The blow-up theorem of a discrete semilinear wave equation

Keisuke Matsuya

1 Introduction

Consider the Cauchy problem for the semilinear wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u + |u|^p \ (p > 1), \\ u(0, \vec{x}) = f(\vec{x}), \\ \dfrac{\partial u}{\partial t}(0, \vec{x}) = g(\vec{x}), \end{cases} \tag{1}$$

where $u := u(t, \vec{x})$ $(t \geq 0,\ \vec{x} := (x_1, \cdots, x_d) \in \mathbb{R}^d)$ and Δ is the d-dimensional Laplacian $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$. When the initial condition $f(\vec{x}),\ g(\vec{x})$ are continuous and unifomly bounded, there is a smooth solution for $t > 0$ and whenever the solution is bounded. It is well known that the solutions of this problem is not necessarily bounded. For instance, considering the spatially uniform initial condition, $f(\vec{x}) \equiv 0,\ g(\vec{x}) \equiv g > 0$, this fact can be understood. In this case, $u(t, \vec{x}) = u(t)$ and (1) becomes an ordinary differential equation,

$$\begin{cases} \dfrac{d^2 u}{dt^2} = |u|^p \\ u(0) = 0 \\ \dfrac{du}{dt}(0) = g > 0 \end{cases} . \tag{2}$$

Because of the initial condition, the solution of (2) is non negative if it is bounded so that $|u|^p = u^p$ is obtained. Multiplying the both sides by $\frac{du}{dt}$ and integrating 0 to t, we get

$$\left(\frac{du}{dt}\right)^2 = \frac{2}{p+1} u^{p+1} + g^2.$$

Owing to $\frac{d^2 u}{dt^2} \geq 0$ and $\frac{du}{dt}(0) = g > 0$, $\frac{du}{dt} \geq 0$ $(t \geq 0)$ is derived. Therefore the differential inequality,

$$\frac{du}{dt} > \sqrt{\frac{2}{p+1}} u^{(p+1)/2} \tag{3}$$

is obtained. Since there exists positive number ε such that $u(\varepsilon) > 0$, the solution of (3) is

$$u(t) > \frac{(\alpha C)^{-1/\alpha}}{\left\{(\alpha C)^{-1} u(\varepsilon)^{-\alpha} + \varepsilon - t\right\}^{1/\alpha}} \quad (t > \varepsilon)$$

where $\alpha = (p-1)/2$ and $C = \sqrt{2/(p+1)}$. Now we see that the right side diverges as $t \to \alpha^{-1}C^{-1}u(\varepsilon)^{-\alpha} + \varepsilon - 0$ so that the solution of (2) is not bounded for all $t \geq 0$. In general, if there exists a finite time $T \in \mathbb{R}_{>0}$ and if the solution of (1) in $(t, \vec{x}) \in [0, T) \times \mathbb{R}^d$ satisfies

$$\limsup_{t \to T-0} \|u(t, \cdot)\|_{L^\infty} = \infty,$$

where

$$\|u(t, \cdot)\|_{L^\infty} := \sup_{\vec{x} \in \mathbb{R}^d} |u(t, \vec{x})|,$$

then we say that the solution of (1) blows up at time T. If such T does not exist for the solution of (1) then we call it a global solution.

The critical exponent $p_{\mathrm{c}}(d) := \frac{d+1+\sqrt{d^2+10d-7}}{2(d-1)}$ $(d \geq 2)$ which characterises the blow up of the solutions for (1) is studied by many researchers [2–8]. F. John [2] proved small data blow up with $1 < p < p_{\mathrm{c}}(3)$ and small data global existence with $p_{\mathrm{c}}(3) < p$. R.T. Glassey [3, 4] proved small data blow up with $1 < p < p_{\mathrm{c}}(2)$ and small data global existence with $p_{\mathrm{c}}(2) < p$. J. Schaeffer [5] proved small data blow up with $p = p_{\mathrm{c}}(d)$ where $d = 2, 3$. T. Sideris [6] proved small data blow up with $1 < p < p_{\mathrm{c}}(d)$ where $d \geq 4$. V. Georgiev, H. Lindblad and C. Sogge [7] proved small data global existence with $p_{\mathrm{c}(d)} < p$ where $d \geq 4$. B. Yordanov and Q.S. Zhang [8] proved small data blow up with $p = p_{\mathrm{c}(d)}$ where $d \geq 4$.
Kato [1] proved the following theorem

Theorem 1.1 *Let u be a generalized solution of*

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} a_{jk}(t, \vec{x}) \frac{\partial}{\partial x_k} u - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_j(t, \vec{x}) u = f(t, \vec{x}, u) \ (t \geq 0,\ \vec{x} \in \mathbb{R}^d)$$

on a time interval $0 \leq t < T \leq \infty$, which is supported on a forward cone

$$K_R = \{(t, \vec{x}); t \geq 0,\ |\vec{x}| \leq t + R\} \ (R > 0).$$

Assume that f satisfies

$$f(t, \vec{x}, s) \geq \begin{cases} b|s|^{p_0} \ (|s| \leq 1), \\ b|s|^{p} \ (|s| \geq 1), \end{cases}$$

where $b > 0$ and $1 < p \leq p_0 = (d+1)/(d-1)$.
(If $d = 1$, p_0 may be any number greater than or equal to p.)
Moreover, assume that, for $w(t) = \int_{\mathbb{R}^d} u(t, \vec{x}) d\vec{x}$, either (a) $\frac{dw}{dt}(0) > 0$, or (b) $\frac{dw}{dt}(0) = 0$ and $w(0) = 0$.
Then one must have $T < \infty$.

From this theorem, we obtain the next corollary.

corollary 1.1 *Let u be the solution of* (1). *Assume that f and g in* (1) *satisfy* $supp(f)\bigcup supp(g) \subset \{\vec{x} \in \mathbb{R}^d; |\vec{x}| \le K\}$ $(K > 0)$ *and* $\int_{\mathbb{R}^d} g d\vec{x} > 0$. *Moreover, assume* $1 < p \le (d+1)/(d-1)$ $(d \ge 2)$.
(If $d = 1$, any assumptions for p but $1 < p$ are not needed.)
Then u blows up at some finite time.

In numerical computation of (1), one has to discretize it and consider a partial difference equation. A naive discretization would be to replace the t-differential and the Laplacian with central differences such that (1) turns into

$$\frac{u_{\vec{n}}^{\tau+1} - 2u_{\vec{n}}^{\tau} + u_{\vec{n}}^{\tau-1}}{\delta^2} = \sum_{k=1}^{d} \frac{u_{\vec{n}+\vec{e}_k}^{\tau} - 2u_{\vec{n}}^{\tau} + u_{\vec{n}-\vec{e}_k}^{\tau}}{\xi^2} + |u_{\vec{n}}^{\tau}|^p,$$

where $u(\tau, \vec{n})(=: u_{\vec{n}}^{\tau}) : \mathbb{Z}_{\ge 0} \times \mathbb{Z}^d \to \mathbb{R}$, for positive constants δ and ξCand where $\vec{e}_k \in \mathbb{Z}^d$ is the unit vector whose kth component is 1 and whose other components are 0. Putting $\lambda := \delta^2/\xi^2$, we obtain

$$u_{\vec{n}}^{\tau+1} = 2d\lambda \hat{M}(u_{\vec{n}}^{\tau}) + (2 - 2d\lambda)u_{\vec{n}}^{\tau} - u_{\vec{n}}^{\tau-1} + \delta^2 |u_{\vec{n}}^{\tau}|^p \quad (p > 1). \tag{4}$$

Here

$$\hat{M}(V_{\vec{n}}) := \frac{1}{2d} \sum_{k=1}^{d} (V_{\vec{n}+\vec{e}_k} + V_{\vec{n}-\vec{e}_k}). \tag{5}$$

For a spatially uniform initial condition, (4) becomes an ordinary difference equation

$$u^{\tau+1} = 2u^{\tau} - u^{\tau-1} + \delta|u^{\tau}|^p.$$

The above equation is a discretization of (2), but the features of its solutions are quite different. In fact, u^{τ} will never blow up at finite time steps. Hence, (4) does not preserve the global nature of the original semilinear wave equation (1).

In this article, we propose and investigate a discrete analogue of (1) which does keep the characteristic of corollary 1.1.
In section 2, we present a partial difference equation with a parameter p whose continuous limit equals to (1), and state the main theorem which shows that this difference equation has exactly the same properties as (1) with respect to p. This theorem is proved in section 3.

2 Discretization of the semilinear wave equation

We consider the following initial value problem for the partial difference equation

$$u_{\vec{n}}^{\tau+1} + u_{\vec{n}}^{\tau-1} = \frac{4v_{\vec{n}}^{\tau}}{2 - \delta^2 v_{\vec{n}}^{\tau} |v_{\vec{n}}^{\tau}|^{p-2}}, \quad (\tau \in \mathbb{Z}_{>0},\ \vec{n} \in \mathbb{Z}^d) \tag{6}$$

where $p > 1$ and $\delta > 0$ are parameters and $v_{\vec{n}}^{\tau}$ is defined by means of $\hat{M}$ (5) as

$$v_{\vec{n}}^{\tau} := \hat{M}(u_{\vec{n}}^{\tau}).$$

If there exists a smooth function $u(t, \vec{x})$ $(t \in \mathbb{R}_{\geq 0},\ \vec{x} \in \mathbb{R}^d)$ that satisfies $u(\tau\delta, \xi\vec{n} = u_{\vec{n}}^{\tau})$ with $\xi := \sqrt{d}\delta$, we find

$$u(t+\delta, \vec{x}) + u(t-\delta, \vec{x}) = v(t, \vec{x})(2 + \delta^2 v(t, \vec{x})|v(t, \vec{x})|^{p-2}) + O(\delta^4),$$

with

$$v(t, \vec{x}) := \frac{1}{2d}\sum_{k=1}^{d}(u(t, \vec{x} + \xi\vec{e}_k) + u(t, \vec{x} - \xi\vec{e}_k)),$$

or

$$\frac{u(t+\delta, \vec{x}) - 2u(t, \vec{x}) + u(t-\delta, \vec{x})}{\delta^2} = \sum_{k=1}^{d}\frac{u(t, \vec{x} + \xi\vec{e}_k) - 2u(t, \vec{x}) + u(t, \vec{x} - \xi\vec{e}_k)}{\xi^2} + |u(t, \vec{x})|^p + O(\delta^2).$$

Taking the limit $\delta \to +0$, we obtain the semilinear wave equation (1)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + |u|^p.$$

Thus (6) can be regarded as a discrete analogue of (1).

Because of the term $2 - \delta^2 v_{\vec{n}}^{\tau}|v_{\vec{n}}^{\tau}|^{p-2}$, if $v_{\vec{n}}^{\tau} \to (2\delta^{-2})^{1/(p-1)}$, then $u_{\vec{n}}^{\tau+1} \to +\infty$. This behaviour may be regarded as an analogue of th blow up of solutions for the semilinear wave equation. Thus we define a blow up of solution for (6) as follow.

Definition 2.1 *Let $u_{\vec{n}}^{\tau}$ be a solution of (6).*

When there exists $\tau_0 \in \mathbb{Z}_{\geq 0}$ such that $v_{\vec{n}}^{\tau} \leq (2\delta^{-2})^{1/(p-1)}$ for all $\tau < \tau_0$ and $\vec{n} \in \mathbb{Z}^d$, and there exists $\vec{n}_0 \in \mathbb{Z}^d$ such that $v_{\vec{n}_0}^{\tau_0} \geq (2\delta^{-2})^{1/(p-1)}$, then we say that the solution $u_{\vec{n}}^{\tau}$ blows up at time $\tau_0 + 1$.

The example of blow-up solutions for (6) is as follow. Considering the spatially uniform initial condition $u_{\vec{n}}^0 \equiv 0,\ u_{\vec{n}}^1 \equiv g > 0,\ u_{\vec{n}}^{\tau} = u^{\tau}$ and (6) becomes an ordinary difference equation,

$$\begin{cases} u^{\tau+1} + u^{\tau-1} = \dfrac{4u^{\tau}}{2 - \delta^2 u^{\tau}|u^{\tau}|^{p-2}} \\ u^0 = 0 \\ u^1 = g > 0 \end{cases}. \tag{7}$$

This is the discrete analogue of (2). One can see that the solution of (7) blows up as follow.

Let the solution of (7) does not blow up at any τ,
i.e., $u^{\tau} < (2\delta^{-2})^{1/(p-1)}$ $({}^{\forall}\tau \in \mathbb{Z}_{\geq 0})$, then we get

$$u^{\tau+1} - 2u^{\tau} + u^{\tau-1} = \frac{2\delta^2|u^{\tau}|^p}{2 - \delta^2 u^{\tau}|u^{\tau}|^{p-2}} > 0.$$

Hence, we obtained a difference inequality $u^{\tau+1} - 2u^{\tau} + u^{\tau-1} > 0$. Solving this inequality with the initial value, $u^{\tau} > g\tau$ is derived. This inequality means that u^{τ} is arbitrarily large for large $\tau \in \mathbb{Z}_{>0}$. This statement contradicts to $u^{\tau} < (2\delta^{-2})^{1/(p-1)}$ $({}^{\forall}\tau \in \mathbb{Z}_{\geq 0})$ so that the solution of (7) blows up at some finite time.

Furthermore, (6) inherits quite similar properties to those of (1). The following theorem is the main result in this article.

Theorem 2.1 *Let $u_{\vec{n}}^{\tau}$ be the solution for* (6). *Assume that*

(A1) $\{\vec{n} \in \mathbb{Z}^d; u_{\vec{n}}^{j} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K\},\ (j = 0, 1\ K > 0)$

(A2) $\sum_{\vec{n}} u_{\vec{n}}^{1} > \sum_{\vec{n}} u_{\vec{n}}^{0}$,

where $\|\vec{n}\| := |n_1| + \cdots + |n_d|$ $(\vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d)$.
Moreover assume $1 < p \leq (d+1)/(d-1)$ $(d \geq 2)$.
(If $d = 1$, any assumptions for p but $1 < p$ are not needed.)
Then $u_{\vec{n}}^{\tau}$ blows up at some finite time.

Remark *The summations in (A2) seem to be infinite series, but owing to (A1), both summations are finite series.*
The author believes that (6) *does keep the characteristic of the critical exponent* $p_{c(d)}$.

3 Proof of the theorem

The idea of the proof is similar to that adopted by Kato [1].

First, to make the equations simply we take the scaling $(2\delta^{-2})^{1/(p-1)} u_{\vec{n}}^{\tau} \to u_{\vec{n}}^{\tau}$ then (6) is changed to

$$u_{\vec{n}}^{\tau+1} + u_{\vec{n}}^{\tau-1} = \frac{2v_{\vec{n}}^{\tau}}{1 - v_{\vec{n}}^{\tau}|v_{\vec{n}}^{\tau}|^{p-2}}. \tag{8}$$

We shall deduce a contradiction by assuming that $u_{\vec{n}}^{\tau}$ does not blow up at any finite time, i.e., $v_{\vec{n}}^{\tau} < 1$ $({}^{\forall}(\tau, \vec{n}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d)$.

Put

$$U^{\tau} := \sum_{\vec{n}} u_{\vec{n}}^{\tau}. \tag{9}$$

Because of (A1), $\{\vec{n} \in \mathbb{Z}^d; u_{\vec{n}}^{\tau} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + \tau - 1\}$ so that the summation of (9) is well-defined. Moreover, from $\{\vec{n} \in \mathbb{Z}^d; v_{\vec{n}}^{\tau} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + \tau\}$, $U^{\tau} = \sum_{\vec{n}} v_{\vec{n}}^{\tau}$ and $v_{\vec{n}}^{\tau} < 1$, we obtain the inequality as follow

$$U^{\tau} < T^{\tau}, \tag{10}$$

where $T^{\tau} := \#\{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + \tau\}$. From (8), we get

$$\sum_{\vec{n}} (u_{\vec{n}}^{\tau+1} - 2v_{\vec{n}}^{\tau} + u_{\vec{n}}^{\tau-1}) = \sum_{\vec{n}} \frac{2|v_{\vec{n}}^{\tau}|}{1 - v_{\vec{n}}^{\tau}|v_{\vec{n}}^{\tau}|^{p-2}} \tag{11}$$

The left hand of (11) is same to $U^{\tau+1} - 2U^{\tau} + U^{\tau-1}$ and the right hand is nonnegative because that all terms of summation are nonnegative.
Hence we get $U^{\tau+1} - 2U^{\tau} + U^{\tau-1} \geq 0$. From this inequality, there exists some positive number C_0 which satisfies the inequality as follow

$$U^{\tau} \geq C_0 \tau \tag{12}$$

for sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Note that $U^{\tau} \geq 0$ for sufficiently large $\tau \in \mathbb{Z}_{>0}$.

To get another inequality about U^{τ}, we need the next lemma.

Lemma 3.1 *Put*

$$h(x) = \frac{2|x|^p}{1 - x|x|^{p-2}} \ (x < 1).$$

Let $0 \leq x_0 < 1,\ x_{j-1} \leq x_j\ (j = 1, \cdots, s)$ *and* $\lambda_j \geq 0\ (j = 0, \cdots, s),\ \lambda_0 + \cdots + \lambda_s = 1$.
If $\lambda_0 x_0 + \cdots + \lambda_s x_s \geq 0$ *then the inequality as follow*

$$\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) \geq h(\lambda_0 x_0 + \cdots + \lambda_s x_s)$$

is satisfied.

Proof *We get*

$$\begin{aligned} &\frac{\partial}{\partial x_0}(\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s)) \\ &\qquad = \lambda_0 (h'(x_0) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s)), \end{aligned} \tag{13}$$

where $h'(x) := \frac{dh}{dx}(x)$.
Since $h(x)$ *is convex on the interval* $[0, 1)$, $h'(x)$ *increases monotonically on the interval* $[0, 1)$. *On the other hand,* $0 \leq \lambda_0 x_0 + \cdots + \lambda_s x_s \leq x_0 < 1$ *by the definitions.*
Then we get that (13) *is nonnegative and*

$$\begin{aligned} &\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s) \\ &\quad \geq \lambda_0 h(-(\lambda_1 x_1 + \cdots \lambda_s x_s)/\lambda_0) + \lambda_1 h(x_1) + \cdots + \lambda_s h(x_s) - h(0) \\ &\quad \geq 0 \end{aligned}$$

is obtained.
Now the proof of lemma is completed. ∎

Since $\{\vec{n} \in \mathbb{Z}^d; v^{\tau}_{\vec{n}} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + \tau\}$ and $U^{\tau} = \sum_{\vec{n}} v^{\tau}_{\vec{n}}$ is nonnegative for sufficiently large $\tau \in \mathbb{Z}_{\geq 0}$, this lemma is adopted to right hand of (11) as follow,

$$\begin{aligned} \frac{2|v^{\tau}_{\vec{n}}|^p}{1 - v^{\tau}_{\vec{n}}|v^{\tau}_{\vec{n}}|^{p-2}} \quad &\geq \quad T^{\tau} \frac{2|\frac{1}{T^{\tau}} \sum_{\vec{n}} v^{\tau}_{\vec{n}}|^p}{1 - \frac{1}{T^{\tau}} \sum_{\vec{n}} v^{\tau}_{\vec{n}} |\frac{1}{T^{\tau}} \sum_{\vec{n}} v^{\tau}_{\vec{n}}|^{p-2}} \\ &= \quad \frac{2(T^{\tau})^{1-p}(U^{\tau})^p}{1 - (T^{\tau})^{1-p}(U^{\tau})^{p-1}}. \end{aligned}$$

Here we put $\lambda_j = 1/T^\tau$ $(j = 1, \cdots, T^\tau)$.
We note that there exists positive number C_T which satisfies $T^\tau < C_T \tau^d$ for sufficiently large $\tau \in \mathbb{Z}_{>0}$. Considering this statement and $U^\tau < T^\tau$, we get

$$U^{\tau+1} - 2U^\tau + U^{\tau-1} \geq 2C_T^{1-p} \tau^{-d(p-1)} (U^\tau)^p$$

with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Since $1 < p \leq (d+1)/(d-1)$, i.e., $-d(p-1) \geq -(p+1)$, we get

$$U^{\tau+1} - 2U^\tau + U^{\tau-1} \geq C_2 \tau^{-(p+1)} (U^\tau)^p, \tag{14}$$

where $C_2 := 2C_T^{1-p}$.
Moreover, using (3),

$$U^{\tau+1} - 2U^\tau + U^{\tau-1} \geq C_2 C_0^{1-p} \tau^{-1}$$

with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Solving this difference inequality, it is found that U^τ increases monotonically and there exists some positive number C_1' which satisfies inequality

$$U^\tau \geq C_1' \tau \log \tau, \tag{15}$$

with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Now we consider about

$$E^\tau := (U^{\tau+1} - U^\tau)^2 - \frac{C_2}{p+1} \tau^{-(p+1)} (U^\tau)^{p+1}.$$

Since (14) and U^τ is monotonically increasing, we get

$$\begin{aligned}
&E^{\tau+1} - E^\tau \\
&\quad = \{(U^{\tau+1} - U^\tau)^2 - (U^\tau - U^{\tau-1})^2\} \\
&\qquad - \frac{C_2}{p+1} \{\tau^{-(p+1)} (U^\tau)^{p+1} - (\tau-1)^{-(p+1)} (U^{\tau-1})^{p+1}\} \\
&\quad \geq 2\tau^{-(p+1)} (U^\tau)^p (U^{\tau+1} - U^{\tau-1}) - \frac{C_2}{p+1} \tau^{-(p+1)} \{(U^\tau)^{p+1} - (U^{\tau-1})^{p+1}\} \\
&\quad \geq C_2 \tau^{-(p+1)} (U^\tau)^{p+1} \left\{ 1 - \frac{U^{\tau-1}}{U^\tau} - \frac{1}{p+1} + \frac{1}{p+1} \left(\frac{U^{\tau-1}}{U^\tau} \right)^{p+1} \right\}.
\end{aligned}$$

It is known that $\frac{1}{p+1}\lambda^{p+1} - \lambda + 1 - \frac{1}{p+1} > 0$ $(0 \leq \lambda \leq 1)$ so that we get $E^{\tau+1} - E^\tau > 0$ with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Due to $U^\tau/\tau \geq C_1' \log \tau$ by (15), there exists some positive number C_3 which satisfies

$$(U^{\tau+1} - U^\tau)^2 \geq C_3 \tau^{-(p+1)} (U^\tau)^{p+1}$$

with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Owing to (15), we get

$$\begin{aligned}
U^{\tau+1} - U^\tau &\geq C_3 \left(\frac{U^\tau}{\tau} \right)^{(p-1)/2} \frac{U^\tau}{\tau} \\
&\geq C_3 C_1'^{(p-1)/2} (\log \tau)^{(p-1)/2} \frac{U^\tau}{\tau},
\end{aligned}$$

with sufficiently large $\tau \in \mathbb{Z}_{>0}$.
Since $(\log \tau)^{(p-1)/2}$ is arbitrarily large for large $\tau \in \mathbb{Z}_{>0}$, the following linear difference inequality

$$U^{\tau+1} - U^{\tau} \geq C\frac{U^{\tau}}{\tau}$$

with any positive number C and $\tau \geq {}^{\exists}\tau_0$ where τ_0 depends on C is held.
Solving this difference inequality, we get

$$U^{\tau} \geq \prod_{s=\tau_0}^{\tau-1} \frac{s+C}{s} U^{\tau_0} \ (\tau \geq \tau_0 + 1).$$

Let $C > d+1$, then

$$U^{\tau} \geq U^{\tau_0} \prod_{k=0}^{d} \frac{\tau+k}{\tau_0+k} \ (\tau \geq \tau_0 + 1).$$

This inequality means that there exists some positive number C' which satisfies inequality $U^{\tau} \geq C'\tau^{d+1}$ with sufficiently large $\tau \in \mathbb{Z}_{>0}$ but this statement contradicts to $U^{\tau} < T^{\tau}$.
Now the contradiction is deduced and the proof of the theorem is completed.

Acknowledgement

I am deeply grateful to Prof. Tokihiro who provided helpful comments and suggestions.

References

[1] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* 32(1980), 501–505,

[2] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28(1980), 235–268,

[3] R.T. Glassey, Finite-time blow up for solutions to nonlinear wave equations, *Math. Z.* 177(1981), 323–340,

[4] R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.* 178(1981), 233–261,

[5] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh* 101A(1985), 31–44,

[6] T.C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations* 52(1984), 378–406,

[7] V. Georgiev, H. Lindblad, C. Sogge, Weighted strichartz estimates and global existence for semilinear wave equations, *Amer. J. Math* 119(6)(1997), 1291–1319,

[8] B.T. Yordanov, Q.S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.* 231(2006), 361–374.