relatively compact in  $L^p_{loc}(\mathbb{R}^2)$ .  $\square$

**Lemma 9.** *Let  $H_\lambda(t)$  be defined in (2.10) and  $h_R$  be defined in (2.13). For every  $p \in (2, \infty)$ , we have that  $\lim_{R \rightarrow \infty} \|h_R EH_\lambda(1)\|_{L^p(\mathbb{R}^2)} = 0$  uniformly in  $\lambda \geq 1$ .*

*Proof.* Let  $\varepsilon > 0$  be an arbitrary small constant and  $R_0 = R_0(\varepsilon)$  be a large constant to be chosen later. We estimate the  $L^p$ -norm of  $h_R EH_\lambda(1)$  using the decomposition of  $EH_\lambda(1)$  from (2.16) with  $R = R_0$ . First, repeating the calculations from (2.15) we have

$$\|h_R ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_{R_0} H_{\Omega_\lambda})\|_{L^p(\mathbb{R}^2)} \leq C(p, \Omega)R_0^{\frac{2}{p}-1}.$$

Since the right-hand side tends to 0 as  $R_0 \rightarrow \infty$  uniformly in  $\lambda \geq 1$ , there exists  $R_0$  independent of  $\lambda$  such that

$$\|h_R ES_\lambda(1)\mathbb{P}_{\Omega_\lambda}(h_{R_0} H_{\Omega_\lambda})\|_{L^p(\mathbb{R}^2)} \leq \varepsilon \quad \text{for all } \lambda \geq 1.$$

Now, for fixed  $R_0$ , we show that

$$\lim_{R \rightarrow \infty} h_R E v_\lambda(1) = 0 \quad \text{with} \quad v_\lambda(t) = S_\lambda(t)\mathbb{P}_{\Omega_\lambda}[(1-h_{R_0})H_{\Omega_\lambda}]$$

where the convergence is in the norm of  $L^p(\mathbb{R}^2)$  and is uniform with respect to  $\lambda \geq 1$ .

First, it follows from relation (2.14) that  $v_\lambda(0) = \mathbb{P}_{\Omega_\lambda}[(1-h_{R_0})H_{\Omega_\lambda}]$  is bounded in  $L^q(\Omega_\lambda)$  for each  $1 < q < 2$ , uniformly in  $\lambda \geq 1$ . Using the decay estimates for the Stokes equation stated in (2.4) and (2.7), we infer that  $v_\lambda$  verifies

$$(2.17) \quad \|v_\lambda(t)\|_{L^q(\Omega_\lambda)} \leq C(q) \quad \text{and} \quad \|\nabla v_\lambda(t)\|_{L^2(\Omega_\lambda)} \leq C(\eta)t^{-\eta}, \quad \text{uniformly in } \lambda \geq 1,$$

for each  $q \in (1, 2)$ ,  $\eta = 1/q \in (1/2, 1)$ , and all  $t > 0$ .

Let  $\omega_\lambda$  denote the curl of  $v_\lambda$ . The quantity  $Eh_R \omega_\lambda$  verifies the following equation in the full plane

$$\partial_t(Eh_R \omega_\lambda) - \Delta(Eh_R \omega_\lambda) = E[\Delta h_R \omega_\lambda - 2 \operatorname{div}(\nabla h_R \omega_\lambda)] \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^2,$$

supplemented with the zero initial datum, because

$$\begin{aligned} Eh_R \omega_\lambda(0) &= Eh_R \operatorname{curl} v_\lambda(0) = Eh_R \operatorname{curl} \mathbb{P}_{\Omega_\lambda}[(1-h_{R_0})H_{\Omega_\lambda}] \\ &= Eh_R \operatorname{curl}[(1-h_{R_0})H_{\Omega_\lambda}] = -Eh_R H_{\Omega_\lambda} \cdot \nabla^\perp h_{R_0} = 0 \end{aligned}$$

for  $R > 2R_0$ . In these calculations, we used the fact that, for any vector field  $w$ , the quantity  $w - \mathbb{P}_{\Omega_\lambda} w$  is a gradient, that  $H_{\Omega_\lambda}$  is curl free, that  $\operatorname{supp} h_R \subset \{|x| > R\}$  and that  $\operatorname{supp} \nabla h_{R_0} \subset \{R_0 < |x| < 2R_0\}$ .

The Duhamel principle for the inhomogeneous heat equation in the full plane implies now that

$$(2.18) \quad Eh_R \omega_\lambda(1) = \int_0^1 \frac{1}{4\pi(1-s)} e^{-\frac{|x|^2}{4(1-s)}} * E[\Delta h_R \omega_\lambda - 2 \operatorname{div}(\nabla h_R \omega_\lambda)](s) ds.$$

Let  $q \in (1, 2)$  satisfy  $1/q = 1/2 + 1/p$ . We estimate the  $L^q$ -norm of  $Eh_R\omega_\lambda(1)$  using relation (2.18) in the following way

$$\begin{aligned}
\|Eh_R\omega_\lambda(1)\|_{L^q(\mathbb{R}^2)} &\leq C \int_0^1 \frac{1}{1-s} \|e^{-\frac{|\cdot|^2}{4(1-s)}}\|_{L^1(\mathbb{R}^2)} \|\Delta h_R\omega_\lambda\|_{L^q(\mathbb{R}^2)} ds \\
&\quad + C \int_0^1 \frac{1}{1-s} \|\nabla[e^{-\frac{|\cdot|^2}{4(1-s)}}]\|_{L^1(\mathbb{R}^2)} \|\nabla h_R\omega_\lambda\|_{L^q(\mathbb{R}^2)} ds \\
&\leq C \int_0^1 \|\Delta h_R\|_{L^p(\mathbb{R}^2)} \|\omega_\lambda\|_{L^2(\Omega_\lambda)} + C \int_0^1 \frac{1}{\sqrt{1-s}} \|\nabla h_R\|_{L^p(\mathbb{R}^2)} \|\omega_\lambda\|_{L^2(\Omega_\lambda)} ds \\
&\leq C \int_0^1 (\|\Delta h_R\|_{L^p(\mathbb{R}^2)} + \|\nabla h_R\|_{L^p(\mathbb{R}^2)}) (1 + (1-s)^{-\frac{1}{2}}) s^{-\frac{3}{4}} ds \\
&\leq CR^{\frac{2}{p}-1},
\end{aligned}$$

where we used (2.17). We conclude, using again (2.17), that

$$\begin{aligned}
\|\operatorname{curl}(Eh_Rv_\lambda(1))\|_{L^q(\mathbb{R}^2)} &\leq \|Eh_R\omega_\lambda(1)\|_{L^q(\mathbb{R}^2)} + \|Ev_\lambda(1) \cdot \nabla^\perp h_R\|_{L^q(\mathbb{R}^2)} \\
&\leq \|Eh_R\omega_\lambda(1)\|_{L^q(\mathbb{R}^2)} + \|v_\lambda(1)\|_{L^q(\Omega_\lambda)} \|\nabla h_R\|_{L^\infty(\mathbb{R}^2)} \\
&\leq CR^{\frac{2}{p}-1}.
\end{aligned}$$

On the other hand, we can also bound

$$\|\operatorname{div}(Eh_Rv_\lambda(1))\|_{L^q(\mathbb{R}^2)} = \|Ev_\lambda(1) \cdot \nabla h_R\|_{L^q(\mathbb{R}^2)} \leq \|v_\lambda(1)\|_{L^q(\Omega_\lambda)} \|\nabla h_R\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{R}.$$

Finally, putting together these estimates, we obtain

$$\begin{aligned}
\|Eh_Rv_\lambda(1)\|_{L^p(\mathbb{R}^2)} &\leq C \|\nabla(Eh_Rv_\lambda(1))\|_{L^q(\mathbb{R}^2)} \\
&\leq C \|\operatorname{div}(Eh_Rv_\lambda(1))\|_{L^q(\mathbb{R}^2)} + C \|\operatorname{curl}(Eh_Rv_\lambda(1))\|_{L^q(\mathbb{R}^2)} \leq CR^{\frac{2}{p}-1} \xrightarrow{R \rightarrow \infty} 0
\end{aligned}$$

uniformly in  $\lambda \geq 1$ . This completes the proof of Lemma 9.  $\square$

### 3 Proof of the main result

The proof of Theorem 1 proceeds in two steps. First, we reduce the problem to the study of initial velocities, which are small in the  $L^{2,\infty}$ -norm. In the second step, we assume that  $u_0$  is sufficiently small in  $L^{2,\infty}(\Omega)$  and we show that if the solution of the Stokes problem (2.1)-(2.3) converges towards the Lamb-Oseen vortex, then so does the solution of the nonlinear problem. Once these two steps are completed, Theorem 1 follows from Theorem 6.

#### 3.1 Reduction to the case of small initial velocity.

We begin by recalling a classical result on the  $L^2$ -decay of weak solutions to problem (1.2)-(1.4).

**Theorem 10** (Borchers & Miyakawa [2, Thm. 1.2]). *For every  $\tilde{u}_0 \in L^2_\sigma(\Omega)$  there is a unique weak solution  $\tilde{u} \in L^\infty((0, \infty); L^2(\Omega)) \cap L^2_{loc}([0, \infty); H^1(\Omega))$ , of problem (1.2)-(1.4) with  $u_0 = \tilde{u}_0$  as an initial datum, such that  $\lim_{t \rightarrow \infty} \|\tilde{u}(t)\|_{L^2} = 0$ .*

We show now the following auxiliary result.



**Lemma 11.** *Let  $u$  be a solution to (1.2)-(1.4) with  $u_0$  of the form (1.5) with arbitrary  $\tilde{u}_0 \in L^2_\sigma(\Omega)$  and  $\alpha \in \mathbb{R}$ . Denote by  $\tilde{u}$  the weak solution from Theorem 10. For every  $t_0 > 0$ , we have that*

$$\sup_{[0, t_0]} \|u(t) - \tilde{u}(t) - \alpha S(t)H_\Omega\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \alpha \rightarrow 0.$$

*Proof.* We show a  $L^2$ -estimate for the function  $z(t) \equiv u(t) - \tilde{u}(t) - \alpha S(t)H_\Omega$  which satisfies the following equation

$$(3.1) \quad \partial_t z - \Delta z + (\tilde{u} + z + \alpha H_1) \cdot \nabla(\tilde{u} + z + \alpha H_1) - \tilde{u} \cdot \nabla \tilde{u} + \nabla \bar{p} = 0,$$

where  $H_1(t) = S(t)H_\Omega$ .

We multiply equation (3.1) by  $z$  and integrate in the space variable to obtain, after some integrations by parts,

$$(3.2) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|z\|_{L^2(\Omega)}^2 + \|\nabla z\|_{L^2(\Omega)}^2 &= \alpha \int \tilde{u} \cdot \nabla z \cdot H_1 - \int z \cdot \nabla \tilde{u} \cdot z + \alpha \int z \cdot \nabla z \cdot H_1 \\ &+ \alpha \int H_1 \cdot \nabla z \cdot \tilde{u} + \alpha^2 \int H_1 \cdot \nabla z \cdot H_1 \\ &\equiv I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Using the following interpolation inequality

$$\|f\|_{L^4(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\Omega)}^{\frac{1}{2}} \quad \text{for every } f \in H_0^1(\Omega),$$

we bound each term on the right-hand side of (3.2) in the following way

$$\begin{aligned} I_1 &\leq |\alpha| \|\tilde{u}\|_{L^2(\Omega)} \|\nabla z\|_{L^2(\Omega)} \|H_1\|_{L^\infty(\Omega)} \leq \frac{1}{6} \|\nabla z\|_{L^2(\Omega)}^2 + C\alpha^2 \|\tilde{u}\|_{L^2(\Omega)}^2 \|H_1\|_{L^\infty(\Omega)}^2, \\ I_2 &\leq \|z\|_{L^4(\Omega)}^2 \|\nabla \tilde{u}\|_{L^2(\Omega)} \leq C \|z\|_{L^2(\Omega)} \|\nabla z\|_{L^2(\Omega)} \|\nabla \tilde{u}\|_{L^2(\Omega)} \leq \frac{1}{6} \|\nabla z\|_{L^2(\Omega)}^2 + C \|z\|_{L^2(\Omega)}^2 \|\nabla \tilde{u}\|_{L^2(\Omega)}^2, \\ I_3 &\leq |\alpha| \|z\|_{L^2(\Omega)} \|\nabla z\|_{L^2(\Omega)} \|H_1\|_{L^\infty(\Omega)} \leq \frac{1}{6} \|\nabla z\|_{L^2(\Omega)}^2 + C\alpha^2 \|z\|_{L^2(\Omega)}^2 \|H_1\|_{L^\infty(\Omega)}^2, \\ I_4 &\leq |\alpha| \|H_1\|_{L^\infty(\Omega)} \|\nabla z\|_{L^2(\Omega)} \|\tilde{u}\|_{L^2(\Omega)} \leq \frac{1}{6} \|\nabla z\|_{L^2(\Omega)}^2 + C\alpha^2 \|H_1\|_{L^\infty(\Omega)}^2 \|\tilde{u}\|_{L^2(\Omega)}^2, \\ I_5 &\leq \alpha^2 \|H_1\|_{L^4(\Omega)}^2 \|\nabla z\|_{L^2(\Omega)} \leq \frac{1}{6} \|\nabla z\|_{L^2(\Omega)}^2 + C\alpha^4 \|H_1\|_{L^4(\Omega)}^4. \end{aligned}$$

Plugging the above inequalities into (3.2) yields

$$\begin{aligned} \frac{d}{dt} \|z\|_{L^2(\Omega)}^2 + \frac{1}{3} \|\nabla z\|_{L^2(\Omega)}^2 &\leq C \|z\|_{L^2(\Omega)}^2 (\|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \alpha^2 \|H_1\|_{L^\infty(\Omega)}^2) \\ &+ C\alpha^2 \|H_1\|_{L^\infty(\Omega)}^2 \|\tilde{u}\|_{L^2(\Omega)}^2 + C\alpha^4 \|H_1\|_{L^4(\Omega)}^4. \end{aligned}$$

Recall that  $z_0 = 0$  and  $H_1(t) = S(t)H_\Omega$ . Thus, the Gronwall inequality implies

$$\begin{aligned} \sup_{[0, t_0]} \|z\|_{L^2(\Omega)}^2 &\leq C\alpha^2 \left( \int_0^{t_0} \|S(\tau)H_\Omega\|_{L^\infty(\Omega)}^2 \|\tilde{u}(\tau)\|_{L^2(\Omega)}^2 d\tau + \alpha^2 \int_0^{t_0} \|S(\tau)H_\Omega\|_{L^4(\Omega)}^4 d\tau \right) \\ &\times \exp \left( C \int_0^{t_0} \|\nabla \tilde{u}(\tau)\|_{L^2(\Omega)}^2 d\tau + C\alpha^2 \int_0^{t_0} \|S(\tau)H_\Omega\|_{L^\infty(\Omega)}^2 d\tau \right). \end{aligned}$$

Since  $\tilde{u} \in L^\infty((0, t_0); L^2(\Omega)) \cap L^2((0, t_0); H^1(\Omega))$  and since  $H_\Omega \in L^p(\Omega)$  for all  $p \in (2, \infty]$ , we infer from the decay estimate (2.4) that the right-hand side of the above inequality is finite and tends to zero as  $\alpha \rightarrow 0$ . This completes the proof of Lemma 11.  $\square$

In the following, we need a simple consequence of this lemma.

**Corollary 12.** *Under the assumptions of Lemma 11, for every  $\varepsilon > 0$ , there exists  $\alpha_0 = \alpha_0(\Omega, \tilde{u}_0, \varepsilon) > 0$  and  $T_0 = T_0(\Omega, \tilde{u}_0, \varepsilon) \geq 0$  such that if  $|\alpha| \leq \alpha_0$  then  $\|u(T_0)\|_{L^2, \infty(\Omega)} \leq \varepsilon$ .*

*Proof.* Let  $\varepsilon > 0$  be arbitrary. First, by Theorem 10, we choose  $T_0$  so large to have  $\|\tilde{u}(T_0)\|_{L^2(\Omega)} \leq \varepsilon/3$ . Next, by (2.6), we have the following bound

$$\|\alpha S(t)H_\Omega\|_{L^2, \infty(\Omega)} \leq K_2|\alpha|\|H_\Omega\|_{L^2, \infty(\Omega)} \leq \frac{\varepsilon}{3},$$

provided that  $\alpha_0 \leq \varepsilon/(3K_2\|H_\Omega\|_{L^2, \infty(\Omega)})$ . Finally, we infer from Lemma 11 that if  $\alpha_0$  is sufficiently small, then

$$\sup_{[0, T_0]} \|u(t) - \tilde{u}(t) - \alpha S(t)H_\Omega\|_{L^2(\Omega)} \leq \frac{\varepsilon}{3}.$$

Consequently,

$$\begin{aligned} \|u(T_0)\|_{L^2, \infty(\Omega)} &\leq \|u(T_0) - \tilde{u}(T_0) - \alpha S(T_0)H_\Omega\|_{L^2, \infty(\Omega)} + \|\tilde{u}(T_0)\|_{L^2, \infty(\Omega)} \\ &\quad + \|\alpha S(T_0)H_\Omega\|_{L^2, \infty(\Omega)} \leq \varepsilon. \end{aligned}$$

□

### 3.2 Large time asymptotics for small velocities.

Now, we show that, for sufficiently small initial conditions, if the linear evolution converges to the Lamb-Oseen vortex, then so does the nonlinear evolution. This result is stated in the following proposition.

**Proposition 13.** *Let  $u_0 \in L_\sigma^{2, \infty}(\Omega)$  and denote by  $u = u(t, x)$  the corresponding solution to (1.2)–(1.4). There exists  $\varepsilon = \varepsilon(\Omega) > 0$  such that if  $\max\{\|u_0\|_{L^2, \infty(\Omega)}, |\alpha|\} \leq \varepsilon$  and if*

$$(3.3) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|S(t)u_0 - \alpha\Theta(t)\|_{L^p(\Omega)} = 0 \quad \text{for every } p \in (2, \infty)$$

then

$$(3.4) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|u(t) - \alpha\Theta(t)\|_{L^p(\Omega)} = 0 \quad \text{for every } p \in (2, \infty).$$

*Proof.* It follows from the results in [15, Thm. 3] that for every  $p \in (2, \infty)$  there exists a constant  $C(p) > 0$  such that

$$(3.5) \quad \sup_{t > 0} t^{\frac{1}{2} - \frac{1}{p}} \|u(t)\|_{L^p(\Omega)} \leq C(p)\varepsilon,$$

provided  $\varepsilon > 0$  is sufficiently small.

First, we show relation (3.4) for  $p = 4$ . The Duhamel principle allows to rewrite problem (1.2)–(1.4) as the integral equation

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\mathbb{P}_\Omega \operatorname{div}(u \otimes u)(s) ds.$$

Subtracting the Lamb-Oseen vortex  $\Theta$  on the both sides of the above relation we get

$$\begin{aligned} u(t) - \alpha\Theta(t) &= S(t)u_0 - \alpha\Theta(t) - \int_0^t S(t-s)\mathbb{P}_\Omega \operatorname{div}(u \otimes u)(s) ds \\ &= S(t)u_0 - \alpha\Theta(t) - \int_0^t S(t-s)\mathbb{P}_\Omega \operatorname{div}[(u - \alpha\Theta) \otimes u + \alpha\Theta \otimes (u - \alpha\Theta)](s) ds, \end{aligned}$$

because  $\mathbb{P}_\Omega \operatorname{div}(\Theta \otimes \Theta) = 0$ . This is a consequence of the fact that the vector field  $\Theta$  is orthogonal to the gradient of a radial function so that  $\operatorname{div}(\Theta \otimes \Theta)$  is a gradient.

Now, computing the  $L^4$ -norm of the above equality, using the decay estimates for the Stokes semigroup (2.8), the Hölder inequality, the assumption on  $\alpha$ , and estimate (3.5) we obtain

$$\begin{aligned} \|u(t) - \alpha\Theta(t)\|_{L^4(\Omega)} &\leq \|S(t)u_0 - \alpha\Theta(t)\|_{L^4(\Omega)} \\ &\quad + C \int_0^t (t-s)^{-\frac{3}{4}} \|[(u - \alpha\Theta) \otimes u + \alpha\Theta \otimes (u - \alpha\Theta)](s)\|_{L^2(\Omega)} ds \\ &\leq \|S(t)u_0 - \Theta(t)\|_{L^4(\Omega)} \\ &\quad + C \int_0^t (t-s)^{-\frac{3}{4}} \|(u - \alpha\Theta)(s)\|_{L^4(\Omega)} (\|u(s)\|_{L^4(\Omega)} + \|\alpha\Theta(s)\|_{L^4(\Omega)}) ds \\ &\leq \|S(t)u_0 - \alpha\Theta(t)\|_{L^4(\Omega)} + C\varepsilon \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} \|(u - \alpha\Theta)(s)\|_{L^4(\Omega)} ds. \end{aligned}$$

Hence, denoting  $\zeta(t) = t^{\frac{1}{4}}\|u(t) - \alpha\Theta(t)\|_{L^4(\Omega)}$  we infer that

$$\begin{aligned} \zeta(t) &\leq t^{\frac{1}{4}}\|S(t)u_0 - \alpha\Theta(t)\|_{L^4(\Omega)} + C\varepsilon t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} \zeta(s) ds \\ &\leq t^{\frac{1}{4}}\|S(t)u_0 - \alpha\Theta(t)\|_{L^4(\Omega)} + C\varepsilon \int_0^1 (1-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} \zeta(t\tau) d\tau. \end{aligned}$$

Now, we compute  $\limsup_{t \rightarrow \infty}$  of both sides of this inequality and we use (3.3) for  $p = 4$ . By the Lebesgue dominated convergence theorem, we obtain

$$\limsup_{t \rightarrow \infty} \zeta(t) \leq C\varepsilon \limsup_{t \rightarrow \infty} \zeta(t) \int_0^1 (1-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau = C_1\varepsilon \limsup_{t \rightarrow \infty} \zeta(t),$$

where  $C_1$  is a constant independent of  $\varepsilon$ . If  $C_1\varepsilon < 1$ , this inequality implies immediately that  $\limsup_{t \rightarrow \infty} \zeta(t) = 0$ , which is the relation (3.4) for  $p = 4$ .

The same argument as above works for  $p \neq 4$  but the constant  $C_1$  will depend on  $p$  so the smallness condition  $C_1\varepsilon < 1$  cannot hold true unless  $\varepsilon = 0$ . To get around this difficulty, we show that if (3.4) holds true for  $p = 4$  then it holds true for all  $p \in (2, \infty)$ . Using similar computations as above we obtain

$$\begin{aligned} \|u(t) - \alpha\Theta(t)\|_{L^p(\Omega)} &\leq \|S(t)u_0 - \alpha\Theta(t)\|_{L^p(\Omega)} \\ &\quad + C(p)\varepsilon \int_0^t (t-s)^{-1+\frac{1}{p}} s^{-\frac{1}{4}} \|u(s) - \alpha\Theta(s)\|_{L^4(\Omega)} ds \\ &\leq \|S(t)u_0 - \alpha\Theta(t)\|_{L^p(\Omega)} + C(p)\varepsilon t^{\frac{1}{p}-\frac{1}{2}} \int_0^1 (1-\tau)^{-1+\frac{1}{p}} \tau^{-\frac{1}{2}} \zeta(\tau t) d\tau. \end{aligned}$$

Multiplying both sides of this inequality by  $t^{\frac{1}{2}-\frac{1}{p}}$ , computing  $\limsup_{t \rightarrow \infty}$ , and using the already-proved decay for  $p = 4$  completes the proof of Proposition 13.  $\square$

### 3.3 Proof of Theorem 1.

We fix  $\varepsilon > 0$  required in Proposition 13 and choose  $\alpha_0 \in (-\varepsilon, \varepsilon)$  and  $T_0$  as in Corollary 12 to have that  $\|u(T_0)\|_{L^{2,\infty}(\Omega)} \leq \varepsilon$ . Let us observe that  $u(T_0)$  verifies

$$(3.6) \quad u(T_0) - \alpha H_\Omega \in L^2_\sigma(\Omega).$$

Indeed, it follows from Lemma 11 that  $u(T_0) - \tilde{u}(T_0) - \alpha S(T_0)H_\Omega \in L_\sigma^2(\Omega)$ . Clearly  $\tilde{u}(T_0) \in L_\sigma^2(\Omega)$ , because  $\tilde{u}$  is a square integrable weak solution of the Navier-Stokes equations. Moreover, we have  $S(t)H_\Omega - H_\Omega \in L_\sigma^2(\Omega)$  as was shown in [14]. Thus, the proof of (3.6) is complete.

In particular, using Corollary 5 we have

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|S(t)(u(T_0) - \alpha H_\Omega)\|_{L^p(\Omega)} = 0 \quad \text{for every } p \in (2, \infty).$$

Thus, we infer from Theorem 6 that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|S(t)u(T_0) - \alpha \Theta(t)\|_{L^p(\Omega)} = 0 \quad \text{for every } p \in (2, \infty).$$

Apply now Proposition 13 starting from time  $T_0$  to obtain

$$\lim_{t \rightarrow \infty} (t - T_0)^{\frac{1}{2} - \frac{1}{p}} \|u(t) - \alpha \Theta(t - T_0)\|_{L^p(\Omega)} = 0 \quad \text{for every } p \in (2, \infty).$$

A calculation using the explicit formula for  $\Theta$  given in (1.1) shows that

$$(t - T_0)^{\frac{1}{2} - \frac{1}{p}} \|\Theta(t) - \Theta(t - T_0)\|_{L^p(\Omega)} = \|\Theta(1) - \Theta\left(\frac{t}{t - T_0}\right)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

by the dominated convergence theorem (observe that  $|\Theta(\frac{t}{t - T_0})| \leq |\Theta(1)|$ ). This completes the proof of Theorem 1.

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