ON THE GRAVITATIONAL FIELDS CREATED BY THE ELECTROMAGNETIC WAVES

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Abstract. We show that the Maxwell equations describing an electromagnetic wave are a mathematical consequence of the Einstein equations for the same wave. This fact is significant for the problem of the Einsteinian metrics corresponding to the electromagnetic waves.

Summary – Introduction – **1**. On a consequence of the fact that the light-rays are null geodesics in any spacetime manifold. – **2**. The Maxwell equations of an electromagnetic wave are a consequence of the Einstein equations for the same wave. – **2bis**. An example. – **3**. A result analogous to that of sect.**2** holds in the linear version of GR. – **3bis**, **3ter**. An example. – **4**. A final remark. – Appendix.

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Introduction. – Very rarely this subject has been approached from a really physical standpoint. On the contrary, there are in the literature many interesting papers of a mathematical character about metrics which are linked *in some way* to the propagation of electromagnetic (e.m.) waves. We recall sect. **8** of a famous paper (1923) by Eddington [1], the paper of 1926 by Baldwin and Jeffery [2], the paper by Bonnor of 1969 [3], and the overflowing of geometrical articles on the plane-fronted waves with parallel propagation (briefly, p-p waves) [4].

The classic treatment by Tolman [5], which is limited to the linear version of GR and to very particular models, is not fully satisfying, because it neglects the important role of the equation of the characteristic surfaces of Maxwell theory.

A new approach to the topic is given in the present paper.

1. – In any spacetime the e.m. rays are null *geodesics*, as it is well known. Consequently, *no* undulatory, purely gravitational, and autonomous field is created by the propagation of any e.m. wave in any spacetime manifold. (This propagation is a "natural" one, like that of any e.m. wave in a Minkowski spacetime). And clearly, *no gravitational interaction exists among the various portions of an e.m. wave.*

2. – Einstein field equations in a space devoid of bodies and of charges are ($c = G = 1$):

$$R_{jk} = -8\pi E_{jk} \quad ; \quad (E^j_j = 0) \quad ; \quad (j,k = 0,1,2,3) \quad , \tag{1}$$

if E_{jk} is the e.m. energy tensor, given by

$$4\pi E^{jk} = -F^j_{\ m}\, F^{km} + \frac{1}{4}\, g^{jk}\, F_{mn}\, F^{mn} \quad , \tag{1$'$}$$

where F_{mn}, $(m, n = 0,1,2,3)$, is the e.m. field of the considered e.m. wave. We have from Maxwell theory:

$$F_{mn} = \Phi_{m:n} - \Phi_{n:m} = \Phi_{m,n} - \Phi_{n,m} \quad , \tag{1$''$}$$

if Φ_m is the e.m. potential; the colon and the comma denote respectively a covariant and an ordinary derivative. Besides eqs. ($1''$), we have Maxwell equations

$$\left(F^{jk}\sqrt{-g}\right)_{,k} = 0 \quad . \tag{2}$$

Eqs. (1) tell us that

$$E^{jk}_{\ :k} = 0 \quad , \tag{3}$$

from which, taking into account ($1'$) and ($1''$), we get

$$F^{jk}_{\ :k} = 0 \quad , \tag{4}$$

which coincide with eqs. (2).

This means that if we express E_{jk} *as a function of* Φ_m, *Einstein eqs.* (1) *have Maxwell eqs.* (2) *as a a mathematical consequence.* Gravitation has “absorbed” the e.m. properties of the e.m. wave. This result has as a necessary condition that the differential equation of the characteristic surfaces $z(x^0, x^1, x^2, x^3) = 0$, *i.e.*

$$g^{jk}\,\frac{\partial z(x)}{\partial x^j}\,\frac{\partial z(x)}{\partial x^k} = 0 \quad , \tag{5}$$

is the *same* for both Maxwell and Einstein fields (Whittaker and Levi-Civita).

At this point, it is very natural to specify the reference frame in such a way that four components of the metric tensor g_{jk} are functionally *identical* to the four components of the e.m. potential Φ_m, which describes our e.m. wave.

2bis. – Let us consider, *e.g.*, a continuous flow of e.m. waves described by the following four-potential Φ_m:

(6)
$$\Phi_0 = \Phi_1 = \Phi_3 = 0 \quad ; \quad \Phi_2(t+x) = A \sin[\omega(t+x)] \quad ; \quad (t \equiv x^0; x \equiv x^1) \quad ;$$

in a *Minkowskian* spacetime, eqs. (6) represent *an ordinary plane* wave ([1]). We have:

$$\begin{cases} F_{21} = \Phi_{2,1} = A\omega \cos[\omega(t+x)] \quad , \\ F_{20} = \Phi_{2,0} = A\omega \cos[\omega(t+x)] \quad . \end{cases} \tag{7}$$

The e.m. energy tensor E_{jk} reduces to ([2]):

$$E_{jk} = -g^{rs} F_{jr} F_{ks} \quad , \tag{8}$$

and eqs. (1) give:

$$R_{jk} = 8\pi g^{rs} F_{jr} F_{ks} \quad ; \tag{9}$$

a solution of the problem can be obtained by putting, for instance:

$$g_{00} = g_{01} = g_{03} = 0 \quad ; \quad g_{02} = A \sin[\omega(t+x)] \quad ; \tag{10}$$

the other components of the metric tensor, $g_{\alpha\beta}(t,x)$, $(\alpha, \beta = 1,2,3)$, are the solutions of eqs. (9) in which we have substituted the values (10). The e.m. field of the e.m. wave is thus fully described by its own gravitational field.

3. – In the linear version (LV) of GR we have approximately:

$$g_{jk} = \eta_{jk} + h_{jk} \quad , \tag{11}$$

where η_{jk} is the Minkowski tensor $(1,-1,-1,-1)$, and the h_{jk}'s are small deviations from it. LV is a Lorentz-invariant theory. Its equations are also invariant under the following gauge transformation of the symmetric tensor h_{jk}:

$$h_{jk} \rightarrow h_{jk} + \xi_{j,k} + \xi_{k,j} \quad , \tag{12}$$

where $\xi_j(x)$ is an infinitesimal vector function of $(x^0, x^1, x^2 x^3)$. Equivalently, formula (12) can be viewed as the result of a transformation of metric (11) under an infinitesimal change of the Lorentzian coordinates x^j:

$$x^j \rightarrow x^j + \xi^j(x) \quad . \tag{13}$$

In lieu of the exact eqs. (1), we have, as it is known:

$$\frac{1}{2} \Box h_{jk} = -8\pi E_{jk} \quad ; \quad h^{jk}_{\ \ ,k} = 0 \quad ; \tag{14}$$

in the following equations of sect. **2**, we must now substitute g_{jk} with η_{jk}, and the covariant derivatives with the ordinary ones. Eq.(5) becomes:

$$\eta^{jk} \frac{\partial z(x)}{\partial x^j} \frac{\partial z(x)}{\partial x^k} = 0 \quad ; \tag{15}$$

the light-rays are *rectilinear* null-geodesics; the e.m. waves and the field h_{jk} are propagated in *Minkowski* spacetime.

Four components of tensor h_{jk} can be identified with the four components of the e.m. potential Φ_m.

3bis. – Let us consider in the LV a continuous flow of *plane* e.m. waves, described by the four-potential Φ_m of eqs. (6). We have ([1]):

$$E_{00} = E_{01} (= E_{10}) = E_{11} = A^2 \omega^2 \cos[\omega(t+x)] \quad ; \tag{16}$$

the other components of E_{jk} are equal to zero.

Accordingly,

$$\Box\, h_{00}(t,x) = \Box\, h_{01}(t,x) = \Box\, h_{11}(t,x) = -16\pi A^2 \omega^2 \cos^2[\omega(t+x)] \quad ; \tag{17}$$

$$\Box\, h_{jk} = 0 \quad , \quad \text{for } (j,k) \neq [(00),(01),(11)] \quad . \tag{17$'$}$$

The e.m. field does not appear in eqs. (17′), and therefore these h_{jk}'s can be put equal to zero: they do not "feel" the action of the e.m. waves. However, if we prefer to follow the procedure of sect. **2bis**, we can put, *e.g.*, (with gauge transformed h_{jk}'s):

$$h_{02}(t+x) = \Phi_2(t+x) = A \sin[\omega(t+x)] \quad ; \tag{18}$$

$$h_{03} = \Phi_3 = 0 \quad ; \quad h_{12} = \Phi_1 = 0 \quad ; \quad h_{13} = \Phi_0 = 0 \quad ; \tag{18$'$}$$

$$h_{22} = h_{23} = h_{33} = 0 \quad . \tag{18$''$}$$

Then, eqs. (17) can be re-written as follows:

$$\Box\, h_{00} = \Box\, h_{01} = \Box\, h_{11} = \left(\frac{\partial h_{02}}{\partial \zeta} \right)^2 \quad , \tag{19}$$

if $\xi := t + x$; the e.m. field is thus fully described by its own gravitational field.

3ter. – One finds easily the solution of eqs. (17). Indeed, the solution of d'Alembert inhomogeneous equation

$$\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} = -16\pi A^2 \omega^2 \cos^2[\omega(t+x)] \tag{20}$$

is given by $F(t,x) = F_0(t,x) + F_1(t,x)$, where

$$F_0(t,x) = \varphi(t+x) + \chi(t-x) \tag{21}$$

is the general solution of the homogeneous equation, with φ and χ any functions of their arguments, and $F_1(t,x)$ is given by

$$F_1(t,x) = -16\pi A^2\omega^2 \cdot (t-x)\left\{\frac{1}{2}(t+x) + \frac{1}{4\omega}\sin[2\pi(t+x)]\right\} \quad . \tag{22}$$

Of course, only $F_1(t,x)$ concerns the gravitational field generated by our e.m. waves.

4. – *A final remark.* Hilbert [6] considered the coupled equations of Einstein and Mie (with the gravitational and e.m. potentials, g_{jk} and q_j, as unique dynamical variables) in lieu of the coupled equations of Einstein and Maxwell. According to Mie's theory [7] (which has revealed itself as unpractical), the electric charges would emerge as solutions of the field equations for the potential q_j, while in Einstein's theory the point-masses emerge as singularities of the metric tensor g_{jk}. Hilbert remarked that Mie's equations, referred to the general-relativistic metric, are an analytical *consequence* of Einstein's equations with Mie's e.m. energy tensor. Therefore, for the 14 components of the potentials g_{jk} and q_j, we have only the 10 functionally independent Einstein's equations.

In our previous treatment of the gravitational field created by an e.m. wave we had a formalism which is quiete analogous, from the *mathematical* standpoint, to the above Hilbertian formalism, with Maxwell's e.m. potential Φ; instead of q_j. We have applied and developed Hilbert's remark.

APPENDIX

Our approach is a *direct* one (like that of Tolman for the linear version of GR ([5]): we start from a *given* e.m.–wave potential and give a prescription to compute the generated gravitational field according to Einstein field equations. On the contrary, we find in the literature a clear prevalence of an *indirect* method: one postulates intuitively the more or less detailed structure of a $\mathrm{d}s^2$ – or of a gravitational potential g_{jk}; then, from the corresponding expression for $R_{jk} - (1/2)g_{jk}R$, one derives the tensor T_{jk} and one verifies whether it can be interpreted as the energy tensor E_{jk} of an e.m. wave. This method has a weak point: it depends on an interpretation, and thus its physical meaning can be dubious. This adjective is appropriate also for those mixed procedures that make a partial use of both the above mentioned methods.

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