AN AMPLENESS CRITERION WITH THE EXTENDABILITY OF SINGULAR POSITIVE METRICS.

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Abstract. Coman, Guedj and Zeriahi proved that, for an ample line bundle L on a projective manifold X, any singular positive metric on the line bundle $L|_V$ along a subvariety $V \subset X$ can be extended to a global singular positive metric of L. In this paper, we prove that the extendability of singular positive metrics on a line bundle along a subvariety implies the ampleness of the line bundle.

1. Introduction

Throughout this paper, let us denote by X a projective manifold of dimension n, by L a holomorphic line bundle on X. It is fundamental to consider a singular metric on L whose Chern curvature is a positive $(1,1)$-current, in the theory of several complex variables and algebraic geometry. A singular metric on L with positive curvature current corresponds with a θ-plurisubharmonic function, where θ is a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$ of the line bundle L. (For simplicity of notation, we will abbreviate "θ-plurisubharmonic" to "θ-psh".) Here a function $\varphi : X \longrightarrow [-\infty, \infty)$ is called a *θ-psh function* if φ is upper semi-continuous on X and the Levi form $\theta + dd^c\varphi$ is a positive $(1,1)$-current. We will denote by $\mathrm{Psh}(X,\theta)$ the set of θ-psh functions on X. That is,

$$\mathrm{Psh}(X,\theta) := \left\{ \varphi : X \longrightarrow [-\infty,\infty) \;\middle|\; \varphi \text{ is upper semi-continuous and } \theta + dd^c\varphi \geq 0 \right\}.$$

It is of interest to know when a $\theta|_V$-psh function on a (closed) subvariety $V \subseteq X$ can be extended to a global θ-psh function on X. Coman, Guedj and Zeriahi proved that a $\theta|_V$-psh function on any subvariety V can be extended a global θ-psh function if L is an ample line bundle (see [CGZ, Theorem B]). Note that a $\theta|_V$-psh function can be defined even if V has singularities (see [CGZ, Section 2] for the precise definition).

Theorem 1.1 ([CGZ, Theorem B]). *Let $V \subseteq X$ be a subvariety and θ be a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$. Assume that L is an ample line bundle. Then L satisfies the following property : For any subvariety V and any $\theta|_V$-psh function $\varphi \in \mathrm{Psh}(V,\theta|_V)$, there exists a global θ-psh function $\widetilde{\varphi} \in \mathrm{Psh}(X,\theta)$ such that $\widetilde{\varphi}|_V = \varphi$.*

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The following theorem asserts that the converse implication of Theorem 1.1 holds, which is a main result in this paper. Theorem 1.2 gives an ampleness criterion by the extendability of singular metrics (θ-psh functions).

Theorem 1.2. *Let L be a psuedo-effective line bundle whose first Chern class $c_1(L)$ is not zero. Assume that L satisfies the following property : For any subvariety V and any $\theta|_V$-psh function $\varphi \in \mathrm{Psh}(V, \theta|_V)$, there exists a global θ-psh function $\widetilde{\varphi} \in \mathrm{Psh}(X, \theta)$ such that $\widetilde{\varphi}|_V = \varphi$. Here θ means a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$. Then L is an ample line bundle.*

A line bundle L is called *pseudo-effective* if $\mathrm{Psh}(X, \theta)$ is not empty. We can easily check that the definition of pseudo-effective line bundles does not depend on the choice of $\theta \in c_1(L)$. In the proof of Theorem 1.2, we consider only the case when V is a strongly movable curve (see section 3). Here a curve C is called a strongly movable curve if

$$C = \mu_*\big(A_1 \cap \cdots \cap A_{n-1}\big)$$

for suitable very ample divisors A_i on $\tilde{X}$, where $\mu : \tilde{X} \to X$ is a birational morphism. See [BDPP, Definition 1.3] for more details.

Thus for an ampleness criterion, it is sufficient to check the extendability from a complete intersection of the complete linear system of a very ample line bundle. It is important to emphasize that even if a given $\theta|_V$-psh function φ is smooth at some point on V, the extended function $\widetilde{\varphi}$ may not be smooth at the point. The fact seems to make the proof of Theorem 1.2 difficult.

Remark that the assumption that the first Chern class $c_1(L)$ is not zero is necessary. Indeed, when the first Chern class $c_1(L)$ is zero and θ is equal to zero as a $(1,1)$-form, a θ-psh function is always constant, from the maximal principle of psh functions. Hence any $\theta|_V$-psh function can be extended. However L is not an ample line bundle. In other words, a line bundle which satisfies the extendability condition in Theorem 1.2 must be ample or numerically trivial (that is, $c_1(L)$ is zero).

This paper is organized in the following way: In section 2, we collect materials to prove Theorem 1.2. Section 3 is devoted to give the proof of Theorem 1.2. In section 4, there are two examples which give ideas for the proof of Theorem 1.2.

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2. Preliminaries

In this section, we collect materials for the proof of Theorem 1.2. The propositions in this section may be known facts. However we give comments or references for the readers' convenience.

Lemma 2.1. *Let L be a psuedo-effective line bundle whose first Chern class $c_1(L)$ is not zero. Then the intersection number $(L \cdot B^{n-1})$ is positive for any ample line bundle B on X.*

Proof. Let θ be a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$. We take an arbitrary smooth $(n-1,n-1)$-form γ on X. Further we take a Kähler form ω which represents the first Chern class $c_1(B)$. Since the $(n-1,n-1)$-form ω^{n-1} is strictly positive, there exists a large constant $C > 0$ such that

$$-C\omega^{n-1} \leq \gamma \leq C\omega^{n-1}.$$

Here we implicitly used the compactness of X. Since L is pseudo-effective, we can take a function φ in $\mathrm{Psh}(X,\theta)$. The Levi form $\theta + dd^c\varphi$ is a positive $(1,1)$-current. It gives the following inequality:

$$-C\int_X (\theta + dd^c\varphi) \wedge \omega^{n-1} \leq \int_X (\theta + dd^c\varphi) \wedge \gamma \leq C\int_X (\theta + dd^c\varphi) \wedge \omega^{n-1}.$$

The $(1,1)$-form θ (reps. ω) represents the first Chern class of L (resp. B). Thus the integral of the right and left hands

$$\int_X (\theta + dd^c\varphi) \wedge \omega^{n-1}$$

agrees with the intersection number $(L \cdot B^{n-1})$. If the intersection number is zero,

$$\int_X (\theta + dd^c\varphi) \wedge \gamma = 0$$

for any smooth $(n-1,n-1)$-form γ from the above inequality. It means that the $(1,1)$-current $\theta + dd^c\varphi$ is a zero current. This is a contradiction to the assumption that $c_1(L)$ is not zero. Hence the intersection number $(L \cdot B^{n-1})$ is positive for any ample line bundle B. $\square$

Theorem 2.2 ([Zh09, Theorem 1.3])**.** *Let X be a projective manifold of dimension at least two and p, q be points on X. Fix an ample line bundle B on X. Then there exists a smooth curve C with the following properties*:
(1) *C is a complete intersection of the complete linear system $|mB|$ for some $m > 0$.*
(2) *C contains points p and q.*

Proof. Take an embedding of X into the projective space $\mathbb{P}^N$. Then two points p, q are always in general position in $\mathbb{P}^N$ (see [Zh09, Definition 1.1] for the definition). [Zh09, Theorem1.3] asserts that a general member of $|mB|_{x,y}$ is irreducible and smooth, where $|mB|_{x,y}$ is a linear system in $|mB|$ passing through p and q. Then by taking a complete intersection of general members of $|mB|_{x,y}$, we can construct a curve with the above properties. $\square$

Lemma 2.3. *Let C be an irreducible curve on X and p be a non-singular point on C. Assume that the intersection number $(L \cdot C)$ is positive (that is, the restriction $L|_C$ is ample). Then there exists a function φ on C with the following properties:*
(1) $\varphi \in \mathrm{Psh}(C, \theta|_C)$
(2) *The function φ has pole at p (that is, $\varphi(p) = -\infty$) and is smooth except p .*

Proof. By the assumption, $L|_C$ is ample on C. Therefore we can obtain a smooth function φ_1 such that $\theta|_C + dd^c\varphi_1$ is a (strictly) positive $(1,1)$-form. (Even if C has singularities, we can obtain such function.) Let z be a local coordinate on C centered at p. We define a function φ_2 on C to be $\varphi_2 := \rho \log |z|^2$, where ρ is a smooth function on C whose support is contained in some neighborhood of p. Then φ_2 has a pole only at p. Further φ_2 is an almost psh function (that is, there exists a smooth function ψ such that $dd^c\varphi_2 \geq dd^c\psi$). Then a function φ which is defined to be $\varphi = (1-\varepsilon)\varphi_1 + \varepsilon\varphi_2$ satisfies the above condition for a sufficiently small $\varepsilon > 0$. In fact, the property (1) follows from the strictly positivity of the Levi-form of φ_1. The function φ has a pole only at p thanks to φ_2. $\square$

Lemma 2.1, 2.3 says that there exists many $\theta|_C$-psh functions on a complete intersection of very ample divisors.

Lemma 2.4. *Let γ be a smooth d-closed $(1,1)$-form on X and a function φ_i (for $i = 1, 2$) be γ-psh function on X. Then the function $\max(\varphi_1, \varphi_2)$ is also a γ-psh function on X.*

Proof. First we remark γ-plurisubharmonicity is a local property. We can locally take a smooth potential function of γ since γ is a d-closed $(1,1)$-form. Thus we can locally write $\gamma = dd^c\psi$ for some function ψ. By the assumption, $dd^c(\psi + \varphi_i)$ is a positive current. Therefore the Levi form of

$$\max(\psi + \varphi_1, \psi + \varphi_2) = \psi + \max(\varphi_1, \varphi_2)$$

is also a positive current. It means that $\gamma + dd^c \max(\varphi_1, \varphi_2) \geq 0$. Upper semi-continuity of functions is preserved. Hence the function $\max(\varphi_1, \varphi_2)$ is a γ-psh function. $\square$

For the proof of Theorem 1.2, it is important to obtain strictly positivity from extended θ-psh functions. The main idea for the purpose is to use the volume of a line bundle and its expression formula with current integrations, which is proved in [Bou02].

Definition 2.5. Let M be a line bundle on a projective variety Y of dimension d. Then the volume of M on Y is defined to be

$$\mathrm{vol}_Y(M) = \limsup_{k\to\infty} \frac{\dim H^0\big(Y, \mathcal{O}_Y(kM)\big)}{k^d/d!}.$$

The volume asymptotically measures the number of global holomorphic sections. The volume of a line bundle can be defined for a $\mathbb{Q}$-line bundle and, depends only on the numerical class (the first Chern class) of the line bundle. Moreover the volume is a continuous function on the $\mathbb{Q}$-vector space $N^1(Y)_{\mathbb{Q}}$ of the numerical equivalent classes of $\mathbb{Q}$-line bundles. (See [La04, Proposition 2.2.35, 2.2.41] for the precise statement.) The above properties are used in the proof of Theorem 1.2.

The following proposition gives a relation between the volume and curvature currents of a line bundle. It is proved by using singular Morse inequalities (which is proved in [Bon98]) and approximations of θ-psh functions (see [Bou02] for details).

Proposition 2.6 ([Bou02, Proposition 3.1]). *Let M be a psuedo-effective line bundle on a projective manifold Y of dimension d and η a smooth $(1,1)$-form which represents the first Chern class $c_1(M)$ of M. Then for any η-psh function φ on Y, we have*

$$\mathrm{vol}_Y(M) \geq \int_Y \big(\eta + dd^c\varphi\big)^d_{\mathrm{ac}}.$$

Here $\big(\eta + dd^c\varphi\big)_{\mathrm{ac}}$ means the absolutely continuous part of a positive current $\big(\eta + dd^c\varphi\big)$ by the Lebesgue decomposition. (See [Bou02, Section2] for the precise definition.) We use only property that when φ is smooth on an open set, the equality

$$\big(\eta + dd^c\varphi\big)_{\mathrm{ac}} = \big(\eta + dd^c\varphi\big)$$

holds on the open set.

Actually, the above inequality would be an equality by taking supremum of the right-hand side over all η-psh functions (see [Bou02, Theorem 1.2]). It is generalized to the restricted volume along a subvariety (cf. [Mat10, Theorem 1.2]). These expressions of the volume and restricted volume with current integrations give an example, which show us that there exist a θ-psh function on some subvariety which can not be extended a global θ-psh function even if L is a big line bundle (see Example 4.2).

In section 3, we need to approximate a given θ-psh function by almost psh functions with mild singularities. For the purpose, we use Theorem 2.7. Theorem 2.7 says that it is possible to approximate a given almost psh function with the same singularities as a logarithm of a sum of squares of holomorphic functions without a large loss of positivity of the Levi form.

Theorem 2.7 ([Dem, Theorem 13.12]). *Let φ be an almost psh function on a compact complex manifold X such that $dd^c\varphi > \gamma$ for some continuous $(1,1)$-form γ. Fix a hermitian form ω on X. Then there exists a sequence of almost psh functions φ_k and a decreasing sequence $\delta_k > 0$ converging to 0 with the following properties*:

(A) $\varphi(x) < \varphi_k(x) \leq \sup_{|\zeta - x| < r} \varphi(\zeta) + C\Big(\dfrac{|\log r|}{k} + r + \delta_k\Big)$
with respect to coordinate open sets covering X.

(B) *φ_k has the same singularities as a logarithm of a sum of squares of holomorphic functions. In particular, φ_k is smooth except the polar set of φ.*

(C) $dd^c\varphi_k \geq \gamma - \delta_k\omega$.

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Let L be a line bundle with the assumption of Theorem 1.2 and θ be a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$. According to the Nakai-Moishezon-Kleiman criterion (cf. [La04, Theorem 1.2.23]), in order to show that L is ample, it is enough to see the self-intersection number $(L^d \cdot V)$ along V is positive for any irreducible subvariety V.

For the purpose, we first show the following proposition which implies that the self-intersection number along an irreducible curve is always positive.

Proposition 3.1. *Let L be a line bundle with the assumption in Theorem 1.2 and V be an (irreducible) subvariety on X. Then*
(1) *The restriction $L|_V$ to V is pseudo-effective.*
(2) *The restriction $L|_V$ to V is not numerically trivial.*

Remark 3.2. When V is non-singular, the property (2) means that the first Chern class $c_1(L|_V)$ is not zero.

Proof. First we take two different points p, q on V_{reg}. Here V_{reg} means the regular locus of V. Then we can take a smooth curve C on X such that C contains p, q by Theorem 2.2. By the construction, the curve C is a complete intersection of the linear system of some very ample line bundles. It follows that the intersection number $(L \cdot C)$ along C is positive from Lemma 2.1. Lemma 2.3 asserts that there exists a function $\varphi \in \mathrm{Psh}(C, \theta|_C)$ such that φ has a pole at p and is smooth at q. The $\theta|_C$-function φ on C can be extended to a global θ-function on X by the assumption of the extendability. The extended function to X does not have a pole at $q \in V$. It means that the restriction to V of the function is well-defined (that is, the function is not identically $-\infty$ on V). We denote by $\widetilde{\varphi}$ the restriction to V of the function. The function $\widetilde{\varphi}$ gives an element in $\mathrm{Psh}(V, \theta|_V)$. Hence $L|_V$ is pseudo-effective.

From now on, we show that $L|_V$ is not numerically trivial. For a contradiction, we assume that $L|_V$ is numerically trivial. First we consider the case when V is non-singular. Then there exists a function on V such that

$$\theta|_V + dd^c\widetilde{\varphi} = dd^c\psi.$$

from the $\partial\bar{\partial}$-Lemma, since $L|_V$ is numerically trivial (that is, first Chern class $c_1(L|_V)$ is zero). Since the function $\widetilde{\varphi}$ is a $\theta|_V$-psh, ψ is a psh function on X. It follows that ψ is actually a constant by the maximal principle of psh functions. Therefore $\theta|_V + dd^c\widetilde{\varphi}$ is a zero current. We have known that θ-pluriharmonic functions are always smooth. Hence the function $\widetilde{\varphi}$ is smooth on V. However $\widetilde{\varphi}$ has a pole at p by the construction. This is a contradiction.

We need to consider the case when V has singularities. Then we take an embedded resolution

$$\mu : \widetilde{V} \subseteq \widetilde{X} \longrightarrow V \subseteq X$$

of $V \subseteq X$. That is, $\mu : \widetilde{X} \longrightarrow X$ is a birational morphism and the restriction of μ to $\widetilde{V}$ gives a resolution of singularities of V. Since p is contained in the regular locus of V, μ is an isomorphism on some neighborhood of p. Further the pull-back $(\mu^*L)|_{\widetilde{V}}$ is also numerically trivial since $L|_V$ is numerically trivial. The same argument asserts that any function in $\mathrm{Psh}(\widetilde{\mathrm{V}}, (\mu^*\theta)|_{\widetilde{\mathrm{V}}})$ is always smooth. Note that the pull-back $\mu^*\widetilde{\varphi}$ is a $(\mu^*\theta)|_{\widetilde{V}}$-psh function on V. It shows that the pull-back $\mu^*\widetilde{\varphi}$ is smooth on $\widetilde{V}$. The function $\widetilde{\varphi}$ is also smooth at p since μ is an isomorphism on some neighborhood of p. However $\widetilde{\varphi}$ has a pole at p by the construction. This is a contradiction. Thus $L|_V$ is not numerically trivial even if V has singularities. □

Corollary 3.3. *Let L be a line bundle with assumption in Theorem 1.2. Then the intersection number $(L \cdot C)$ is positive for any irreducible curve C on X.*

Proof. Any pseudo-effective line bundle on a curve which is not numerically trivial is always ample. Thus, the corollary follows from Proposition 3.1. □

In oder to show $(L^d \cdot V) > 0$ for any subvariety, we need only consider the case when the dimension of V is larger than or equal to two from the above corollary. Moreover the above corollary asserts that L is a nef line bundle on X. It is well-known that the volume of $L|_V$ is equal to the self-intersection number $(L^d \cdot V)$ along V for a nef line bundle. (see [La04, Section 2.2 C]). That is, the equality holds

$$\mathrm{vol}_V(L) = (L^d \cdot V)$$

for any irreducible subvariety V of dimension d. (Note the restriction of a nef line bundle is also nef.) Therefore for the proof of Theorem 1.2, it is enough to show that the volume $\mathrm{vol}_V(L)$ is always positive for any irreducible subvariety V of dimension $d \geq 2$. From now on, we will show that the volume $\mathrm{vol}_V(L)$ is positive for a subvariety V of dimension $d \geq 2$, by using Proposition 2.6.

We first consider the case when V is non-singular. Even if V has singularities, the same argument can be justified by taking an embedded resolution of $V \subseteq X$. We argue the case at the end of this section.

Fix a point p on V. Let $(z_1, z_2, \ldots, z_d)$ be a local coordinate centered at p. We consider an open ball B, which is defined by

$$B := \big\{(z_1, z_2, \ldots, z_d) \bigm| |z|^2 < 1\big\}.$$

Since $dd^c|z|^2$ is a strictly positive $(1,1)$-form on B, there exists a large positive number A such that

$$A dd^c|z|^2 + \theta|_V > 0 \quad \text{on } B. \tag{3.1}$$

For every point y on the boundary ∂B of B, we can take a curve C_y on V such that C_y contains p and y. We can take such curve from Theorem 2.2 and the assumption that the dimension of V is larger than or equal to two. By Lemma 2.1 and the property (1) in Theorem 2.2, the restriction of L to C_y is ample. Therefore there exists a function φ_y on C_y with following properties:

$$\varphi_y \in \mathrm{Psh}(C_y, \theta|_{C_y}), \tag{3.2}$$

$$\varphi_y(p) = -\infty, \tag{3.3}$$

$$\varphi_y(y) = 0, \tag{3.4}$$

Indeed, we can take a function $\varphi_y \in \mathrm{Psh}(C_y, \theta|_{C_y})$ such that φ_y has a pole at p and φ_y is smooth at y by Lemma 2.3. After replacing φ_y by $\varphi_y - \varphi_y(y)$, the function satisfies the property (3.4). Note that the function is a $\theta|_{C_y}$-psh function even if we replace φ_y by $\varphi_y - \varphi_y(y)$.

Now the function φ_y on C_y can be extended to a $\theta|_V$-psh function $\widetilde{\varphi}_y$ on V by the assumption of Theorem 1.2. In fact, we can extend φ_y to a global θ-psh function on X by the assumption in Theorem 1.2. From the property (3.4), the extended function does not have pole at y. Thus we can restrict the function to V. The function gives the extension of φ_y to V, which we denote by $\widetilde{\varphi}_y$. Then the function $\widetilde{\varphi}_y$ satisfies the following properties:

$$\widetilde{\varphi}_y \in \mathrm{Psh}(V, \theta|_V), \tag{3.5}$$

$$\widetilde{\varphi}_y(p) = -\infty, \tag{3.6}$$

$$\widetilde{\varphi}_y(y) = 0. \tag{3.7}$$

In the following step, we approximate the function $\widetilde{\varphi}_y$ with the same singularities as a logarithm of a sum of squares of holomorphic functions. If the extended function $\widetilde{\varphi}_y$ is continuous on some neighborhood of y, this step is not necessary. However the function $\widetilde{\varphi}_y$ may not be continuous at y even if φ_y is smooth at y on C_y. Thus the following step seems to be necessary in general.

Lemma 3.4. *Fix a hermitian form ω on X. For every positive number ε and a point $y \in \partial B$, there exist a neighborhood U_y of p which is independent of ε and an almost psh $\widetilde{\varphi}_{y,\varepsilon}$ with following properties*:

$$\theta|_V + dd^c\widetilde{\varphi}_{y,\varepsilon} \geq -\varepsilon\omega, \tag{3.8}$$

$$\widetilde{\varphi}_{y,\varepsilon}(y) > 0 \textit{ and } \widetilde{\varphi}_{y,\varepsilon} \textit{ is smooth on some neighborhood of } y. \tag{3.9}$$

$$-A > \widetilde{\varphi}_{y,\varepsilon} \quad \text{on } U_y. \tag{3.10}$$

Proof. By applying Theorem 2.7 to $\varphi = \widetilde{\varphi}_y$ and $\gamma = -\theta|_V$, we obtain almost psh functions $\{\widetilde{\varphi}_{y,k}\}_{k=1}^{\infty}$ with the properties in Theorem 2.7. For a given ε, by taking a sufficiently large $k = k(\varepsilon, y)$, the property (3.8) holds from the property (C).

From the left side inequality of the property (A) in Theorem 2.7 and (3.7), we can easily check the property (3.9) for every positive integer k. In fact the property (B) implies that if $\widetilde{\varphi}_{y,k}$ does not have a pole at a point, $\widetilde{\varphi}_{y,k}$ is smooth at the point . In particular, $\widetilde{\varphi}_{y,k}$ is smooth on some neighborhood of y. In order to show the existence of U_y with the property (3.10), we estimate the right hand inequality of the property (A). We can easily show that there exists a sufficiently small $r_1 > 0$ which does not depend on ε such that

$$0 < C(\frac{|\log r_1|}{k} + r_1 + \delta_k) < A \quad \text{for any } k \geq \lceil \frac{1}{r_1} \rceil. \tag{3.11}$$

Here $\lceil \cdot \rceil$ means round up of a real number. Indeed, for any $k \geq \lceil \frac{1}{r_1} \rceil$, we have

$$\frac{|\log r_1|}{k} + r_1 + \delta_k \leq r_1 |\log r_1| + r_1 + \delta_k.$$

Now C depends on the choice of coordinate open sets covering V. However C is independent of ε. (We may assume that the coordinate open set $\big(B, (z_1, z_2, \ldots, z_d)\big)$ is a member of coordinate open sets covering V.) Therefore the inequality (3.11) holds for a sufficiently small r_1 which is independent of ε.

On the other hand, $\widetilde{\varphi}_y$ has a pole at p by (3.6). Thus we have

$$\sup_{|z - z(p)| < r_2} \widetilde{\varphi}_y(z) < -2A$$

for a sufficiently small $r_2 > 0$. Here we used upper semi-continuity of $\widetilde{\varphi}_y$. Then we define U_y to be

$$U_y := \big\{ z \in B \ \big| \ |z - z(p)| < r_3 \big\},$$

where r_3 is $\min\{r_1, r_2\}$. Then the right hand of the property (A) in Theorem 2.7 is strictly smaller than A for any $k \geq \lceil \frac{1}{r_3} \rceil$. We emphasize that r_1 and r_2 does not depend on ε. Therefore we obtain U with the property (3.10). $\square$

By using these functions, we construct an almost psh function whose value at p smaller than values on the boundary ∂B of B. From the property (3.9), there exists a neighborhood W_y of y such that

$$\widetilde{\varphi}_{y,\varepsilon} > 0 \quad \text{on } W_y.$$

Since ∂B is a compact set, we can cover ∂B by finite members $\big\{ W_{y_i} \big\}_{i=1}^N$. Now we define a function Φ_ε to be

$$\Phi_\varepsilon := \max_{i=1,\ldots,N} \big\{ \widetilde{\varphi}_{y_i,\varepsilon} \big\}.$$

Lemma 3.5. *Then the function Φ_ε satisfies the following properties:*

$$\theta|_V + dd^c \Phi_\varepsilon \geq -\varepsilon\omega. \tag{3.12}$$

$$\Phi_\varepsilon > 0 \text{ on some neighborhood } V_\varepsilon \text{ of } \partial B. \tag{3.13}$$

$$-A > \Phi_\varepsilon \text{ on some neighborhood } U \text{ of } p. \tag{3.14}$$

Proof. The property (3.12) follows from Lemma 2.4 and the property (3.8). The property (3.13) is clear by the definition of Φ_ε and the property (3.9). If a neighborhood U is defined to be $U := \cap_{i=1}^N U_{y_i}$, the property (3.14) holds from the property (3.10). Here U_{y_i} is a neighborhood of p with the property (3.10) in Lemma 3.4. $\square$

Remark 3.6. We can assume that a neighborhood U in the property (3.14) does not depend on ε. It follows from the definition of U in the proof of Lemma 3.5. The fact is essentially important in the estimation of the volume $\mathrm{vol}_V(L)$ with current integrations.

We want to construct a almost psh function whose Levi-form is strictly positive on some neighborhood of p. The integral of the Levi-form would imply that the volume $\mathrm{vol}_V(L)$ is positive. Then we define a new function Ψ_ε on V as follows:

$$\Psi_\varepsilon := \begin{cases} \Phi_\varepsilon & \text{on } V \setminus B \\ \max\left\{\Phi_\varepsilon, A|z|^2 - A\right\} & \text{on } B. \end{cases} \tag{3.15}$$

Then the function Ψ_ε satisfy the following properties:

Lemma 3.7. *The function Ψ_ε satisfies the following properties:*

$$\theta|_V + dd^c\Psi_\varepsilon \geq -\varepsilon\omega, \tag{3.16}$$

$$\Psi_\varepsilon = A|z|^2 - A \text{ on } U, \tag{3.17}$$

where U is a neighborhood of p which is independent of ε.

Proof. From the property (3.14), $\Phi_\varepsilon < -A$ for some neighborhood U of p which is independent of ε. Further the function $(A|z|^2 - A)$ is continuous and the value at the origin is $-A$. Hence we can assume

$$A|z|^2 - A > \Phi_\varepsilon \quad \text{on } U$$

by shrinking the neighborhood U of p if we need. Thus the property (3.17) holds. Further by the choice of A, the $(1,1)$-form $Add^c|z|^2 + \theta|_V$ is strictly positive on the neighborhood B of p. In particular,

$$Add^c|z|^2 + \theta|_V \geq -\varepsilon\omega$$

holds. Hence we have

$$\theta|_V + dd^c \max\left\{\Phi_\varepsilon, A|z|^2 - A\right\} \geq -\varepsilon\omega \quad \text{on } B$$

from Lemma 2.4 and the property (3.12).

On the other hand, we obtain

$$\max\left\{\Phi_\varepsilon, A|z|^2 - A\right\} = \Phi_\varepsilon$$

on some neighborhood of ∂B from the property (3.13). Therefore the function Ψ_ε satisfies

$$\theta|_V + dd^c\Psi_\varepsilon = \theta|_V + dd^c\Phi_\varepsilon \geq -\varepsilon\omega$$

on the neighborhood of ∂B from the property (3.12). Thus the property (3.16) holds on X. □

Finally, we estimate the volume $\mathrm{vol}_V(L)$ of L with current integrations for the computation of the intersection number $(L^d \cdot V)$. The function Ψ_ε is a $(\theta|_V + \varepsilon\omega)$-psh function by the property (3.16). Here ω can be assumed to be a Kähler form which represents the first Chern class $c_1(B)$ of B, where B is an ample line bundle on V. The d-closed

$(1,1)$-form $(\theta|_V+\varepsilon\omega)$ represents the first Chern class $c_1(L)+\varepsilon c_1(B)$. Thus by Proposition 2.6, we have

$$\mathrm{vol}_V\big(L+\varepsilon B\big) \geq \int_V \big(\theta|_V+\varepsilon\omega+dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}}.$$

Since $\big(\theta|_V+\varepsilon\omega+dd^c\Psi_\varepsilon\big)$ is a positive current, the absolute continuous part is (semi)-positive. It shows

$$\mathrm{vol}_V\big(L+\varepsilon B\big) \geq \int_U \big(\theta|_V+\varepsilon\omega+dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}},$$

U is a neighborhood of p which satisfies the properties in Lemma 3.7. If we converge ε to zero, the left hand of the above inequality converges to $\mathrm{vol}_V(L)$ from the continuity of the volume. Thus we have

$$\begin{aligned}\mathrm{vol}_V\big(L\big) &\geq \liminf_{\varepsilon\to 0}\int_U \big(\theta|_V+\varepsilon\omega+dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}}\\ &\geq \int_U \liminf_{\varepsilon\to 0}\big(\theta|_V+\varepsilon\omega+dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}}\\ &= \int_U \big(\theta|_V+dd^cA|z|^2\big)^d_{\mathrm{ac}}.\end{aligned}$$

The second inequality follows form Fatou's lemma. Here we used the fact that U does not shrink even if ε goes to 0, since U is independent of ε, The equality follows from the property (3.17). By the choice of A (see (3.1)), the right hand of the above inequality

$$\int_U \big(\theta|_V+dd^cA|z|^2\big)^d_{\mathrm{ac}} = \int_U \big(\theta|_V+dd^cA|z|^2\big)^d$$

is positive. Hence we proved the volume $\mathrm{vol}_V(L)$ is positive for a non-singular subvariety V.

When V has singularities, we take an embedded resolution $\mu:\widetilde{V}\subseteq\widetilde{X}\longrightarrow V\subseteq X$. Then we can show that $\mathrm{vol}_{\widetilde{V}}(\mu^*L)>0$ by the same argument as above. Note that we used only the following property on the line bundle L in the above argument:

$(*)$ For every point $y\in\partial B$, there exists a $\theta|_V$-psh function $\widetilde{\varphi}_y$ such that $\widetilde{\varphi}_y(p)=-\infty$ and $\widetilde{\varphi}_y(y)=0$.

We can easily show that the property $(*)$ holds for the pull-back μ^*L of L as follows: We first choose a point p on $\widetilde{V}$ such that μ is an isomorphism on a neighborhood B of p. For every point $y\in\partial B$, we consider a curve C_y on $\widetilde{V}$ which contains the points p and y. Since μ is an isomorphism on B, the push-forward $\mu(C_y)$ is not a point. Therefore it follows that the intersection number $\big(L\cdot\mu(C_y)\big)$ is positive from Proposition 3.1. Lemma 2.3 implies that there exists a $\theta|_{\mu(C_y)}$-psh function φ_y such that $\varphi_y\big(\mu(p)\big)=-\infty$ and $\varphi_y(\mu(y))=0$. By the assumption of Theorem 1.2 on L, we can extend φ_y to a global θ-psh function $\widetilde{\varphi}_y$ on X. Then the pull-back $\mu^*\widetilde{\varphi}_y$ of $\widetilde{\varphi}_y$ satisfies property $(*)$ which we want to obtain. By the same argument as above, we obtain $\mathrm{vol}_{\widetilde{V}}(\mu^*L)>0$. Since the restriction $\mu|_{\widetilde{V}}$ a birational morphism from $\widetilde{V}$ to V, $\mathrm{vol}_V(L)=\mathrm{vol}_{\widetilde{V}}(\mu^*L)$ (see [La04, Proposition 2.2.43]). Hence we proved the volume $\mathrm{vol}_V(L) > 0$ for any subvariety V, even if V has singularities.

Since L is a nef line bundle, the volume of L on V coincides with the intersection number $(L^d \cdot V)$. Therefore L is an ample line bundle by the Nakai-Moishezon-Kleiman criterion.

4. Example

Example 4.1. (This example shows us that there exists a θ-psh function on some subvariety which can not extended to a global θ-psh function even if L is semi-ample and big.) Let $\pi : X := \mathrm{Bl}_{\mathrm{p}}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ be a blow up along a point $p \in \mathbb{P}^2$ and L the pull-back of the hyperplane bundle by π. Then L is a semi-ample and big. (However L is not ample.) We denote by θ the pull-back of the Fubini-Study form on $\mathbb{P}^2$. Note that θ represents the first Chern class $c_1(L)$ of L. By the definition of θ, the restriction of θ to E is zero $(1,1)$-form, where E is the exceptional divisor. Therefore any $\theta|_E$-psh on E is constant by the maximal principle of psh functions. It says that a global θ-psh function has the same value along E.

Now we denote by C an irreducible curve on X which intersect the exceptional divisor E with more two points. Then C is not contractive by π. Therefore the degree of L on C is positive by the projection formula (that is, $L|_C$ is ample on C). It implies that there exist many $\theta|_C$-psh functions on C. In particular, there exists a $\theta|_C$-psh function which has different values at intersection points with E. Indeed, we can such function by using Lemma 2.3. Such function can not extend to a global θ-psh function.

Example 4.2. (Relations between the restricted volume of a line bundle and the extendability of θ-psh functions.)
The restricted volume of L along a subvariety V is defined to be

$$\mathrm{vol}_{X|V}(L) = \limsup_{k\to\infty} \frac{\dim H^0\big(X|V, \mathcal{O}(kL)\big)}{k^d/d!},$$

where

$$H^0\big(X|V, \mathcal{O}(kL)\big) = \mathrm{Im}\Big(H^0\big(X, \mathcal{O}_X(kL)\big) \longrightarrow H^0\big(V, \mathcal{O}_V(kL)\big)\Big).$$

The restricted volume measures the number of sections of $\mathcal{O}_V(kL)$ which can be extended to X. Due to [Mat10], restricted volumes can be expressed with current integrations as follows (see [Mat10] for details).

Theorem 4.3 ([Mat10, Theorem 1.2])**.** *Assume that V is not contained in the augmented base locus $\mathbb{B}_+(L)$. Then the restricted volume of L along V satisfies the following equality :*

$$\mathrm{vol}_{X|V}(L) = \sup_{\varphi\in \mathrm{Psh}(X,\theta|_X)} \int_{V_{\mathrm{reg}}} (\theta|_{V_{\mathrm{reg}}} + \varphi|_{V_{\mathrm{reg}}})_{\mathrm{ac}},$$

for φ ranging among θ-psh functions on X with analytic singularities whose singular locus does not contain V.

The right hand integral measures Monge-Ampère products of $\theta|_V$-psh functions which can be extended to X. On the other hand, the volume of the line bundle $L|_V$ can also be expressed with current integrations (see Proposition 2.6 and [Bou02]). If any $\theta|_V$-psh function can be extended to a global θ-psh function, the restricted volume along V and the volume on V coincides. However there exist an example such that they are different even if V is not contained in the augmented base locus.

For example, when X is a surface, a big line bundle L admits a Zariski decomposition. That is, there exist nef $\mathbb{Q}$-divisor P and effective $\mathbb{Q}$-divisor N such that

$$H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \longrightarrow H^0(X, \mathcal{O}_X(kL)).$$

The map is multipling the section e_k, where e_k is the standard section of the effective divisor $\lceil kN \rceil$. Here $\lfloor G \rfloor$ (resp. $\lceil G \rceil$) denotes round down (resp. round up) of an $\mathbb{R}$-divisor G. Let V be a irreducible curve which is not contained in the augmented base locus $\mathbb{B}_+(L)$. Then $L|_V$ is an ample line bundle. Further, the restricted volume along V is computed by the self-intersection number of the positive part P along V when L admits a Zariski decomposition (see [Mat10, Proposition 3.1]). That is,

$$\begin{aligned} &\mathrm{vol}_{X|V}(L) = (P \cdot V), \\ &\mathrm{vol}_V(L) = (L \cdot V) = (P \cdot V) + (N \cdot V). \end{aligned}$$

Therefore, the volume and restricted volume may be different value unless $(N \cdot V)$ is not equal to zero. When L is not nef (that is, N is non-zero divisor) and V is an ample divisor, $(N \cdot V)$ is not zero.

References

[BDPP] Boucksom S, Demailly J P, Paun M, Peternell T. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* preprint, arXiv:math/0405285v1.

[Bon98] Bonavero L. *Inégalités de Morse holomorphes singuliéres.* J. Geom. Anal. **8** (1998), no. 3, 409–425.

[Bou02] Boucksom S. *On the volume of a line bundle.* Intern. J. Math. **13** (2002), no. 10, 1043–1063.

[CGZ] Coman D., Guedj V., Zeriahi A. *Extension of plurisubharmonic functions with growth control.* preprint, arXiv:math/1007.0262v1.

[Dem] Demailly J.-P. *Analytic methods in algebraic geometry.* Compilation of Lecture Notes on the web page of the author, December 2009.

[La04] Lazarsfeld R. *Positivity in Algebraic Geometry I-II.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49., Springer Verlag, Berlin, 2004.

[Mat10] Matsumura S. *Restricted volumes and divisorial Zariski decompositions.* preprint, arXiv:1005.1503v1.

[Zh09] Zhang J. *Bertini Type Theorems.* preprint, arXiv:math/0910.4105v1.

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