STANLEY CONJECTURE ON INTERSECTION OF THREE MONOMIAL PRIMARY IDEALS

ANDREI ZAROJANU

Abstract. We show that the Stanley's Conjecture holds for an intersection of three monomial primary ideals of a polynomial algebra S over a field.

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Introduction

Let K be a field and $S = K[x_1, ..., x_n]$ be the polynomial ring over K in n variables. Let $I \subset S$ be a monomial ideal of $S, u \in I$ a monomial and $uK[Z], Z \subset \{x_1, ..., x_n\}$ the linear K-subspace of I of all elements $uf,\ f \in K[Z]$. A presentation of I as a finite direct sum of spaces $\mathcal{D} : I = \bigoplus_{i=1}^{r} u_i K[Z_i]$ is called a Stanley decomposition of I. Set $\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, ..., r\}$ and

$$\text{sdepth } I := \max\{\text{sdepth } (\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I\}.$$

The Stanley's Conjecture [11] says that sdepth $I \geq$ depth I. This is proved if either I is an intersection of four monomial prime ideals by [6, Theorem 2.6] and [8, Theorem 4.2], or I is the intersection of two monomial irreducible ideals by [10, Theorem 5.6], or a square free monomial ideal of $K[x_1, \dots, x_5]$ by [7] (a short exposition on this subject is given in [9]). It is the purpose of our paper to show that the Stanley's Conjecture holds for intersections of three monomial primary ideals (see Theorem 2.2).

1. Computing depth

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^{s} Q_i$ an irredundant primary decompostion of I, where the Q_i are monomial primary ideals. Set $P_i = \sqrt{Q_i}$. According to Lyubeznik [5] $\text{size}\, I$ is the number $v + (n - h) - 1$, where $h = \text{height } \sum_{j=1}^{s} Q_j$ and v is the minimum number t such that there exist $1 \leq j_1 < ... < j_t \leq s$ with

$$\sqrt{\sum_{k=1}^{t} Q_{j_k}} = \sqrt{\sum_{j=1}^{s} Q_j}.$$

In [5] it shows that $\text{depth}_S\, I \geq 1 + \text{size}\, I$.

In the study of the Stanley's Conjecture, we may always assume that $h = n$, that is $\sum_{i=1}^{s} P_i = m =: (x_1, \ldots, x_n)$, because each free variable on I increases depth and sdepth with 1.

Lemma 1.1. *Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^{3} Q_i$ an irredundant primary decomposition of I, where each Q_i is P_i - primary. Suppose that $P_i \neq m$ for all $i \in [3]$. Then*

(a) *If $Q_1 \subset Q_2 + Q_3$ and $P_1 \not\subset P_i$ for $i = 2, 3$, then* $\operatorname{depth}_S S/I = 1 + \min\{\dim S/(P_1 + P_2), \dim S/(P_1 + P_3)\}$.

(b) *If $Q_1 \subset Q_2 + Q_3$ and $P_1 \subset P_2$,$P_1 \not\subset P_3$, then* $\operatorname{depth}_S S/I = \min\{\dim S/P_2, 1 + \dim S/(P_1 + P_3)\}$.

(c) *If $Q_1 \subset Q_2 + Q_3$ and $P_1 \subset P_i$ for $i = 2, 3$ then* $\operatorname{depth}_S S/I = \min\{\dim S/P_2, \dim S/P_3\}$.

(d) *If $Q_i \not\subset \sum_{j=1,\ j\neq i}^{3} Q_j$, for all i then* $\operatorname{depth}_S S/I = 1$ *if and only if* $\operatorname{size} I = 1$.

(e) *If $Q_i \not\subset \sum_{j=1,\ j\neq i}^{3} Q_j$, for all i then* $\operatorname{depth}_S S/I = 2$ *if and only if* $\operatorname{size} I = 2$.

Proof. As $\operatorname{Ass}_S S/I = \{P_1, P_2, P_3\}$ we get $\operatorname{depth}_S S/I > 0$ by assumptions. We have the following exact sequences

(1)
$$0 \to \frac{S}{I} \to \frac{S}{Q_1 \cap Q_2} \oplus \frac{S}{Q_1 \cap Q_3} \to \frac{S}{Q_1} \to 0,$$

(2)
$$0 \to \frac{S}{Q_1 \cap Q_2} \to \frac{S}{Q_1} \oplus \frac{S}{Q_2} \to \frac{S}{Q_1 + Q_2} \to 0,$$

(3)
$$0 \to \frac{S}{Q_1 \cap Q_3} \to \frac{S}{Q_1} \oplus \frac{S}{Q_3} \to \frac{S}{Q_1 + Q_3} \to 0.$$

Apply Depth Lemma in (2) and (3). If P_1 is not properly contained in P_2 or P_3 then $\operatorname{depth} \frac{S}{Q_1 \cap Q_3} = 1 + \operatorname{depth} \frac{S}{Q_1 + Q_3}$ and $\operatorname{depth} \frac{S}{Q_1 \cap Q_2} = 1 + \operatorname{depth}_S \frac{S}{Q_1 + Q_2}$. If $P_1 \subset P_2$ then $\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2} \geq \operatorname{depth}_S \frac{S}{Q_2} = \dim \frac{S}{P_2}$. But $\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2} \leq \dim \frac{S}{Q_2}$, that is $\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2} = \dim \frac{S}{P_2}$. Similarly, $\operatorname{depth}_S \frac{S}{Q_1 \cap Q_3} = \dim \frac{S}{P_3}$ if $P_1 \subset P_3$.

The statements (a),(b), (c) follow if we show that

$$\operatorname{depth}_S S/I = \min\{\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2}, \operatorname{depth}_S \frac{S}{Q_1 \cap Q_3}\}.$$

If $\operatorname{depth}_S \frac{S}{Q_1} > \min\{\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2}, \operatorname{depth}_S \frac{S}{Q_1 \cap Q_3}\}$ then by Depth Lemma applied in (1) we get the above equality. If $\operatorname{depth}_S \frac{S}{Q_1} = \min\{\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2}, \operatorname{depth}_S \frac{S}{Q_1 \cap Q_3}\}$ then we get similarly $\operatorname{depth}_S S/I \geq \operatorname{depth}_S S/Q_1 = \operatorname{depth}_S S/P_1$. As $P_1 \in \operatorname{Ass} S/I$ then $\operatorname{depth}_S S/I \leq \dim S/P_1 = \operatorname{depth}_S S/Q_1$. Thus $\operatorname{depth}_S S/I = \operatorname{depth}_S \frac{S}{Q_1}$, which is enough.

(d) If $\operatorname{depth}_S S/I = 1$ then $2 = \operatorname{depth}_S I \geq 1 + \operatorname{size} I$, that is $1 \geq \operatorname{size} I \geq 0$. But $\operatorname{size} I \neq 0$ because the primary decomposition is irredundant. Conversely, if $\operatorname{size} I = 1$ then $v = 2$ and we may assume that $P_2 + P_3 = P_1 + P_2 + P_3 = m$. We consider the exact sequences

(4)
$$0 \to \frac{S}{I} \to \frac{S}{Q_1 \cap Q_2} \oplus \frac{S}{Q_3} \to \frac{S}{Q_3 + (Q_1 \cap Q_2)} \to 0,$$

(5)
$$0 \to \frac{S}{Q_1 \cap Q_2} \to \frac{S}{Q_1} \oplus \frac{S}{Q_2} \to \frac{S}{Q_1 + Q_2} \to 0.$$

From (5) we have $\operatorname{depth}_S \frac{S}{Q_1 \cap Q_2} = 1 + \operatorname{depth}_S \frac{S}{Q_1 + Q_2} \geq 1$ by Depth Lemma. Note that $\operatorname{depth}_S S/Q_3 \geq 1$ and $\operatorname{depth}_S \frac{S}{Q_3 + (Q_1 \cap Q_2)} = \operatorname{depth}_S \frac{S}{(Q_1+Q_3)\cap(Q_2+Q_3)} = 0$ because $\sqrt{Q_2 + Q_3} = m$, and $Q_1 \not\subset Q_2 + Q_3$. Thus Depth Lemma applied in (4) gives $\operatorname{depth}_S S/I = 1$.

(e) If $\operatorname{depth}_S S/I = 2$, then $\operatorname{depth}_S I = 3 \geq 1 + \operatorname{size} I$. But $\operatorname{size} I \leq 1$ was the subject of (d), so $\operatorname{size} I = 2$. Conversely, suppose that $\operatorname{size} I = 2$, that is $v = 3$. Then $P_i \not\subset \sum_{j=1,\ j\neq i}^{3} P_j$, for all i and by [4, Proposition 2.1] we get $\operatorname{depth}_S I \leq 3$. As $\operatorname{depth}_S I \geq 1 + \operatorname{size} I$ we get $\operatorname{depth}_S S/I = 2$. □

2. Stanley's depth

In this section we introduce a new way of splitting, inspired from [4], that helps us to prove the Stanley Conjecture when $I = \bigcap_{i=1}^{3} Q_i$ is an irredundant primary decomposition of I.

Theorem 2.1. *Let I be a monomial ideal and $I = Q_1 \cap Q_2$ an irredundant primary decomposition of I , where Q_i is P_i primary. Then the Stanley conjecture holds for I.*

Proof. As usual we my suppose that $P_1 + P_2 = m$. Also we may suppose that $P_i \neq m$ for all i, because otherwise $\operatorname{depth}_S I = 1$ and there exists nothing to show. Applying Depth Lemma in the above exact sequence (2) we get $\operatorname{depth}_S S/I = 1$, so $\operatorname{depth}_S I = 2 = 1 + \operatorname{size} I$. By [3, Theorem 3.1] we have $\operatorname{sdepth}_S I \geq \operatorname{depth}_S I$. □

Theorem 2.2. *Let I be a monomial ideal and $I = \bigcap_{i=1}^{3} Q_i$ an irredundant primary decomposition of I , where Q_i is P_i primary. Then the Stanley conjecture holds for I.*

Proof. We may suppose as above $P_1 + P_2 + P_3 = m$ and $P_i \neq m$ for all i. If $Q_i \not\subset \sum_{j=1,\ j\neq i}^{3} Q_j$, for all $i \in [3]$ we have according to Lemma 1.1 minimal depth that is $\operatorname{depth} I = 1 + \operatorname{size} I$. Then by [3, Theorem 3.1] we get $\operatorname{sdepth}_S I \geq \operatorname{depth}_S I$. Now suppose that $Q_1 \subset Q_2 + Q_3$. It follows that $\operatorname{size} I = 1$. If $P_1 + P_2 = m$ or $P_1 + P_3 = m$ then $\dim \frac{S}{Q_1+Q_2} = 0$ or $\dim \frac{S}{Q_1+Q_3} = 0$ therefore $\operatorname{depth}_S S/I = 1$ that is $\operatorname{depth}_S I = 2$. Then again we get $\operatorname{sdepth}_S I \geq 1 + \operatorname{size} I = 2 = \operatorname{depth}_S I$ by by [3, Theorem 3.1].

Otherwise $P_1 + P_2 \neq m \neq P_1 + P_3$. Let $P_1 = (x_1, ..., x_r)$ and $P_3 = (x_{e+1}, ..., x_t)$, $2 \leq r \leq n-1, e+1 \leq r$. If $r = 1$ then $Q_1 \subset Q_2$ or $Q_1 \subset Q_3$ because $Q_1 \subset Q_2 + Q_3$. This is false since the primary decomposition is irredundant. If $r = n$ then $P_1 = m$, which is not possible. If $e+1 > r$ then $Q_1 \subset Q_2$, also a contradiction. We will prove this case by induction on n. If $n = 3$, then $\operatorname{sdepth}_S I \geq 1 + \operatorname{size} I = 2 \geq \operatorname{depth}_S I$, because I is not principal. Assume now $n > 3$. We set $S' = K[x_1, ..., x_r]$, $\bar{S} := K[x_1, ..., x_e, x_{r+1}, ..., x_n]$ and $J_3 = \bigoplus_w w((I : w) \cap \bar{S})$, where w runs in the finite set of monomials of $K[x_{e+1}, ..., x_r] \setminus Q_3$.

We claim that $I = Q_1 \cap Q_2 \cap (Q_3 \cap S')S \oplus J_3$. It is enough to see the inclusion " $\subset$ ". Let $a \in I$ be a monomial, then $a = uv$, where $u \in \bar{S}$ and $v \in K[x_{e+1}, ..., x_r]$ are monomials. If $v \notin Q_3$ then $u \in (I : v) \cap \bar{S}$, so $a \in J_3$. If $v \in Q_3$ then $a \in (Q_3 \cap S')S$. As $a \in I$ we get $a \in Q_1 \cap Q_2$ therefore a $\in Q_1 \cap Q_2 \cap (Q_3 \cap S')S$. The above sum is direct. Indeed, let $a = uv \in Q_1 \cap Q_2 \cap (Q_3 \cap S')S \cap J_3$ be as above. Then $v \notin Q_3$ because $a \in J_3$. But v must be in $(Q_3 \cap S')S$. Contradiction!

The ideal $I' := Q_1 \cap Q_2 \cap (Q_3 \cap S')S \subset P_1 + P_2 \neq m$ and so is an extension of an ideal from less than n-variables and we may apply the induction hypothesis for I', that is $\operatorname{sdepth}_S I' \geq \operatorname{depth}_S I'$. Since $\operatorname{sdepth}_S I \geq \min\{\operatorname{sdepth}_S I', \{\operatorname{sdepth}_{\bar{S}}((I : w) \cap \bar{S})\}_w\}$ it remains to show that $\operatorname{depth}_S I' \geq \operatorname{depth}_S I$ and $\operatorname{depth}_{\bar{S}}((I : w) \cap \bar{S}) \geq \operatorname{depth}_S I$, applying again the induction hypothesis since $\bar{S}$ has less than n-variables. The first inequality follows because $\dim S/(P_3 \cap S')S \geq \dim S/P_3$, $\dim S/(P_1 + (P_3 \cap S')S) \geq \dim S/P_1 + P_3$ using Lemma 1.1 (a), (b), (c).

For the second inequality note that for $w \notin Q_1 \cup Q_2 \cup Q_3$ we have $(Q_i : w)$ primary and so $L_i := (Q_i : w) \cap \bar{S}$ is $\bar{P}_i := P_i \cap \bar{S}$-primary too. We have $\dim \bar{S}/\bar{P}_i = \dim S/P_i$ for $i = 1, 3$ because $(x_{e+1}, \ldots, x_r) \subset P_1 \cap P_3$. Thus $\dim \bar{S}/(\bar{P}_1 + \bar{P}_i) = \dim S/(P_1 + P_i)$ for all $i = 2, 3$. Using Lemma 1.1 we are done because $\dim S/P_2$ appears in the formulas only when $P_1 \subset P_2$, that is when $\dim \bar{S}/\bar{P}_2 = \dim S/P_2$.

If $w \in Q_2 \setminus (Q_1 \cup Q_3)$ then

$$\operatorname{depth}_{\bar{S}} \bar{S}/(L_1 \cap L_3) = 1 + \dim \bar{S}/(\bar{P}_1 + \bar{P}_3) = 1 + \dim S/(P_1 + P_3) \geq \operatorname{depth}_S S/I$$

by the same lemma, the only problem could appear when $P_1 \subset P_3$, but in this case

$$\dim \bar{S}/(\bar{P}_1 + \bar{P}_3) = \dim S/(P_1 + P_3) = \bar{S}/\bar{P}_3 = \dim S/P_3$$

and it follows

$$\operatorname{depth}_{\bar{S}} \bar{S}/(L_1 \cap L_3) = 1 + \dim \bar{S}/(\bar{P}_1 + \bar{P}_3) > \dim S/P_3 \geq \operatorname{depth}_S S/I.$$

If $w \in (Q_1 \cap Q_2) \setminus Q_3$ then $\operatorname{depth}_{\bar{S}} \bar{S}/L_3 = \dim S/P_3 \geq \operatorname{depth}_S S/I$ by [1]. $\square$

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Andrei Zarojanu , Faculty of Mathematics and Computer Sciences, University of Bucharest, Str. Academiei 14, Bucharest, Romania

E-mail address: `andrei.zarojanu@yahoo.com`