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ALIGNED CP-SEMIGROUPS

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ABSTRACT. A CP-semigroup is aligned if its set of trivially maximal subordinates is totally ordered by subordination. We prove that aligned spatial E_0-semigroups are prime: they have no non-trivial tensor product decompositions up to cocycle conjugacy. As a consequence, we establish the existence of uncountably many non-cocycle conjugate E_0-semigroups of type II_0 which are prime.

Let $\mathfrak{H}$ be a Hilbert space, which we will always assume to be separable and infinite-dimensional, and let $\mathfrak{B}(\mathfrak{H})$ denote the $*$-algebra of all bounded operators over $\mathfrak{H}$. A *CP-semigroup acting on* $\mathfrak{B}(\mathfrak{H})$ is a point-σ-weakly continuous semigroup $\alpha = \{\alpha_t : \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})\}_{t\geq 0}$ of normal completely positive contractions such that $\alpha_0 = \mathrm{id}$. When α_t is an endomorphism and $\alpha_t(I) = I$ for all $t \geq 0$, then α is called an *E_0-semigroup*. In the special case when $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$, a CP-semigroup α acting on $\mathfrak{B}(\mathfrak{H})$ is called a *CP-flow over* $\mathfrak{K}$ if $\alpha_t(A)S_t = S_tA$ for all $A \in \mathfrak{B}(\mathfrak{H})$, $t \geq 0$ where $\{S_t : t \geq 0\}$ is the right shift semigroup. We will say that a CP-semigroup β is a *subordinate* of α if it also acts on $\mathfrak{B}(\mathfrak{H})$ and $\alpha_t - \beta_t$ is completely positive. If in addition β is a CP-flow, then it is called a *flow subordinate* of α. We direct the reader to [Arv03] for a general reference on the theory of CP-semigroups and to [Pow03b] for the basic theory of CP-flows.

In this paper we study a class of CP-semigroups which has a set of subordinates which is minimal in a natural sense. We call such CP-semigroups aligned. This class is shown to include examples considered previously in [Pow03a, APP06, Jan10, JMP11]. We prove that aligned E_0-semigroups have a notable property: they are prime in the sense that they have no non-trivial tensor product decompositions up to cocycle conjugacy. As a consequence, we establish the existence of uncountably many non-cocycle conjugate E_0-semigroups of type II_0 which are prime. The previously known non-trivial examples of prime E_0-semigroups were obtained independently in [MP09] (type II_1), [Tsi08] (type II_1) and [Lie09] (type II_k for $k = 1, 2, \dots$).

Powers introduced in [Pow03a] the concept of q-purity which has been valuable for the study of E_0-semigroups (see for example [APP06, Jan10]). The notion of q-purity was refined in [JMP11]: a CP-flow is *q-pure* if and only if its set of flow subordinates is totally ordered by subordination. It is clear that an E_0-semigroup which is in addition a q-pure CP-flow must have index 0 or 1, and must be of type I_1, II_0 or II_1. The case of type I_0 is excluded because an automorphism group cannot be a CP-flow.

The restriction to flow subordinates in the definition of q-pure CP-flows can obscure some useful properties of these CP-semigroups. In order to circumvent this difficulty we consider an alternative concept which is inspired by the notion of q-purity.

Definition 1. Let α be a CP-semigroup and let β be a CP-semigroup subordinate of α. We will say that β is *trivially maximal* if the semigroup $e^{kt}\beta_t$ is not a subordinate of α for $k > 0$.

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We will denote by $\mathfrak{S}(\alpha)$ the set of all trivially maximal subordinates of α partially ordered by subordination. We will say α is *aligned* if $\mathfrak{S}(\alpha)$ is totally ordered.

Let β be a trivially maximal subordinate of α. We will denote by $\mathfrak{S}(\alpha;\beta)$ the set of all trivially maximal CP-semigroup subordinates of α which dominate β, partially ordered by subordination. We will say that α is *aligned relative to* β if $\mathfrak{S}(\alpha;\beta)$ is totally ordered.

Notice that if an aligned E_0-semigroup is spatial, then it must have index zero.

Lemma 2. *A unital CP-semigroup is aligned if and only if its minimal dilation is aligned.*

Proof. Suppose that α is a unital CP-semigroup acting on $\mathfrak{H}$ with minimal dilation α^d acting on $\mathfrak{B}(\mathfrak{H}_1)$, i.e. there exists an isometry $W : \mathfrak{H} \to \mathfrak{H}_1$ such that

$$\alpha_t(x) = W^*\alpha_t^d(WxW^*)W$$

and WW^* is increasing for α^d. In order to prove the statement it suffices to show that there is an order isomorphism between $\mathfrak{S}(\alpha)$ and $\mathfrak{S}(\alpha^d)$. As proved in [Bha01] or by Theorem 3.5 of [Pow03b], there exists an order isomorphism between the set of CP-semigroup subordinates of α and the set of CP-semigroup subordinates of α^d. This isomorphism is described as follows. For every subordinate β of α there exists a unique subordinate β' of α^d such that

$$\beta_t(x) = W^*\beta_t'(WxW^*)W. \tag{1}$$

It is clear that if β is trivially maximal (with respect to α), then β' is trivially maximal (with respect to α^d). Conversely, suppose that β is not trivially maximal, so that there exists $k > 0$ such that $e^{kt}\beta_t$ is a subordinate of α. Then there exists γ a unique subordinate of α^d such that

$$e^{kt}\beta_t(x) = W^*\gamma_t(WxW^*)W, \forall x \in \mathfrak{B}(\mathfrak{H}), t \geq 0.$$

Therefore, by dividing by e^{kt} we obtain that β is also the compression of $e^{-kt}\gamma_t$ which is obviously also a subordinate since $k > 0$. By uniqueness of the correspondence, $\beta_t' = e^{-kt}\gamma_t$ for all t. Hence β' is not trivially maximal. □

CP-flow subordinates of a CP-flow are always trivially maximal, therefore if a CP-flow is aligned then it is automatically q-pure. We also note that a CP-flow is q-pure if and only if it is aligned with respect to the subordinate $x \mapsto S_t x S_t^*$. We omit the elementary proof of this fact.

We now show that unital CP-flows are aligned if and only if they are q-pure and induce E_0-semigroups of type II_0. First let us approach the case when the CP-flow is in also a semigroup of endomorphisms.

Proposition 3. *Let α be an E_0-semigroup which is in addition a CP-flow over $\mathfrak{K}$. If α has type II_0, then $\mathfrak{S}(\alpha)$ is precisely the set of flow subordinates of α. Therefore, α is aligned if and only if it is type II_0 and it is q-pure.*

Proof. Suppose that α is an E_0-semigroup of type II_0 which is also a CP-flow over $\mathfrak{K}$. Note that every flow subordinate of α is clearly an element of $\mathfrak{S}(\alpha)$, as flow subordinates are always trivially maximal. Conversely, let β be a trivially maximal CP-semigroup subordinate of α. By Theorem 3.4 of [Pow03b] there exists a family of operators $(C(t))_{t\geq 0}$ in $\mathfrak{B}(\mathfrak{H})$ such that

$$\beta_t(x) = C(t)\alpha_t(x),$$

where the family $(C(t))_{t\geq 0}$ is a contractive positive local cocycle, i.e. $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ and $0 \leq C(t) \leq I$ for all $t > 0$, $C(t+s) = C(t)\alpha_t(C(s))$ for all $t, s \geq 0$ and $t \mapsto C(t)$ is strongly continuous for $t \geq 0$ with $C(t) \to I$ as $t \to 0+$.

Let S_t denote as usual the right shift semigroup on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and let $V_t = C(t)S_t$. Note that V_t is strongly continuous and furthermore it is a semigroup: for all $t, r \geq 0$,

$$V_tV_r = C(t)S_tC(r)S_r = C(t)\alpha_t(C(r))S_tS_r = C(t+r)S_{t+r} = V_{t+r}.$$

Moreover, V_t is a unit of α_t, since for all $x \in \mathfrak{B}(\mathfrak{H})$,

$$\alpha_t(x)V_t = \alpha_t(x)C(t)S_t = C(t)\alpha_t(x)S_t = C(t)S_t x = V_t x.$$

Notice, however, that α is type II_0, therefore there exists $\kappa \in \mathbb{C}$ such that $V_t = e^{-\kappa t}S_t$ for all $t \geq 0$. Furthermore, since $C(t)$ is a contraction, we have that V_t is a contraction for all $t > 0$, hence we must have $\mathrm{Re}(\kappa) \geq 0$. We now show that in fact κ must be a real number. Recall that β_t is a CP-semigroup, and

$$0 \leq S_t^* \beta_t(I) S_t = S_t^* C(t) S_t = e^{-\kappa t} I,$$

hence κ is real and satisfies $\kappa \geq 0$.

We will now prove that $\kappa = 0$. Let $\gamma_t(x) = e^{\kappa t}\beta_t(x)$. Notice that S_t is an intertwiner semigroup for γ: for all $t > 0$ and $x \in \mathfrak{B}(\mathfrak{H})$,

$$\gamma_t(x)S_t = e^{\kappa t}\beta_t(x)S_t = e^{\kappa t}C(t)\alpha_t(x)S_t = e^{\kappa t}C(t)S_t x = S_t x.$$

It follows that γ is a CP_κ-flow in the sense of Definition 4.0 of [Pow03b], that is to say, $e^{-\kappa t}\gamma_t$ is a CP-semigroup and γ is intertwined by the shift. However, by Theorem 4.15 of [Pow03b], every CP_κ-flow must be a CP-flow. But a CP-flow is contractive, hence we must have $\gamma_t(I) \leq I$, thus

$$e^{\kappa t}\beta_t(I) = e^{\kappa t}C(t) \leq I$$

for all $t > 0$. Therefore, we have that for all positive $t > 0$,

$$e^{kt}\beta_t(x) = e^{\kappa t}C(t)\alpha_t(x) \leq \alpha_t(x).$$

Since β is trivially maximal, we must have that $\kappa = 0$. Thus we have shown that every element of $\mathfrak{S}(\alpha)$ is a CP-flow.

Therefore, if α is q-pure and of type II_0 then it is aligned. On the other hand, it is clear that if α is an aligned E_0-semigroup, then it is type II_0 and q-pure as discussed before the proposition. □

Theorem 4. *Let α be a unital CP-flow over $\mathfrak{K}$. If the minimal dilation of α is type II_0, then $\mathfrak{S}(\alpha)$ is precisely the set of flow subordinates of α. Therefore, α is aligned if and only if it is q-pure and its minimal dilation is type II_0.*

Proof. By Lemma 4.50 of [Pow03b], there exists a minimal dilation of α to an E_0-semigroup α^d which is a CP-flow over the Hilbert space $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$, i.e. there exists an isometry $W : \mathfrak{H} \to \mathfrak{H}_1$ such that

$$\alpha_t(x) = W^*\alpha_t^d(WxW^*)W,$$

WW^* is increasing for α^d and if S_t^d denotes the right shift semigroup on the space $\mathfrak{H}_1$, then $WS_t = S_t^d W$ for all $t > 0$. We use the order isomorphism established in the proof of Lemma 2 that associates to each subordinate $\beta \in \mathfrak{S}(\alpha)$ a unique subordinate $\beta' \in \mathfrak{S}(\alpha^d)$ satisfying (1). If α^d is type II_0 and $\beta \in \mathfrak{S}(\alpha)$, then it follows from the previous proposition that β' is a CP-flow. Hence for all $t > 0$,

$$\beta_t(x)S_t = W^*\beta_t'(WxW^*)WS_t = W^*\beta_t'(WxW^*)S_t^d W = W^*S_t^d WxW^*W = S_t x.$$

In other words, β is a CP-flow. Thus we proved that all elements of $\mathfrak{S}(\alpha)$ are CP-flows. On the other hand, every flow subordinate of α is clearly an element of $\mathfrak{S}(\alpha)$.

Thus, if α is a unital q-pure CP-flow and its minimal dilation has type II_0, then it is aligned. Conversely, it is clear that if α is aligned then its minimal dilation α^d as discussed above is also aligned, thus it has index zero. Since α^d is a CP-flow, it cannot be an automorphism group, hence it has type II_0. Moreover, since α is aligned, it is clearly q-pure. □

Prime E_0-semigroups

Definition 5. An E_0-semigroup α is called *prime* if, whenever α is cocycle conjugate to $\beta \otimes \gamma$ where β and γ are E_0-semigroups, then β or γ is type I_0.

It follows from the complete classification of E_0-semigroups of type I in terms of the index, and the additivity of the index with respect to tensoring, that a prime E_0-semigroup of type I is cocycle conjugate either to an automorphism group or to the CAR/CCR flow of index 1. It is a corollary of the work on the gauge group of an E_0-semigroup by Markiewicz-Powers in [MP09] or Tsirelson in [Tsi08], that prime E_0-semigroups of type II_1 exist. And, as it has belatedly come to our attention, earlier Liebscher had proven that prime E_0-semigroups of type II_k exist for $k \geq 1$ (see Proposition 4.32 and Note 4.33 in [Lie09]). We establish that there exist uncountably many prime E_0-semigroups of type II_0.

Theorem 6. *Aligned spatial E_0-semigroups are prime.*

Proof. We prove the contrapositive. Suppose that α is an E_0-semigroup and $\alpha = \beta \otimes \gamma$ where β and γ are two spatial E_0-semigroups neither of which has type I_0. Without loss of generality, by applying an appropriate conjugacy, we may assume that both act on $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{H}$ is infinite-dimensional and separable. Let U and V be normalized units of β and γ, respectively. Let us denote by θ^U and θ^V the semigroups given by $\theta_t^U(x) = U_t x U_t^*$ and $\theta_t^V(x) = V_t x V_t^*$, which are E-semigroup subordinates of β and γ, respectively (notice these are not unital since β and γ are not automorphism groups). Notice that $\sigma^V = \beta \otimes \theta^V$ is a subordinate of α. We show that it is trivially maximal. Suppose that $k \geq 0$ and $e^{kt}\sigma_t^V$ is also a subordinate of α. Then we have that for all $x \in \mathfrak{B}(\mathfrak{H})$,

$$e^{kt}(I \otimes V_t V_t^*) = e^{kt}(\beta_t \otimes \theta_t^V)(I) = e^{kt}\sigma_t^V(I) \leq \alpha_t(I) = I$$

However $\|V_t\| = 1$, hence by taking norms on both sides we conclude $e^{kt} \leq 1$ for all $t > 0$. Thus $k = 0$. Analogously, $\sigma^U = \theta^U \otimes \gamma$ is trivially maximal.

We prove that σ^V and σ^U are incomparable. Suppose that $\sigma^V \geq \sigma^U$. Notice that for all $x, y \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$,

$$\sigma_t^V(x \otimes y)(U_t \otimes I) = \alpha_t(x)U_t \otimes \theta_t^V(x) = (U_t \otimes I)(x \otimes \theta_t^V(y))$$
$$\sigma_t^U(x \otimes y)(U_t \otimes I) = \theta_t^U(x)U_t \otimes \gamma_t(x) = (U_t \otimes I)(x \otimes \gamma_t(y))$$

Therefore we have that for all $x, y \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$,

$$(U_t \otimes I)^* \Big[\sigma_t^V(x \otimes y) - \sigma_t^U(x \otimes y)\Big](U_t \otimes I) = x \otimes [\theta_t^V(y) - \gamma_t(y)]$$

Thus in the special case when $x = I$, we have that the map $y \mapsto I \otimes [\theta_t^V(y) - \gamma_t(y)]$ is completely positive for all $t > 0$, hence $\theta^V \geq \gamma$. However θ^V is a pure element in the cone of completely positive maps, therefore we have that for every $t > 0$, γ_t is a multiple of θ_t^V, and we have a contradiction since γ is not type I_0. By symmetry, we obtain that σ^U and σ^V are incomparable as asserted. Thus we proved that α is not aligned. □

Corollary 7. *There exist uncountably many non-cocycle conjugate E_0-semigroups of type II_0 which are prime.*

Proof. By Theorem 6, it suffices to exhibit an uncountable family of non-cocycle conjugate aligned E_0-semigroups. In order to do so, we consider a class of E_0-semigroups arising from boundary weight doubles as in [Jan10]. Let $g(x)$ be a fixed complex-valued measurable function such that $g \notin L^2(0,\infty)$ yet $(1 - e^{-x})^{1/2}g(x) \in L^2(0,\infty)$, and for each $t > 0$ let $g_t = \chi_{(t,\infty)}g$, which is an element of $L^2(0,\infty)$. Define the weight on $\mathfrak{B}(L^2(0,\infty))$ given by $\nu(A) = \lim_{t\to 0+}(g_t, Ag_t)$. For every $0 < \lambda < 1/2$, let $\mu_\lambda : M_2(\mathbb{C}) \to \mathbb{C}$ be given by

$$\mu_\lambda(X) = \lambda x_{11} + (1 - \lambda)x_{22}.$$

Let us define the boundary weight map from $M_2(\mathbb{C})_*$ to boundary weights on $\mathfrak{B}(\mathbb{C}^2 \otimes L^2(0,\infty))$ given by $\omega(\rho)(A) = \rho(I)\mu_\lambda\big((I \otimes \nu)(A)\big)$ for all $\rho \in M_2(\mathbb{C})$ and A in the domain of finiteness of $I \otimes \nu$ (the so called null boundary algebra of definition 4.16 in [Pow03b]). Then by Corollary 3.3 of [Jan10] and the assumptions on $g(x)$, we have that ω gives rise to an E_0-semigroup of type II_0. Once one applies Theorem 4.4 of [JMP11] to reconcile the earlier definition of q-purity with the one in this paper, we obtain from Lemma 4.3 and Proposition 5.2 of [Jan10] that ω gives rise to a q-pure unital CP-flow. Therefore by Theorem 4 it gives to a aligned E_0-semigroup α^λ. Finally, by Theorem 5.4 of [Jan10], given $\lambda, \zeta \in (0, 1/2)$, we have that α^λ is cocycle conjugate to α^ζ if and only if $\lambda = \zeta$. □

We should point out that it is possible to obtain a different uncountable family of non-cocycle conjugate E_0-semigroups by using Theorem 3.22 of [Pow03a]. For details see the discussion in the end of section III therein. This would be indeed be a different family from the one exhibited above in the sense that, by Corollary 5.5 of [Jan10], the E_0-semigroups constituting both families are not cocycle conjugate.

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