CALDERÓN COUPLES OF p-CONVEXIFIED BANACH LATTICES

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Abstract. We deal with the question of whether or not the p-convexified couple $\left(X_0^{(p)}, X_1^{(p)}\right)$ is a Calderón couple under the assumption that (X_0, X_1) is a Calderón couple of Banach lattices on some measure space. In this preliminary version of the paper we find that the answer is affirmative in the simple case where X_0 and X_1 are sequence spaces and an additional "positivity" assumption is imposed regarding (X_0, X_1). We also prove a quantitative version of the result with appropriate norm estimates. In future versions of this paper we plan to deal with other and more general cases of these results. .

1. Preliminaries, definitions, notations, conventions, and one auxilary result

Definition 1. A **Banach lattice of measurable functions** X is a Banach space of (equivalence classes of) measurable functions defined on a certain measure space (Ω, Σ, μ) and taking values in $\mathbb{R}$ or $\mathbb{C}$ (in this paper, in $\mathbb{R}$), with the following property: if $f, g : \Omega \to \mathbb{R}$ are two measurable functions, and if $f \in X$ and $|g| \leq |f|$ almost everywhere then we also have $g \in X$ and $\| g \| \leq \| f \|$.

In this paper we will usually use the shorter terminology "Banach lattice" although in other settings this is used in a more abstract context (see e.g. [9] Definition 1.a.1 p. 1).

Definition 2. For each Banach lattice X of measurable functions on a measure space (Ω, Σ, μ) and each $p \in (1, \infty)$ we recall that the **p-convexification of** X is the set $X^{(p)}$ of all measurable functions $f : \Omega \to \mathbb{R}$ for which $|f|^p \in X$. When endowed with the norm $\|f\|_{X^{(p)}} = \left(\| |f|^p \|^{1/p} \right)$ it is also a Banach lattice.

Definition 3. Whenever X_0, X_1 are two Banach lattices with the same underlying measure space (Ω, Σ, μ) we define $X_0 + X_1$ to be the space of all measurable functions $f : \Omega \to \mathbb{R}$ for which there are $a_j \in X_j$ $(j = 0, 1)$ such that $f = a_0 + a_1$. This is a Banach space (in fact a Banach lattice), when endowed with the following norm:

$$\|f\|_{X_0+X_1} = \inf \left\{ \|a_0\|_{X_0} + \|a_1\|_{X_1} \, | a_j \in X_j \, , \, j = 0, 1 \, , \, f = a_0 + a_1 \right\} \tag{1.1}$$

Remark. Proofs that (1.1) is a norm rather than merely a seminorm can be found, e.g. in [5] Remark 1.41 pp. 34-35 or [8] Corollary 1, p. 42. This fact implies that (X_0, X_1) is a **Banach couple**, i.e. that there exists some topological Hausdorff vector space $\mathcal{X}$ such that X_0 and X_1 are both continuously embedded in $\mathcal{X}$ (clearly one can choose $\mathcal{X} = X_0 + X_1$).

Remark. In a more general context, whenever (X_0, X_1) is a Banach couple, the space $X_0 + X_1$ aforementioned is a Banach space in which X_0 and X_1 are continuously embedded (see e.g. [1, 2]).

Definition 4. For each fixed $t>0$ the following functional

$$K(t,f;X_0,X_1)=\inf\left\{\|a_0\|_{X_0}+t\,\|a_1\|_{X_1}\,|a_j\in X_j, j=0,1, f=a_0+a_1\right\}$$

is equivalent to the norm (1.1) and is known as the **Peetre K-functional** (see e.g. [1, 2]).

Definition 5. The statement that "$T:(X_0,X_1)\to(X_0,X_1)$ **is a bounded linear operator**" means that T is a linear operator from X_0+X_1 into itself such that the restriction of T to X_j is a bounded operator from X_j into itself (for $j=0,1$).

Remark. We remark that if $T:(X_0,X_1)\to(X_0,X_1)$ is a bounded linear operator then automatically T is also a bounded linear operator from X_0+X_1 into itself, and the following inequality holds:

$$\|T\|_{X_0+X_1\to X_0+X_1}\le\max\left\{\|T\|_{X_0\to X_0},\|T\|_{X_1\to X_1}\right\}.$$

Definition 6. Whenever X_0 and X_1 are two Banach spaces continuously embedded in some topological Hausdorff vector space $\mathcal{X}$, the statement "A **is an interpolation space with respect to** (X_0,X_1)" is a concise way to say the following: A is a Banach space satisfying $X_0\cap X_1\subseteq A\subseteq X_0+X_1$ where all the inclusions are continuous, and the restriction to A of every bounded linear operator $T:(X_0,X_1)\to(X_0,X_1)$ is a bounded operator from A into itself.

Remark. A Banach space A satisfying $X_0\cap X_1\subseteq A\subseteq X_0+X_1$ where all the inclusions are continuous is also called an **intermediate space** of (X_0,X_1)

Definition 7. The statement "(X_0,X_1) **is a Calderón couple**" means that the Banach couple (X_0,X_1) has the following property: if $f,g\in X_0+X_1$ and if $K(t,g;X_0,X_1)\le K(t,f;X_0,X_1)$ for every $t>0$, then there exists a bounded linear operator $T:(X_0,X_1)\to(X_0,X_1)$ such that $Tf=g$.

The statement "(X_0,X_1) **is a C-Calderón couple**" means that (X_0,X_1) has the above property, and furthermore the operator T with the above properties can also be assumed to satisfy $\|T\|_{X_j\to X_j}\le C$ for $j=0,1$.

Most of the definitions in this section appear extensively in the literature, but the following one is perhaps new. It relates to a notion which has been considered in a so far unpublished paper [4].

Definition 8. The statement "(X_0,X_1) **is a positive Calderón couple**" means that (X_0,X_1) is a Banach couple of Banach lattices on the same underlying measure space with the following property: If $f,g\in X_0+X_1$ and if $K(t,g;X_0,X_1)\le K(t,f;X_0,X_1)$ for every $t>0$ and if also $f,g\ge0$ then there exists a **positive** bounded linear operator $T:(X_0,X_1)\to(X_0,X_1)$ such that $Tf=g$ (positive in the sense that if $h\ge0$ a.e. then $Th\ge0$ a.e.)

Analogously to before, the statement "(X_0,X_1) **is a positive C-Calderón couple**" means that (X_0,X_1) is a positive Calderón couple and, furthermore the operator T with the above properties can also be assumed to satisfy $\|T\|_{X_j\to X_j}\le C$ for $j=0,1$.

Remark. Using the fact that pointwise multiplication by a unimodular measurable function is a norm one linear operator on any Banach lattice, it is clear that if (X_0,X_1) is a positive Calderón couple then it is also a Calderón couple in the usual sense. Similarly a positive C-Calderón couple is a C-Calderón couple.

Remark. Of course the interesting and well known property of Calderón couples is that all their interpolation spaces can be characterized by a simple monotonicity property in terms of the K-functional (see e.g. [1] or [2] or [5]).

We would also like to mention the following result, that we shall resort to later on:

Proposition 9. *Assume X is a Banach lattice defined on a measure space (Ω, Σ, μ) and $G : X \to X$ is a positive linear operator. Then, for every $1 < p < \infty$ and every two measurable functions $h_1, h_2 : \Omega \to \mathbb{R}$ such that $|h_1|^p, |h_2|^p \in X$ we have the pointwise almost everywhere inequality*

$$(G(|h_1 + h_2|^p))^{\frac{1}{p}} \leq (G(|h_1|^p))^{\frac{1}{p}} + (G(|h_2|^p))^{\frac{1}{p}}$$

The proof of this proposition appears in [6, p. 59].

2. THE MAIN PART

For simplicity, in this preliminary version of the paper we explicitly deal only with the case where the underlying measure space has a special property which means that our Banach lattices are in fact sequence spaces. But the same ideas extend readily to more general contexts. In subsequent versions we plan to present the same result for a wider class of measure spaces and Banach lattices of functions defined on them. For example, using the methods of [7] we can extend the following result to the case of an arbitrary underlying measure space if all conditional expectation operators are bounded on $X_0^{(p)}$ and $X_1^{(p)}$. Here, however, a different constant may replace the constant $4C^{\frac{1}{p}}$ which appears below. Furthermore, some Hahn-Banach-Kantorovich type theorems may be applied to prove similar results for other measure spaces and Banach lattices.

It is clear that analogous results can be readily obtained in the context of relative Calderón couples, i.e. where the relevant operators map between two possibly different couples (X_0, X_1) and (Y_0, Y_1). Here, again for simplicity, in this version we have only considered the case where $(X_0, X_1) = (Y_0, Y_1)$.

Theorem 10. *Suppose (X_0, X_1) is a positive Calderón couple of Banach lattices defined on an underlying measure space (Ω, Σ, μ). Suppose that the singleton set $\{\omega\}$ has a positive measure for every $\omega \in \Omega$, and that the set Ω is countable. Then, $(X_0^{(p)}, X_1^{(p)})$ is a Calderón couple for each $p \in (1, \infty)$.*

If, in addition, (X_0, X_1) is a positive C-Calderón couple then $(X_0^{(p)}, X_1^{(p)})$ is a $4C^{\frac{1}{p}}$-Calderón couple.

Before proving the theorem, a few remarks:

For every $f \in X_0 + X_1$ we define the following counterpart of the K-functional:

$$D(t, f; X_0, X_1) = \inf \left\{ \|a_0\|_{X_0} + t \|a_1\|_{X_1} | a_j \in X_j, j = 0, 1, f = a_0 + a_1, a_0 \cdot a_1 = 0 \right\} .$$

It is well known that for every $t > 0$, $f, g \in X_0 + X_1$ the inequality

$$K(t, f; X_0, X_1) \leq D(t, f; X_0, X_1) \leq 2K(t, f; X_0, X_1) \tag{2.1}$$

holds.

The straightforward proof of (2.1) appears essentially as part of the proof of Lemma 4.3 on p. 310 of [10] and is also given on pp. 280-281 of [3]. (The additional assumptions made in the context of Lemma 4.3 of [10] do not effect the validity of the argument in a more general setting.)

Claim 11. For a measurable function $f:\Omega\to\mathbb{R}$, $f\in X_0^{(p)}+X_1^{(p)}$ iff $|f|^p\in X_0+X_1$. In addition, for every $1<p<\infty$ the following inequality is valid

$$\left(D(t,|f|^p;X_0,X_1)\right)^{\frac{1}{p}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})\le 2^{1-\frac{1}{p}}\left(D(t,|f|^p;X_0,X_1)\right)^{\frac{1}{p}}.$$

Remark. It has been mentioned without proof in [3, p. 289] that the functionals $K(t,|f|^p;X_0,X_1)$ and $\left(K(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})\right)^p$ are equivalent. In fact, combining Claim 11 with (2.1) immediately gives us

$$K(t,|f|^p;X_0,X_1)\le 2^p\left(K(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})\right)^p\le 2^{2p}K(t,|f|^p;X_0,X_1)\,. \tag{2.2}$$

The proof of Claim 11 and thus of the equivalence (2.2) is indeed quite simple, but we present it here for completeness.

Proof. Fix $t,\epsilon>0$ and assume $f\in X_0^{(p)}+X_1^{(p)}$. There exist elements $a_j\in X_j^{(p)}$ $(j=0,1)$ such that $f=a_0+a_1$ and $a_0\cdot a_1=0$ and

$$\|a_0\|_{X_0^{(p)}}+t^{\frac{1}{p}}\|a_1\|_{X_1^{(p)}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})+\epsilon$$

that is

$$\left(\||a_0|^p\|_{X_0}\right)^{\frac{1}{p}}+\left(t\,\||a_1|^p\|_{X_1}\right)^{\frac{1}{p}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})+\epsilon\,.$$

Since $f=a_0+a_1$ and $a_0\cdot a_1=0$, it is also true that $|f|^p=|a_0|^p+|a_1|^p$ and $|a_0|\cdot|a_1|=0$ (and of course $|a_j|^p\in X_j$). From this we conclude that $|f|^p\in X_0+X_1$. In addition, because $(a+b)^{\frac{1}{p}}\le a^{\frac{1}{p}}+b^{\frac{1}{p}}$ for all $a,b\ge 0$ and every $1<p<\infty$, we have

$$\left(\||a_0|^p\|_{X_0}+t\,\||a_1|^p\|_{X_1}\right)^{\frac{1}{p}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})+\epsilon$$

and so

$$\left(D(t,|f|^p;X_0,X_1)\right)^{\frac{1}{p}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})+\epsilon$$

for every $t,\epsilon>0$, hence

$$\left(D(t,|f|^p;X_0,X_1)\right)^{\frac{1}{p}}\le D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})\,.$$

On the other hand, assuming $|f|^p\in X_0+X_1$, then there are $a_j\in X_j$ such that $|f|^p=a_0+a_1$ and $a_0\cdot a_1=0$ and

$$\|a_0\|_{X_0}+t\,\|a_1\|_{X_1}\le D(t,|f|^p;X_0,X_1)+\epsilon\,.$$

Clearly a_0 and a_1 must both be non-negative. Now we can define $b_j=\operatorname{sgn}(f)\cdot|a_j|^{\frac{1}{p}}$ for $j=0,1$ ($\operatorname{sgn}(f)=f/|f|$ whenever $f\neq 0$ and 0 otherwise). One readily checks that $b_j\in X_j^{(p)}$ and that $f=b_0+b_1$ and $b_0\cdot b_1=0$. Hence $f\in X_0^{(p)}+X_1^{(p)}$ and

$$D(t^{\frac{1}{p}},f;X_0^{(p)},X_1^{(p)})\le\|b_0\|_{X_0^{(p)}}+t^{\frac{1}{p}}\|b_1\|_{X_1^{(p)}}\,.$$

Some manipulations yield

$$\begin{aligned}\|b_0\|_{X_0^{(p)}}+t^{\frac{1}{p}}\|b_1\|_{X_1^{(p)}} &= \left(\||b_0|^p\|_{X_0}\right)^{\frac{1}{p}}+\left(t\,\||b_1|^p\|_{X_1}\right)^{\frac{1}{p}}\\ &= \left(\|a_0\|_{X_0}\right)^{\frac{1}{p}}+\left(t\,\|a_1\|_{X_1}\right)^{\frac{1}{p}}\,.\end{aligned}$$

Since for every $1 < p < \infty$ and every $a, b \geq 0$ it is true that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ we gather that

$$\begin{aligned}\left(\left(\|a_0\|_{X_0}\right)^{\frac{1}{p}} + \left(t\,\|a_1\|_{X_1}\right)^{\frac{1}{p}}\right)^p &\leq 2^{p-1}\left(\|a_0\|_{X_0} + t\,\|a_1\|_{X_1}\right) \\ &\leq 2^{p-1}\left(D(t, |f|^p; X_0, X_1) + \epsilon\right)\end{aligned}$$

hence

$$\left(D(t^{\frac{1}{p}}, f; X_0^{(p)}, X_1^{(p)})\right)^p \leq 2^{p-1}\left(D(t, |f|^p; X_0, X_1) + \epsilon\right)$$

for every $t, \epsilon > 0$, and so

$$D(t^{\frac{1}{p}}, f; X_0^{(p)}, X_1^{(p)}) \leq 2^{1-\frac{1}{p}}\left(D(t, |f|^p; X_0, X_1)\right)^{\frac{1}{p}}.$$

□

We now turn to the proof of Theorem 10.

Proof. We start by assuming that $f, g \in X_0^{(p)} + X_1^{(p)}$ and that $K(t, g; X_0^{(p)}, X_1^{(p)}) \leq K(t, f; X_0^{(p)}, X_1^{(p)})$ for every $t > 0$. We wish to prove there exists a linear operator $L : (X_0^{(p)}, X_1^{(p)}) \to (X_0^{(p)}, X_1^{(p)})$ such that $Lf = g$.

It follows from (2.2) that

$$K(t, |g|^p; X_0, X_1) \leq K(t, 2^{2p}|f|^p; X_0, X_1)$$

for all $t > 0$.

According to our assumption, since (X_0, X_1) is a positive Calderón couple, there exists a bounded linear positive operator $T : (X_0, X_1) \to (X_0, X_1)$ such that $T(2^{2p}|f|^p) = |g|^p$. If, furthermore, (X_0, X_1) is a positive C-Calderón couple then we can also assert that

$$\|T\|_{X_j \to X_j} \leq C \text{ for } j = 0, 1\,. \tag{2.3}$$

Let us now define

$$H : X_0^{(p)} + X_1^{(p)} \to X_0^{(p)} + X_1^{(p)}$$

by setting

$$H(h) = \left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}}$$

for every $h \in X_0^{(p)} + X_1^{(p)}$ (Since T is positive and $|h|^p \geq 0$, the expression $\left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}}$ is meaningful).

According to Claim 11, it is obvious that $H(h) \in X_0^{(p)} + X_1^{(p)}$.

Then we notice that

$$\begin{aligned}H(f) &= \left(T(2^{2p}|f|^p)\right)^{\frac{1}{p}} \\ &= |g|^{p \cdot \frac{1}{p}} \\ &= |g|\,.\end{aligned}$$

It is easy to check that H is sublinear, that is:

- For every $\lambda \in \mathbb{R}$ we have

$$H(\lambda h) = |\lambda| H(h)\,. \tag{2.4}$$

- For every $h_1, h_2 \in X_0^{(p)} + X_1^{(p)}$ we have

$$H(h_1 + h_2) \leq H(h_1) + H(h_2)\,. \tag{2.5}$$

We do so with the intent to apply the Hahn-Banach theorem to the function H later on.

(2.4) is immediate and (2.5) follows from Proposition 9 and the fact that T is positive and linear.

The equality (2.4) and inequality (2.5) should be regarded as valid almost everywhere, but since we assume $\{\omega\}$ has a positive measure for every $\omega \in \Omega$, these last two assertions are actually valid everywhere.

Hence, for every fixed $\omega \in \Omega$ we can define a (clearly sublinear) functional $H_\omega : X_0^{(p)} + X_1^{(p)} \to \mathbb{R}$ by setting

$$H_\omega(h) = H(h)(\omega) \text{ for every } h \in X_0^{(p)} + X_1^{(p)}\,.$$

We now define $l_\omega : \mathrm{Span}\,\{f\} \to \mathbb{R}$ by

$$l_\omega(\lambda f) = \lambda g(\omega) \text{ for every } \lambda \in \mathbb{R}.$$

Since for every $\lambda \in \mathbb{R}$,

$$\begin{aligned} |l_\omega(\lambda f)| &= |\lambda g(\omega)| \\ &= |\lambda|\, H(f)(\omega) \\ &= H(\lambda f)(\omega) \\ &= H_\omega(\lambda f)\,, \end{aligned}$$

the classical Hahn-Banach theorem implies there exists a linear functional $L_\omega : X_0^{(p)} + X_1^{(p)} \to \mathbb{R}$ which extends l_ω such that $|L_\omega(h)| \le H_\omega(h)$ for every $h \in X_0^{(p)} + X_1^{(p)}$.

We now define $L : X_0^{(p)} + X_1^{(p)} \to X_0^{(p)} + X_1^{(p)}$ by

$$L(h)(\omega) = L_\omega(h) \text{ for each } \omega \in \Omega \text{ and each } h \in X_0^{(p)} + X_1^{(p)}.$$

We first note that $L(h)$ is a measurable function, since our assumptions regarding (Ω, Σ, μ) imply that every function $f : \Omega \to \mathbb{R}$ is measurable.

Secondly, the linearity of L_ω for each ω immediately implies the linearity of L.

Furthermore, $Lf = g$, as this simple verification confirms:

$$\begin{aligned} L(f)(\omega) &= L_\omega(f) \\ &= l_\omega(f) \\ &= g(\omega)\,. \end{aligned}$$

Finally, to complete the proof, we will show that the restriction of L to $X_j^{(p)}$ (for $j = 0, 1$) is a bounded linear operator into $X_j^{(p)}$ and estimate its norm.

Let us therefore assume $h \in X_j^{(p)}$. Then, for each $\omega \in \Omega$,

$$\begin{aligned} |L(h)(\omega)| &= |L_\omega(h)| \\ &\le H_\omega(h) \\ &= H(h)(\omega) \\ &= \left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}}(\omega)\,. \end{aligned}$$

Since $h \in X_j^{(p)}$, it is also true that $|h|^p \in X_j$, and thus $T(2^{2p}|h|^p) \in X_j$. It follows from the definition of $X_j^{(p)}$ that $\left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}} \in X_j^{(p)}$, and so, from the lattice property, $L(h) \in X_j^{(p)}$.

Furthermore,

$$\begin{aligned}
\|L(h)\|_{X_j^{(p)}} &\leq \left\| \left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}} \right\|_{X_j^{(p)}} \\
&= \left(\left\| \left| \left(T(2^{2p}|h|^p)\right)^{\frac{1}{p}} \right|^p \right\|_{X_j} \right)^{\frac{1}{p}} \\
&= \left(\left\| T(2^{2p}|h|^p) \right\|_{X_j} \right)^{\frac{1}{p}} \\
&\leq \left(\|T\|_{X_j \to X_j} \cdot \left\| 2^{2p}|h|^p \right\|_{X_j} \right)^{\frac{1}{p}} \\
&= 4 \left(\|T\|_{X_j \to X_j} \right)^{\frac{1}{p}} \cdot \left(\left\| |h|^p \right\|_{X_j} \right)^{\frac{1}{p}} \\
&= 4 \left(\|T\|_{X_j \to X_j} \right)^{\frac{1}{p}} \|h\|_{X_j^{(p)}}
\end{aligned}$$

which proves that $L : (X_0^{(p)}, X_1^{(p)}) \to (X_0^{(p)}, X_1^{(p)})$ is bounded. In addition, if (X_0, X_1) is a positive C-Calderón couple, the preceding estimates and (2.3) show that $\|L\|_{X_j^{(p)} \to X_j^{(p)}} \leq 4C^{\frac{1}{p}}$ and therefore that $(X_0^{(p)}, X_1^{(p)})$ is a $4C^{\frac{1}{p}}$-Calderón couple. This completes the proof of Theorem 10. □

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