Relativistic quantum theory of high harmonic generation on atoms/ions by strong laser fields

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High-order harmonic generation (HHG) by hydrogenlike atoms/ions in the uniform periodic electric field, formed by the two linearly polarized counterpropagating laser beams of relativistic intensities, is studied. The relativistic quantum theory of HHG in such fields, at which the impeding factor of relativistic magnetic drift of a strong wave can be eliminated, is presented arising from the Dirac equation. Specifically, a scheme of HHG in underdense plasma with the copropagating ultraintense laser and fast ion beams is proposed.

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I. INTRODUCTION

The unprecedented progress of laser technology in the recent decade made available the realization of femtosecond laser sources with relativistic: 10^{18} W/cm^2 (in optical domain) and ultrarelativistic intensities up to 10^{22} W/cm^2 by the chirped pulse amplification technique [1]. It is expected that the next generation of ultrarelativistic lasers will significantly exceed these intensities in the near future [2]. The interaction of such powerful electromagnetic radiation with matter opens real possibilities for revelation of ultrafast laser-matter interaction dynamics in supershort time scales with new research fields, such as Relativistic Optics [3–5] and Attoscience [6–8], as well as many nonlinear electrodynamic effects at the ultrarelativistic interaction with the QED vacuum and accelerator beams [9].

At the laser-matter interaction the high-order harmonic generation (HHG) by atoms/ions [10–13] is a subject of extreme interest, since HHG is one of the most promising mechanisms for creation of coherent XUV sources [14, 15] and intense attosecond pulses [8], in particular, for the control of electrodynamic processes on attosecond time scales. To reach a far X-ray region one needs the atoms or ions with a large nuclear charge and laser fields of ultrahigh intensities at which the nondipole interaction and relativistic effects become essential [16–19]. At such intensities the state of ionized electron becomes relativistic already at the distances $l \ll \lambda$ (λ is the wavelength of a laser radiation) irrespective of its initial state, hence the investigation of laser-atom/ion induced processes such as Stimulated Bremsstrahlung (SB), Above Threshold Ionization (ATI) etc., require relativistic consideration [20–22]. Specifically, the relativistic drift of a photoelectron due to the magnetic field of a strong electromagnetic wave becomes the major inhibiting factor in the relativistic regime of HHG. Due to this drift the significant HHG suppression takes place and diverse schemes have been proposed to overcome this negative effect [23–25]. For compensation of the magnetic drift in relativistic domain, in Refs. [23], [24] two counterpropagating laser pulses of linear and circular polarizations were considered, respectively, at which the configuration of a standing wave is formed. The effect of resulting magnetic field of the standing wave is vanished near the stationary maxima at the waves' linear polarizations [23]. While at the circular polarizations of counterpropagating laser pulses one can achieve the fully vanishing of resulting longitudinal magnetic force, responsible for magnetic drift [24]. However, the latter takes place at the adiabatic turn on of the waves [9], which is not valid for short laser pulses of relativistic intensities.

So, a standing wave configuration formed by the two counterpropagating laser beams of linear polarizations is of interest due to the simplicity to realize such field structure providing incomparable large HHG rates in the relativistic regime. At the lengths much smaller than a wavelength of a pump wave the effective field of the standing wave may be approximated by the uniform periodic electric field. Note that the consideration of the HHG in the field of a standing wave in the paper [23] is based on the semiclassical model, and only the first order relativistic effects have been taken into account. The entire relativistic quantum-mechanical consideration of the HHG in a standing wave or in a nonstationary strong electric field on the base of the Dirac equation is still lacking.

To overcome the suppression of the HHG in the relativistic regime caused by the magnetic field of a strong laser pulse, in the present work the relativistic HHG by hydrogenlike ions in the uniform periodic electric field, formed by the two linearly polarized counterpropagating laser beams of relativistic intensities, at the distances much less than a laser wavelength, is studied. We propose also a scheme of relativistic HHG in plasma, where a traveling laser pulse in the own frame of reference (moving with the pulse group velocity in plasma) is transformed into the uniform periodic electric field, i.e. the wave magnetic field in this case vanishes completely, in contrast to a standing wave configuration formed by the counterpropagating laser beams. So, it is also feasible the effective HHG in plasma with the copropagating ul-

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traintense laser and fast ion beams of the same velocities. Note that in Refs. [26, 27] a scheme with counterpropagating relativistic ion and laser beams was considered, where due to the large Doppler upshift the high harmonic frequency may appear in the 10 keV-1 MeV domain.

The organization of the paper is as follows. In Sec. II we present analytic and fully relativistic quantum theory of HHG arising from the Dirac equation. In Sec. III with the help of Fast Fourier Transform algorithm we present HHG spectrum for diverse ion and field parameters. Finally, conclusions are given in Sec. IV.

II. BASIC MODEL AND THEORY

Let two linearly polarized plane electromagnetic waves with carrier frequency ω and amplitude of the electric field $\mathbf{E}_a$

$$\mathbf{E}_1 = \mathbf{E}_a \cos(\omega t - \mathbf{kr}), \quad \mathbf{E}_2 = \mathbf{E}_a \cos(\omega t + \mathbf{kr}), \tag{1}$$

propagating in the opposite directions in vacuum, interplay with the hydrogenlike ions having the charge number of the nucleus Z_a. We will assume that $\lambda >> a$, where a is the characteristic size of the atomic system and λ is the wavelength of a pump wave (for the HHG this condition is always satisfied).

At the photoionization of an atom/ion in the strong traveling wave field taking into account the relativistic drift due to the magnetic field, one can expect that the probability of ionized electron recombination with ionic core could be non-negligible only if the electron is initially born with a nonzero velocity oppositely directed to incident laser beam. The probability of tunneling ionization with nonzero initial velocity is given by quantum mechanical tunneling theory [28] as:

$$W_{ion} \propto e^{-2\left(Z_a^2 + \mathrm{v}^2\right)^{3/2}/(3E)}, \tag{2}$$

where v is the initial velocity, and E is the electric field strength of the wave (here and below, unless stated otherwise, we employ atomic units). However, according to (2) the ionization probability falls off exponentially if this velocity v becomes larger than the characteristic atomic velocities, irrespective of it's direction. Since we study the case of superstrong laser fields with $\xi \equiv E/c\omega \sim 1$ (ξ is the relativistic invariant parameter of the wave intensity) when the energy of the interaction of an electron with the field over a wavelength becomes comparable to the electron rest energy, the required velocities becomes comparable with the light speed c ($c = 137$ a.u.). Hence, the probability (2) in the such fields is practically zero. Therefore, in considering case of a standing wave formed by the laser beams (1), a significant input in the HHG process will be conditioned by the ions situated near the stationary maxima of the standing wave. For this points the magnetic fields of the counterpropagating waves cancel each other. Since the HHG is essentially produced at the lengths $l << \lambda$, we will assume the effective field to be:

$$\mathbf{E}(t) = \widehat{\mathbf{e}} E_0 \cos \omega t, \tag{3}$$

where $E_0 = 2E_a$, and $\widehat{\mathbf{e}}$ is the unit polarization vector.

As is clear from the above consideration for HHG, one needs to exclude the relativistic drift of a photoelectron due to the magnetic field of a traveling wave, i.e. the magnetic component of the wave. This can also be achieved in the plasma- like medium with a refractive index $n_p(\omega) < 1$. Indeed, let a plane, transverse, and linearly polarized electromagnetic wave with a frequency ω propagates in plasma ($\omega > \omega_p$; $\omega_p = \sqrt{4\pi N_e}$ is the plasma frequency). For the electric and magnetic field strengths we have

$$\mathbf{E} = \mathbf{E}_a \cos\left(\omega t - \mathbf{kr}\right), \; \mathbf{H} = \frac{c}{\omega}\left[\mathbf{k} \times \mathbf{E}\right], \tag{4}$$

where $|\mathbf{k}| = n_p(\omega)\omega/c$. Suppose that an ion beam copropagates together a laser beam in such plasma with a mean velocity V equal to a laser beam group velocity: $\mathrm{V} = cn_p(\omega)$. To find out the HHG rate it is convenient to solve the problem in the center-of-mass (C) frame of ions. In this frame the wave vector of the photons is $\mathbf{k}' = 0$. The traveling electromagnetic wave is transformed in the C frame into the field [9]:

$$\mathbf{H}' \equiv 0, \; \mathbf{E}'\left(t'\right) = \mathbf{E}'_a \cos \omega' t',$$

$$\omega' = \omega \sqrt{1 - n_p^2(\omega)} = \omega_p. \tag{5}$$

Hence, in the frame of ions center-of-mass the problem of HHG in a plasma is reduced to one in vacuum in the field of the same configuration (3) with $\mathbf{E}'_a = \mathbf{E}_a \omega_p/\omega$. When $\omega >> \omega_p$, one needs relativistic ion beams. Concerning the ion beams of necessary relativism $\gamma = \left(1 - \mathrm{V}^2/c^2\right)^{-1/2} = \omega/\omega_p$, note that the relativistic ion beams in arbitrary charge states, with the Lorentz factor up to about 30 is supposed to be realized at the new accelerator complex at Gesellschaft für Schwerionenforschung (GSI) (Darmstadt,Germany). It is worthy to note that if a laser frequency ω is close enough to plasma frequency ω_p, one can achieve the implementation of proposed scheme for HHG with nonrelativistic ion beams. In this case one can use quasimonoenergetic and low emittance ions bunches of solid densities generated from ultrathin foils -nanotargets by supershort laser pulses of relativistic intensities [29].

To find out the relativistic probabilities of HHG in the field (3) we arise from the Dirac equation:

$$i\frac{\partial \left|\Psi\right\rangle}{\partial t} = \left(c\widehat{\alpha}\widehat{\mathbf{p}} + \widehat{\beta}c^2 + \frac{Z_a}{r} - \mathbf{rE}\left(t\right)\right)\left|\Psi\right\rangle, \tag{6}$$

where $\widehat{\alpha}$ and $\widehat{\beta}$ are the Dirac matrices in the standard representation, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and

$\hat{\mathbf{p}}$ is the operator of the kinetic momentum ($\hat{\mathbf{p}} = -i\nabla$). Without loss of generality one can take the polarization vector $\hat{\mathbf{e}}$ aligned with the z axis of spherical coordinates.

We denote the atomic bound states by $|\eta\rangle$, where η indicates the set of quantum numbers that characterizes the state $\eta = \{n, j, l, M\}$. Here n is the principal quantum number, j is the whole moment, l is the orbital moment and M is the magnetic quantum number.

Following the ansatz developed in the Ref. [12], the time dependent wave function can be expanded as

$$|\Psi\rangle = \left(C_0(t)|\eta_0\rangle + \sum_{\mu} \int d\mathbf{p} C_\mu(\mathbf{p},t)|\mathbf{p},\mu\rangle \right) e^{-i\varepsilon t}. \quad (7)$$

Here

$$|\mathbf{p},\mu\rangle = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\mathcal{E}(\mathbf{p}) + c^2}{2\mathcal{E}(\mathbf{p})}} \begin{pmatrix} \varphi_\mu \\ \frac{c(\sigma\mathbf{p})}{\mathcal{E}(\mathbf{p})+c^2}\varphi_\mu \end{pmatrix} e^{i\mathbf{pr}}, \quad (8)$$

are the Dirac free solutions [30] with energy $\mathcal{E}(\mathbf{p}) = \sqrt{c^2\mathbf{p}^2 + c^4}$ and polarization states $\mu = 1, -1$:

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9)$$

As an initial bound state wave function $|\eta_0\rangle$ we assume the ground-state bispinor wave function [30] for the hydrogenlike ion with the quantum numbers $n = 1$, $j = 1/2$, $l = 0$, and $M = 1/2$:

$$|\eta_0\rangle = \frac{Z_a^{3/2}}{\sqrt{\pi\Gamma(3-2\epsilon)}} \begin{pmatrix} \sqrt{2-\epsilon} \\ 0 \\ i\cos\theta\sqrt{\epsilon} \\ i\sin\theta e^{i\varphi}\sqrt{\epsilon} \end{pmatrix} (2rZ_a)^{-\epsilon} e^{-Z_a r}. \quad (10)$$

Here $\Gamma(x)$ is the Euler gamma function, θ and φ are the polar and azimutal angles, $\epsilon = 1 - \sqrt{1 - Z_a^2/c^2}$, and $\varepsilon = c^2(1-\epsilon)$ is the energy of the ground state. We assume $Z_a < 137$, and the parameter ϵ can take values lying in the interval $0 < \epsilon < 1$. In the expansion (7) we have excluded the negative energy states, since the input of the particle-antiparticle intermediate states will lead only to small corrections to the processes considered. Neglecting the depletion of the ground state $C_0(t) \simeq 1$ and the free-free transitions due to Coulomb field, the Dirac equation for $C_\mu(\mathbf{p},t)$ reads as:

$$\frac{\partial C_\mu(\mathbf{p},t)}{\partial t} + E(t)\frac{\partial C_\mu(\mathbf{p},t)}{\partial p_z} + i(\mathcal{E}(\mathbf{p}) - \varepsilon) C_\mu(\mathbf{p},t)$$

$$= i\mathcal{D}_\mu(\mathbf{p}) E(t) + \frac{c^2 E(t)(\mu p_x - i p_y)}{2\mathcal{E}(\mathbf{p})(\mathcal{E}(\mathbf{p}) + c^2)} C_{-\mu}(\mathbf{p},t), \quad (11)$$

where

$$\mathcal{D}_\mu(\mathbf{p}) = \langle \mathbf{p},\mu | z | \eta_0 \rangle \quad (12)$$

is the atomic dipole matrix element for the bound-free transition. The latter can be calculated with the help of integral [31]:

$$\int_0^\pi e^{ic_1\cos\theta\cos\Theta} J_m(c_1\sin\theta\sin\Theta) P_l^m(\cos\theta)\sin\theta d\theta$$

$$= \left(\frac{2\pi}{c_1}\right)^{1/2} i^{l-m} P_l^m(\cos\Theta) J_{l+1/2}(c_1),$$

where $P_l^m(\cos\Theta)$ is an associated Legendre function of degree l and order m, J_m is a Bessel function of order m. Thus, for $\mathcal{D}_{\pm\mathbf{1}}(\mathbf{p})$ we have:

$$\mathcal{D}_{\mathbf{1}}(\mathbf{p}) = i\frac{2^{3-\epsilon}}{\pi}\sqrt{\frac{\mathcal{E}+c^2}{2\mathcal{E}}} \frac{Z_a^{5/2-\epsilon}\Gamma(3-\epsilon)}{\sqrt{\Gamma(3-2\epsilon)}} \frac{p_z}{(p^2+Z_a^2)^{3-\epsilon}}$$

$$\times \left[\frac{1}{4c\sqrt{2\epsilon}} \frac{Z_a^{3-\epsilon}}{2p^2} \left(\Upsilon^{3-\epsilon} + \Upsilon^{\dagger 3-\epsilon} - \frac{i(p^2+Z_a^2)}{(2-\epsilon)Z_a p}\left(\Upsilon^{2-\epsilon} - \Upsilon^{\dagger 2-\epsilon}\right) \right) + \frac{1}{\mathcal{E}+c^2}\sqrt{\frac{1}{2(2-\epsilon)}} \frac{iZ_a^{3-\epsilon}}{8p} \right.$$

$$\left. \times \left(\left(\Upsilon^{3-\epsilon} - \Upsilon^{\dagger 3-\epsilon}\right) - \frac{2i(p^2+Z_a^2)}{(2-\epsilon)Z_a p}\left(\Upsilon^{2-\epsilon} + \Upsilon^{\dagger 2-\epsilon}\right) - \frac{2\Gamma(1-\epsilon)}{\Gamma(3-\epsilon)} \frac{(p^2+Z_a^2)^2}{Z_a^2 p^2}\left(\Upsilon^{1-\epsilon} - \Upsilon^{\dagger 1-\epsilon}\right) \right) \right], \quad (13)$$

$$\mathcal{D}_{-\mathbf{1}}(\mathbf{p}) = \frac{i\sqrt{\epsilon} Z_a^{5/2-2\epsilon} c(p_x + ip_y)}{2^{1+\epsilon}\pi p^3 \sqrt{\Gamma(3-2\epsilon)\mathcal{E}(\mathcal{E}+c^2)}} \left[\frac{\Gamma(1-\epsilon)}{(p^2+Z_a^2)^{1-\epsilon}} i\left(\Upsilon^{1-\epsilon} - \Upsilon^{\dagger 1-\epsilon}\right) - \frac{Z_a p\Gamma(2-\epsilon)}{(p^2+Z_a^2)^{2-\epsilon}}\left(\Upsilon^{2-\epsilon} + \Upsilon^{\dagger 2-\epsilon}\right) \right], \quad (14)$$

where

$$\Upsilon = 1 - i\frac{p}{Z_a}, \;\; \Upsilon^\dagger = 1 + i\frac{p}{Z_a}. \quad (15)$$

The last term in Eq. (11) describes spin flip due to free-free transitions and proportional to transverse momentum $p_{x,y}$. As will be seen below, the main contribution to the HHG process is conditioned by the electrons with momentum components $p_{x,y}=0$. Hence, we will neglect the last term in Eq. (11), which is valid for the fields $E << E_S$, where $E_S = m^2c^3/e\hbar$ is the Schwinger field strength. Then, from Eq. (11) for the probability amplitudes $C_\mu(\mathbf{p},t)$ we obtain:

$$C_\mu(\mathbf{p},t) = i\int_0^t dt'\mathcal{D}_\mu\left(\mathbf{p}+\frac{1}{c}\left(\mathbf{A}(t)-\mathbf{A}(t')\right)\right)E(t')$$
$$\times\exp\left\{-i\int_{t'}^t\left[\mathcal{E}\left(\mathbf{p}+\frac{1}{c}\left(\mathbf{A}(t)-\mathbf{A}(t'')\right)\right)-\varepsilon\right]dt''\right\}, \tag{16}$$

where $\mathbf{A}(t) = -\widehat{\mathbf{e}}c^2\xi\sin\omega t$ is the vector potential of resulting electric field.

For the harmonic radiation perpendicular to the polarization direction $\widehat{\mathbf{e}}$ one needs mean value of the z component of the electron current density $J(t) = c\langle\Psi|\,\widehat{\alpha}_z\,|\Psi\rangle$. Using Eqs. (7), (16) and neglecting the contribution by free-free transitions, we obtain:

$$J(t) = i\sum_\mu\int d\mathbf{p}\int_0^t dt'\mathcal{J}_\mu\left(\mathbf{p}-\frac{1}{c}\mathbf{A}(t)\right)$$
$$\times\mathcal{D}_\mu\left(\mathbf{p}-\frac{1}{c}\mathbf{A}(t')\right)E(t')$$
$$\times\exp\left\{-iS(\mathbf{p},t,t')+i\varepsilon(t-t')\right\}+\text{c.c.}, \tag{17}$$

where

$$S(\mathbf{p},t,t') = \int_{t'}^t\mathcal{E}\left(\mathbf{p}-\frac{1}{c}\mathbf{A}(t'')\right)dt''$$
$$= \int_{t'}^t\sqrt{c^2\left(\mathbf{p}+\widehat{\epsilon}c\xi\sin\omega t''\right)^2+c^4}dt'' \tag{18}$$

is the relativistic classical action of an electron in the field and $\mathcal{J}_\mu(\mathbf{p}) = c\langle\eta_0|\,\widehat{\alpha}_z\,|\mathbf{p},\mu\rangle$.

As in the nonrelativistic case, the HHG rate is mainly determined by the exponential in the integrand of Eq. (17) with exact relativistic classical action. The integral over the intermediate momentum $\mathbf{p}$ and time t' can be calculated using saddle-point method. The saddle momentum is determined by the equation:

$$\frac{\partial S(\mathbf{p},t,t')}{\partial\mathbf{p}} = 0, \tag{19}$$

which for momentum components gives $p_{x,y}=0$, and the z component of momentum (p_s) is given by the solution of the equation:

$$\int_{t'}^t\frac{p_s+c\xi\sin\omega t''}{\sqrt{\left(p_s+c\xi\sin\omega t''\right)^2+c^2}}dt'' = 0. \tag{20}$$

In contrast to nonrelativistic [12] and relativistic [17] cases for a traveling wave, this equation can not be solved analytically and requires numerical solution. Integrating the latter over $\mathbf{p}$, for the current density we obtain:

$$J(t) = (2\pi)^{3/2}\sqrt{i}\int_0^t dt'\frac{e^{-i\left(S(\mathbf{p}_s,t,t')-\varepsilon\left(t-t'\right)\right)}}{\sqrt{\left|\det S''_{\mathbf{pp}}\right|}}E(t')$$
$$\times\mathcal{D}_{\mathbf{1}}\left(\mathbf{p}_s-\frac{1}{c}\mathbf{A}(t')\right)\mathcal{J}_1\left(\mathbf{p}_s-\frac{1}{c}\mathbf{A}(t)\right)+\text{c.c.}, \tag{21}$$

where

$$\det S''_{\mathbf{pp}} = \left(\int_{t'}^t\frac{d\tau}{\gamma(t',\tau;\xi)}\right)^2\int_{t'}^t\frac{d\tau}{\gamma^3(t',\tau;\xi)},$$
$$\gamma(t',\tau;\xi) = \left(1+\left(p_s+c\xi\sin\omega\tau\right)^2/c^2\right)^{1/2}. \tag{22}$$

Note that contribution from spin flip due to bound-free transitions in the saddle-point vanishes: $\mathcal{D}_{-\mathbf{1}}|_{p_{x,y}=0}=0$. At $\xi << 1$, for $\det S''_{\mathbf{pp}}$ one can recover nonrelativistic result: $\det S''_{\mathbf{pp}} = (t-t')^3$.

The complex saddle times t_s are the solutions of the following equation:

$$\frac{\partial S(\mathbf{p}_s,t,t')}{\partial t'}+\varepsilon = 0, \tag{23}$$

which may be expressed by the transcendental equation

$$\sqrt{c^2\left(p_s+c\xi\sin\omega t_s\right)^2+c^4}-\varepsilon = 0. \tag{24}$$

Then expressing the saddle time as $t_s = t_b+i\delta$, with $\omega\delta << 1$, one can obtain the saddle momentum

$$p_s = -c\xi\sin\omega t_b, \tag{25}$$

and the imaginary part of the saddle time:

$$\delta = \frac{Z_a}{|E(t_b)|}, \tag{26}$$

where $E(t_b) = c\xi\omega\cos\omega t_b$. Taking into account Eq. (25), from Eq. (20) for the real part of the saddle time we obtain

$$\int_{t_b}^t\frac{\sin\omega t''-\sin\omega t_b}{\sqrt{1+\xi^2\left(\sin\omega t''-\sin\omega t_b\right)^2}}dt'' = 0. \tag{27}$$

As usual, t_b is interpreted as the birth time of the photoelectron which returns at the moment t to the core and generates harmonic radiation. The transition dipole moment has singularity at the saddle times and the integration has been made with the help of the formula:

$$\int g(x)\frac{e^{-\lambda f(x)}}{\left(x-x_0\right)^\nu}dx \simeq i^\nu\sqrt{\pi}g(x_0)\left[2f''(x_0)\lambda\right]^{\frac{\nu-1}{2}}$$
$$\times\frac{\Gamma(\nu/2)\,e^{-\lambda f(x_0)}}{\Gamma(\nu)}. \tag{28}$$

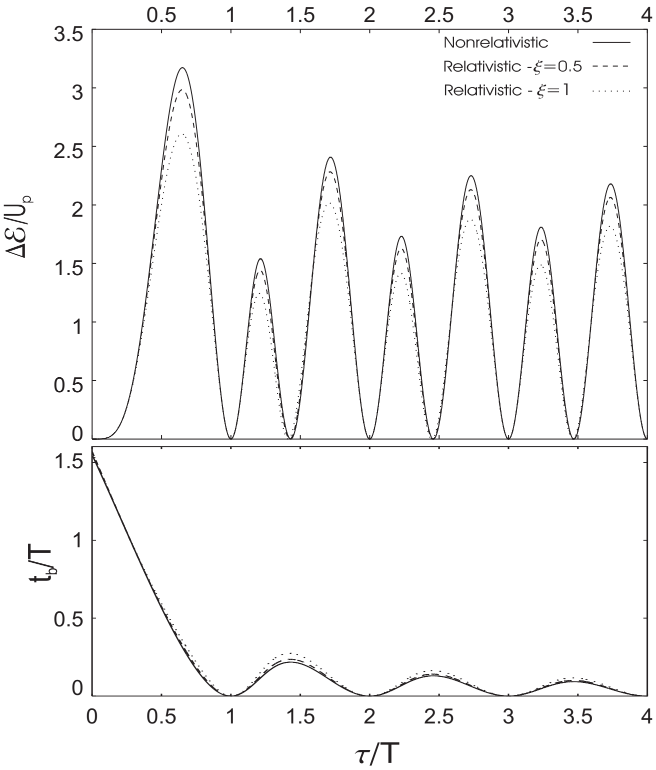

FIG. 1: The photoelectron energy gain in units of ponderomotive energy $U_p = c^2\xi^2/4$ and ionization time versus the electron time evolution in the continuum (return time) for the various laser intensities.

Thus, taking into account Eqs. (19-28), we obtain the ultimate formula for current density:

$$J(t) = \sum_{t_b} C_{\text{ion}}(t_b)\, C_{\text{pr}}(t, t_b)\, C_{\text{rec}}(t, t_b) + \text{c.c.}. \qquad (29)$$

The formula (29) is the analogous to the nonrelativistic formula for the dipole moment in the three step model [13]. Here the summation is carried out over the solutions of Eq. (27). The tunneling ionization amplitude $C_{\text{ion}}(t_b)$ is:

$$C_{\text{ion}}(t_b) = \frac{2i}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3-\epsilon}{2}\right)}{\sqrt{\Gamma(3-2\epsilon)}} \frac{|E(t_b)|^{\frac{\epsilon}{2}}}{2^{\frac{3\epsilon}{2}} E(t_b)}$$

$$\times \frac{Z_a^{\frac{3-3\epsilon}{2}}}{(1-\epsilon)^{\frac{1-\epsilon}{2}}} e^{-\frac{Z_a}{|E(t_b)|}\left(c^2\epsilon - \frac{Z_a^2}{6}\right)}. \qquad (30)$$

The propagation amplitude is given by the expression

$$C_{\text{pr}}(t, t_b) = \frac{(2\pi)^{3/2}}{\sqrt{i}} \frac{\exp\{-iS(\mathbf{p}_s, t, t_b) + i\varepsilon(t - t_b)\}}{\sqrt{\left|\det S''_{\mathbf{pp}}\right|}}, \qquad (31)$$

and the recombination amplitude is: $C_{\text{rec}}(t, t_b) = \mathcal{J}_1\left(\mathbf{p}_s - \frac{1}{c}\mathbf{A}(t)\right)$, where

$$\mathcal{J}_1(\mathbf{p}) = \frac{1}{\pi}\sqrt{\frac{\mathcal{E}+c^2}{2\mathcal{E}}} \frac{Z_a^{7/2-2\epsilon} 2^{-\epsilon}}{\sqrt{2\Gamma(3-2\epsilon)}} \frac{c p_z \Gamma(2-\epsilon)}{(p^2 + Z_a^2)^{2-\epsilon}}$$

$$\times \left[i \frac{c\sqrt{(2-\epsilon)}}{(\mathcal{E}+c^2)\, p} \left(\Upsilon^{2-\epsilon} - \Upsilon^{\dagger 2-\epsilon}\right) - \frac{\sqrt{\epsilon}}{p^2} \left(\Upsilon^{2-\epsilon} + \Upsilon^{\dagger 2-\epsilon} - i \frac{\Gamma(1-\epsilon)}{1-\epsilon} \frac{(p^2 + Z_a^2)}{p Z_a} \left(\Upsilon^{1-\epsilon} - \Upsilon^{\dagger 1-\epsilon}\right) \right) \right]. \qquad (32)$$

III. RELATIVISTIC HARMONIC SPECTRA FOR DIVERSE ION AND FIELD PARAMETERS

In this section we present some numerical simulations for HHG when relativistic effects become essential. In considering relativistic case the saddle time and energy gain essentially depend on the intensity of the pump wave. The latter leads to the modification of the HHG spectrum compared with nonrelativistic one. In Fig. 1 we present the solution of Eq. (27) for the born times t_b which are limited to a quarter of the laser period. The Fig. 1 also illustrated the photoelectron energy gain versus the electron's time evolution in the continuum (return time) for the various laser intensities. It is well known that the cutoff frequency of HHG is defined by the maximal classical energy gain. For this reason in Fig. 2 the maximum energy gain of the photoelectron as a function of the relativistic parameter of the wave intensity ξ^2 is displayed. As we see, the relativistic cutoff essentially differs from the nonrelativistic one for $\xi \gtrsim 1$ and the shift of the cutoff position to the lower values of the harmonic order for the same laser intensity becomes evident. The Fig. 1 also reveals the multiplateau character of the harmonic spectrum like to the nonrelativistic one.

To find out harmonic spectrum, the Fast Fourier Transform algorithm has been used.

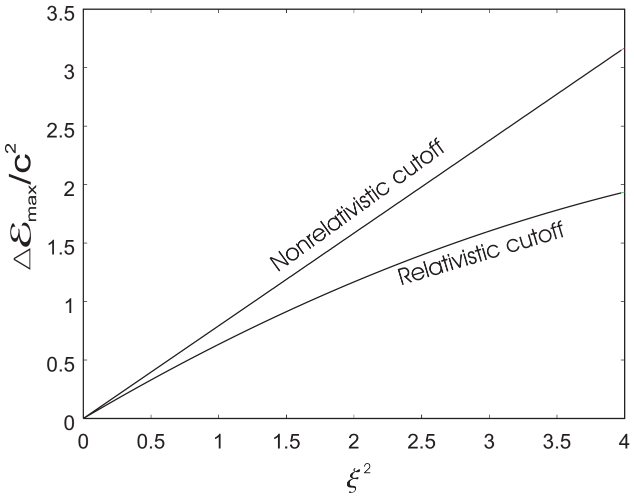

FIG. 2: Maximum energy gain of photoelectron, which defines the cutoff frequency as a function of the relativistic parameter of the wave intensity.

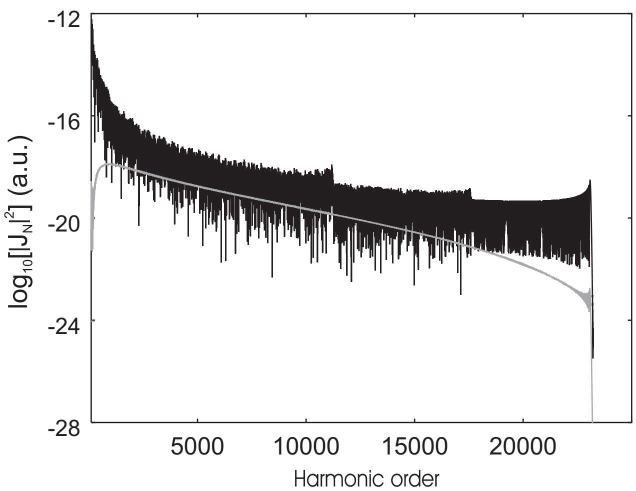

FIG. 3: Harmonic emission rate via $\log_{10}\left(|J_N|^2\right)$, as a function of the harmonic order, for an ion with $Z_a = 3$, $\xi = 0.3$, and frequency $\omega = 0.057$ a.u. (800 nm). The gray curve represents the HHG spectrum for a traveling wave.

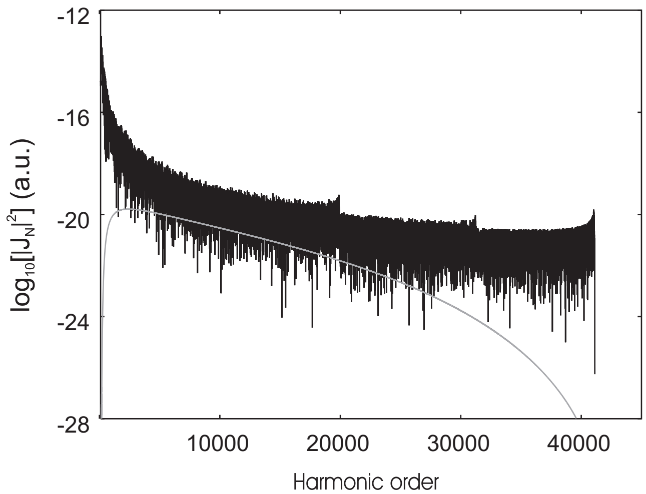

FIG. 4: The same, as in Fig. 3, but for the frequency $\omega = 0.043$ a.u.(1054 nm) and $\xi = 0.35$.

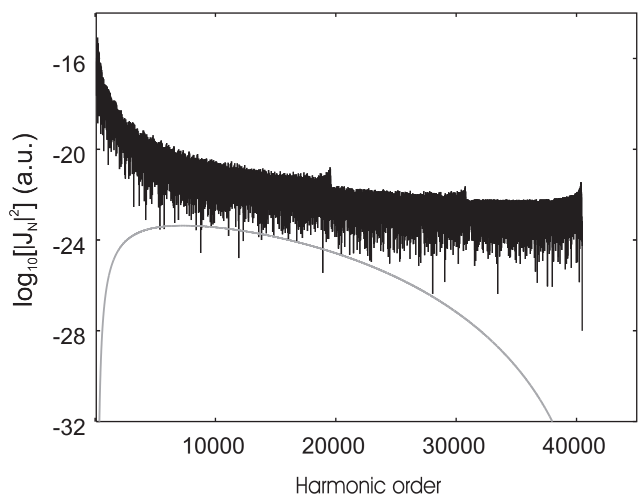

FIG. 5: Harmonic emission rate for an ion with $Z_a = 4$, standing wave intensity 3.4×10^{17} W/cm^2 ($\xi = 0.4$), and frequency $\omega = 0.057$ a.u.(800 nm). The gray curve represents the HHG spectrum for a traveling wave.

Figures 3 and 4 display harmonic emission rate via modulus squared of the Fourier transform of the current density $|J_N|^2$ as a function of the harmonic order N for an ion with $Z_a = 3$. The frequencies and relativistic parameters of intensity are: $\omega = 0.057$ a.u. ($\xi = 0.3$) and $\omega = 0.043$ a.u. ($\xi = 0.35$), respectively. Figure 5 displays harmonic emission rate for an ion with $Z_a = 4$ and field intensity 3.4×10^{17} W/cm^2 ($\xi = 0.4$) with the wave frequency: $\omega = 0.057$ a.u.. For the comparison we have also brought the spectra for a traveling wave which was obtained using the results of Ref. [18]. As we see from these figures, with the increase of the laser intensity the HHG rate with a standing wave field by many orders of magnitude is larger than HHG rate with a traveling wave. Besides, the relativistic quantum mechanical harmonic cutoff is shifted to the lower values compared with nonrelativistic one.

Figures 6, 7, and 8 display harmonic emission rates when the relativistic parameter of the wave intensity is: $\xi = 1$. In this case the relativistic effects are essential. Figure 6 corresponds to $\omega = 0.184$ a.u. and $Z_a = 7$. Figures 7 and 8 correspond to $Z_a = 5$ with $\omega = 0.057$ a.u. and $\omega = 0.043$ a.u., respectively. For the last three figures the cutoff position is $\simeq 2.61 U_p$, which is in good agreement with semiclassical analysis (see Fig. 2). For these parameters the harmonic emission rates with a traveling wave are negligible, meanwhile the relativistic rates with a standing wave field at photons energies approaching to MeV region, are considerable.

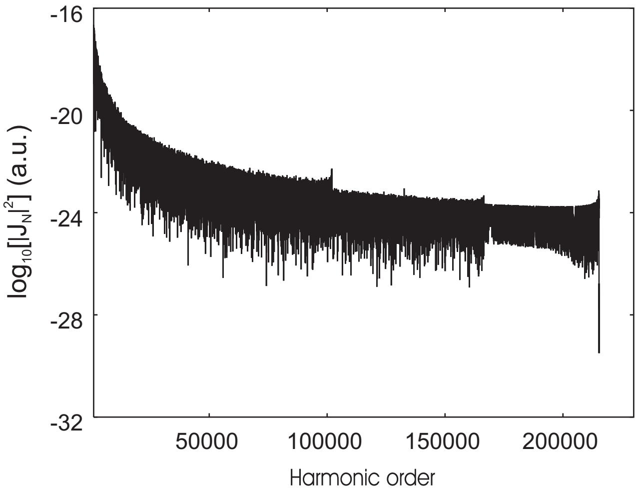

FIG. 6: Harmonic emission rate as a function of the harmonic order for an ion with $Z_a = 7$, $\xi = 1$, and $\omega = 0.184$ (248 nm).

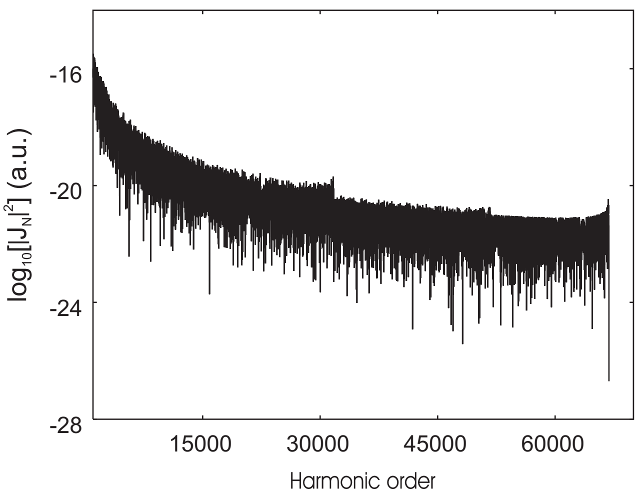

FIG. 7: Harmonic emission rate as a function of the harmonic order for an ion with $Z_a = 5$, $\xi = 1$, and $\omega = 0.057$ a.u..

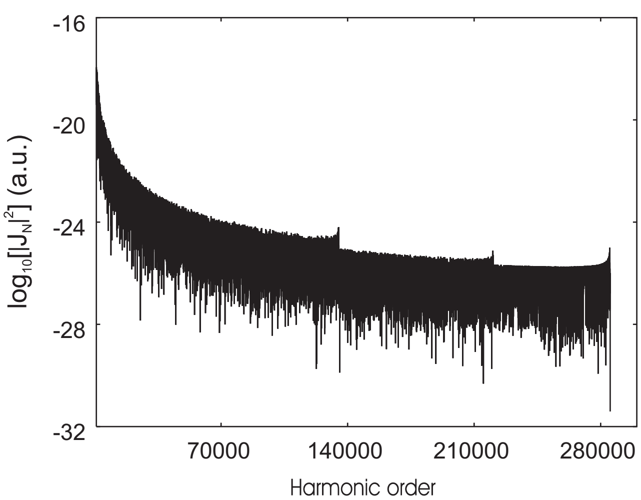

FIG. 8: The same, as in Fig. 7, but for the frequency $\omega = 0.043$ a.u..

IV. CONCLUSION

We have presented a theoretical treatment of the HHG by hydrogenlike ions with a large nuclear charge in the relativistic regime with a standing wave configuration, formed by the two linearly polarized counterpropagating laser beams of relativistic intensities, at the distances much less than a laser wavelength. The investigation of the HHG with the such pump field configuration is conditioned by known fact that in the relativistic regime the significant HHG suppression takes place in the field of a strong traveling wave because of the magnetic drift of a photoelectron along the wave propagation direction. We have proposed a new scheme for relativistic HHG with an ultraintense traveling laser pulse and copropagating ions in a plasma. In this case the relativistic HHG occurs exactly without the magnetic drift of the photoelectron, because the traveling wave in the ions own frame of reference (moving with the laser pulse group velocity) in underdense plasma is transformed into the purely uniform periodic electric field (in contrast to a standing wave – counterpropagating laser beams). Hence, we expect that the efficient HHG with the current ultraintense laser pulses and forthcoming relativistic ion beams (at GSI-Darmstadt accelerator) will be feasible in the plasmas of conventional densities.

On the base of the Dirac equation we have presented the analytic and fully relativistic quantum theory of HHG. With the help of the dynamic solution of the Dirac equation and Fast Fourier Transform algorithm we have calculated the harmonic spectrum. The obtained results have been applied to hydrogenlike ions with a moderate nuclear charge. The harmonic spectrum displays the significant difference compared both to non-relativistic and relativistic spectra with a traveling laser pulse. In particular, the shift of the cutoff position to the lower values of the harmonic order is considerable. In the relativistic regime of interaction the harmonic emission rates for a traveling laser pulse are negligible, meanwhile the relativistic rates with a standing wave field, at photons energies approaching to MeV region, are considerable.

Acknowledgments

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