On generalized Flett's mean value theorem[1]

Jana MOLNÁROVÁ
Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University in Košice, Jesenná 5, 040 01 Košice, Slovakia,
E-mail address: jana.molnarova88@gmail.com

Abstract. We present a new proof of generalized Flett's mean value theorem due to Pawlikowska (from 1999) using only the original Flett's mean value theorem. Also, a Trahan-type condition is established in general case.

Key words and phrases. Flett's mean value theorem, real function, differentiability, Taylor polynomial

1 Introduction

Mean value theorems play an essential role in analysis. The simplest form of the mean value theorem due to Rolle is well-known.

Theorem 1.1 (Rolle's mean value theorem) *If $f : \langle a, b\rangle \to \mathbb{R}$ is continuous on $\langle a, b\rangle$, differentiable on (a, b) and $f(a) = f(b)$, then there exists a number $\eta \in (a, b)$ such that $f'(\eta) = 0$.*

A geometric interpretation of Theorem 1.1 states that if the curve $y = f(x)$ has a tangent at each point in (a, b) and $f(a) = f(b)$, then there exists a point $\eta \in (a, b)$ such that the tangent at $(\eta, f(\eta))$ is parallel to the x-axis. One may ask a natural question: *What if we remove the boundary condition $f(a) = f(b)$?* The answer is well-known as the Lagrange's mean value theorem. For the sake of brevity put

$$ {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) = \frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)}, \quad n \in \mathbb{N} \cup \{0\}, $$

for functions f, g defined on $\langle a, b\rangle$ (for which the expression has a sense). If $g^{(n)}(b) - g^{(n)}(a) = b - a$, we simply write ${}_a^b\mathcal{K}\left(f^{(n)}\right)$.

Theorem 1.2 (Lagrange's mean value theorem) *If $f : \langle a, b\rangle \to \mathbb{R}$ is continuous on $\langle a, b\rangle$ and differentiable on (a, b), then there exists a number $\eta \in (a, b)$ such that $f'(\eta) = {}_a^b\mathcal{K}(f)$.*

Clearly, Theorem 1.2 reduces to Theorem 1.1 if $f(a) = f(b)$. Geometrically, Theorem 1.2 states that given a line ℓ joining two points on the graph of a differentiable function f, namely $(a, f(a))$ and $(b, f(b))$, then there exists a point $\eta \in (a, b)$ such that the tangent at $(\eta, f(\eta))$ is parallel to the given line ℓ.

[1] *Mathematics Subject Classification (2010):* Primary 26A24

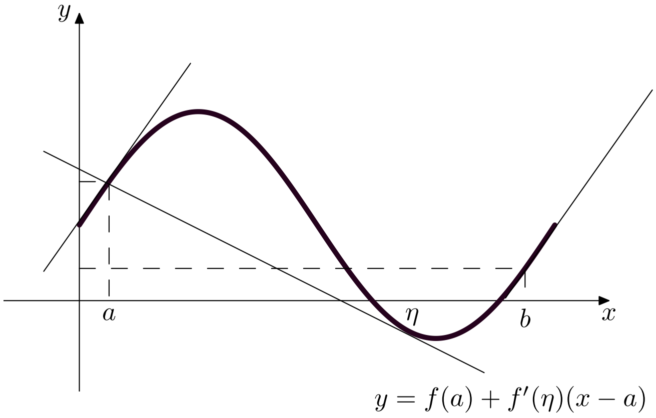

Figure 1: Geometric interpretation of Flett's mean value theorem

In connection with Theorem 1.1 the following question may arise: *Are there changes if in Theorem 1.1 the hypothesis $f(a) = f(b)$ refers to higher-order derivatives?* T. M. Flett, see [3], first proved in 1958 the following answer to this question for $n = 1$ which gives a variant of Lagrange's mean value theorem with Rolle-type condition.

Theorem 1.3 (Flett's mean value theorem) *If $f : \langle a, b\rangle \to \mathbb{R}$ is a differentiable function on $\langle a, b\rangle$ and $f'(a) = f'(b)$, then there exists a number $\eta \in (a, b)$ such that*

$$f'(\eta) = {}_a^\eta\mathcal{K}(f). \tag{1}$$

Flett's original proof, see [3], uses Theorem 1.1. A slightly different proof which uses Fermat's theorem instead of Rolle's can be found in [10]. There is a nice geometric interpretation of Theorem 1.3: if the curve $y = f(x)$ has a tangent at each point in $\langle a, b\rangle$ and if the tangents at $(a, f(a))$ and $(b, f(b))$ are parallel, then there exists a point $\eta \in (a, b)$ such that the tangent at $(\eta, f(\eta))$ passes through the point $(a, f(a))$, see Figure 1.

Similarly as in the case of Rolle's theorem we may ask about possibility to remove the boundary assumption $f'(a) = f'(b)$ in Theorem 1.3. As far as we know the first result of that kind is given in book [11].

Theorem 1.4 (Riedel-Sahoo) *If $f : \langle a, b\rangle \to \mathbb{R}$ is a differentiable function on $\langle a, b\rangle$, then there exists a number $\eta \in (a, b)$ such that*

$$f'(\eta) = {}_a^\eta\mathcal{K}(f) + {}_a^b\mathcal{K}(f') \cdot \frac{\eta - a}{2}.$$

We point out that there are also other sufficient conditions guaranteeing the existence of a point $\eta \in (a, b)$ satisfying (1). First such a condition was published in Trahan's work [13]. An interesting idea is presented in paper [12] where the discrete and integral arithmetic mean is used. We suppose that this idea may be further generalized for the case of means studied e.g. in [5, 6, 7].

In recent years there has been renewed interest in Flett's mean value theorem. Among the many other extensions and generalizations of Theorem 1.3, see e.g. [1], [2], [4], [9], we focus on that of Iwona Pawlikovska [8] solving the question of Zsolt Pales raised at the 35-th International Symposium on Functional Equations held in Graz in 1997.

Theorem 1.5 (Pawlikowska) *Let f be n-times differentiable on $\langle a,b\rangle$ and $f^{(n)}(a)=f^{(n)}(b)$. Then there exists $\eta\in(a,b)$ such that*

$$f(\eta)-f(a)=\sum_{i=1}^{n}\frac{(-1)^{i+1}}{i!}(\eta-a)^i f^{(i)}(\eta). \tag{2}$$

Observe that the Pawlikowska's theorem has a close relationship with the n-th Taylor polynomial of f. Indeed, for

$$T_n(f,x_0)(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

the Pawlikowska's theorem has the following very easy form $f(a)=T_n(f,\eta)(a)$.

Pawlikowska's proof follows up the original idea of Flett, see [3], considering the auxiliary function

$$G_f(x)=\begin{cases} g^{(n-1)}(x), & x\in(a,b\rangle \\ \frac{1}{n}f^{(n)}(a), & x=a \end{cases}$$

where $g(x)={}_a^x\mathcal{K}(f)$ for $x\in(a,b\rangle$ and using Theorem 1.1. In what follows we provide a different proof of Theorem 1.5 which uses only iterations of an appropriate auxiliary function and Theorem 1.3. In Section 3 we give a general version of Trahan condition, cf. [13] under which Pawlikowska's theorem holds.

2 New proof of Pawlikowska's theorem

The key tool in our proof consists in using the auxiliary function

$$\varphi_k(x)=xf^{(n-k+1)}(a)+\sum_{i=0}^{k}\frac{(-1)^{i+1}}{i!}(k-i)(x-a)^i f^{(n-k+i)}(x),\quad k=1,2,\ldots,n.$$

Running through all indices $k=1,2,\ldots,n$ we show that its derivative fulfills assumptions of Flett's mean value theorem and it implies the validity of Flett's mean value theorem for l-th derivative of f, where $l=n-1,n-2,\ldots,1$.

Indeed, for $k=1$ we have $\varphi_1(x)=-f^{(n-1)}(x)+xf^{(n)}(a)$ and $\varphi_1'(x)=-f^{(n)}(x)+f^{(n)}(a)$. Clearly, $\varphi_1'(a)=0=\varphi_1'(b)$, so applying the Flett's mean value theorem for φ_1 on $\langle a,b\rangle$ there exists $u_1\in(a,b)$ such that $\varphi_1'(u_1)(u_1-a)=\varphi_1(u_1)-\varphi_1(a)$, i.e.

$$f^{(n-1)}(u_1)-f^{(n-1)}(a)=(u_1-a)f^{(n)}(u_1). \tag{3}$$

Then for $\varphi_2(x)=-2f^{(n-2)}(x)+(x-a)f^{(n-1)}(x)+xf^{(n-1)}(a)$ we get

$$\varphi_2'(x)=-f^{(n-1)}(x)+(x-a)f^{(n)}(x)+f^{(n-1)}(a)$$

and $\varphi_2'(a)=0=\varphi_2'(u_1)$ by (3). So, by Flett's mean value theorem for φ_2 on $\langle a,u_1\rangle$ there exists $u_2\in(a,u_1)\subset(a,b)$ such that $\varphi_2'(u_2)(u_2-a)=\varphi_2(u_2)-\varphi_2(a)$, which is equivalent to

$$f^{(n-2)}(u_2)-f^{(n-2)}(a)=(u_2-a)f^{(n-1)}(u_2)-\frac{1}{2}(u_2-a)^2f^{(n)}(u_2).$$

Continuing this way after $n-1$ steps, $n \geq 2$, there exists $u_{n-1} \in (a,b)$ such that

$$f'(u_{n-1}) - f'(a) = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!}(u_{n-1}-a)^i f^{(i+1)}(u_{n-1}). \tag{4}$$

Considering the function φ_n we get

$$\begin{aligned}\varphi_n'(x) &= -f'(x) + f'(a) + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!}(x-a)^i f^{(i)}(x) \\ &= f'(a) + \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{i!}(x-a)^i f^{(i+1)}(x).\end{aligned}$$

Clearly, $\varphi_n'(a) = 0 = \varphi_n'(u_{n-1})$ by (4). Then by Flett's mean value theorem for φ_n on $\langle a, u_{n-1}\rangle$ there exists $\eta \in (a, u_{n-1}) \subset (a,b)$ such that

$$\varphi_n'(\eta)(\eta - a) = \varphi_n(\eta) - \varphi_n(a). \tag{5}$$

Since

$$\varphi_n'(\eta)(\eta-a) = (\eta-a)f'(a) + \sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!}(\eta-a)^i f^{(i)}(\eta)$$

and

$$\varphi_n(\eta) - \varphi_n(a) = (\eta-a)f'(a) - n(f(\eta)-f(a)) + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}(n-i)(\eta-a)^i f^{(i)}(\eta),$$

the equality (5) yields

$$\begin{aligned}-n(f(\eta)-f(a)) &= \sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!}(\eta-a)^i f^{(i)}(\eta)\left(1 + \frac{n-i}{i}\right) \\ &= n\sum_{i=1}^{n} \frac{(-1)^i}{i!}(\eta-a)^i f^{(i)}(\eta),\end{aligned}$$

which corresponds to (2). □

It is also possible to state the result which no longer requires any endpoint conditions. If we consider the auxiliary function

$$\psi_k(x) = \varphi_k(x) + \frac{(-1)^{k+1}}{(k+1)!}(x-a)^{k+1} \cdot {}_a^b\mathcal{K}\left(f^{(n)}\right),$$

then the analogous way as in the proof of Theorem 1.5 yields the following result also given in [8] including Riedel-Sahoo's Theorem 1.4 as a special case ($n=1$).

Theorem 2.1 *If $f : \langle a,b\rangle \to \mathbb{R}$ is n-times differentiable on $\langle a,b\rangle$, then there exists $\eta \in (a,b)$ such that*

$$f(a) = T_n(f,\eta)(a) + \frac{(a-\eta)^{n+1}}{(n+1)!} \cdot {}_a^b\mathcal{K}\left(f^{(n)}\right).$$

Note that the case $n = 1$ is used to extend Flett's mean value theorem for holomorphic functions, see [1]. An easy generalization of Pawlikowska's theorem involving two functions is the following one.

Theorem 2.2 *Let f, g be n-times differentiable on $\langle a, b\rangle$ and $g^{(n)}(a) \neq g^{(n)}(b)$. Then there exists $\eta \in (a, b)$ such that*

$$f(a) - T_n(f, \eta)(a) = {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot [g(a) - T_n(g, \eta)(a)].$$

This may be verified applying the Pawlikowska's theorem to the auxiliary function

$$h(x) = f(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot g(x), \quad x \in \langle a, b\rangle.$$

A different proof will be presented in the following section.

3 A Trahan-type condition

In [13] Trahan gave a sufficient condition for the existence of a point $\eta \in (a, b)$ satisfying (1) under the assumptions of differentiability of f on $\langle a, b\rangle$ and inequality

$$(f'(b) - {}_a^b\mathcal{K}(f)) \cdot (f'(a) - {}_a^b\mathcal{K}(f)) \geq 0. \tag{6}$$

Modifying the Trahan's original proof using the Pawlikowska's auxiliary function G_f we are able to state the following condition for validity (2).

Theorem 3.1 *Let f be n-times differentiable on $\langle a, b\rangle$ and*

$$\left(\frac{f^{(n)}(a)(a-b)^n}{n!} + M_f\right)\left(\frac{f^{(n)}(b)(a-b)^n}{n!} + M_f\right) \geq 0,$$

where $M_f = T_{n-1}(f, b)(a) - f(a)$. Then there exists $\eta \in (a, b)$ satisfying (2).

Proof. Since G_f is continuous on $\langle a, b\rangle$ and differentiable on $(a, b\rangle$ with

$$G_f'(x) = g^{(n)}(x) = \frac{(-1)^n n!}{(x-a)^{n+1}}\left(f(x) - f(a) + \sum_{i=1}^{n} \frac{(-1)^i}{i!}(x-a)^i f^{(i)}(x)\right),$$

for $x \in (a, b\rangle$, see [8], then

$$\begin{aligned}(G_f(b) - G_f(a))G_f'(b) &= \left(g^{(n-1)}(b) - \frac{1}{n}f^{(n)}(a)\right) g^{(n)}(b) \\ &= -\frac{n!(n-1)!}{(b-a)^{2n+1}}\left(\frac{f^{(n)}(a)(a-b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right) \\ &\cdot \left(\frac{f^{(n)}(b)(a-b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right) \leq 0.\end{aligned}$$

According to Lemma 1 in [13] there exists $\eta \in (a, b)$ such that $G_f'(\eta) = 0$ which corresponds to (2). □

Now we provide an alternative proof of Theorem 2.2 which does not use original Pawlikowska's theorem.

Proof of Theorem 2.2. For $x \in (a,b\rangle$ put $\varphi(x) = {}_a^x\mathcal{K}(f)$ and $\psi(x) = {}_a^x\mathcal{K}(g)$. Define the auxiliary function F as follows

$$F(x) = \begin{cases} \varphi^{(n-1)}(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n-1)}(x), & x \in (a,b\rangle \\ \frac{1}{n}\left[f^{(n)}(a) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot g^{(n)}(a)\right], & x = a. \end{cases}$$

Clearly, F is continuous on $\langle a,b\rangle$, differentiable on $(a,b\rangle$ and for $x \in (a,b\rangle$ there holds

$$\begin{aligned} F'(x) &= \varphi^{(n)}(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n)}(x) \\ &= \frac{(-1)^n n!}{(x-a)^{n+1}}\left[T_n(f,x)(a) - f(a) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot (T_n(g,x)(a) - g(a))\right]. \end{aligned}$$

Then

$$F'(b)[F(b) - F(a)] = -\frac{n!(n-1)!}{(b-a)^{2n+1}}(F(b) - F(a))^2 \leq 0,$$

and by Lemma 1 in [13] there exists $\eta \in (a,b)$ such that $F'(\eta) = 0$, i.e.,

$$f(a) - T_n(f,\eta)(a) = {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot (g(a) - T_n(g,\eta)(a)).$$

□

References

[1] R. Davitt, R. C. Powers, T. Riedel, P. K. Sahoo, Flett mean value theorem for holomorphic functions, Math. Mag. 72 (1999) 304–307.

[2] M. Das, T. Riedel, P. K. Sahoo, Flett's mean value theorem for approximately differentiable functions, Rad. Mat. 10 (2001) 157–164.

[3] T. M. Flett, A mean value theorem, Math. Gazette 42 (1958) 38-39.

[4] I. Jędrzejewska, B. Szkopińska, On generalizations of Flett's Theorem, Real Anal. Exchange 30(1) (2004) 75–86.

[5] O. Hutník, On Hadamard type inequalities for generalized weighted quasi-arithmetic means, J. Ineq. Pure Appl. Math. 7(3) (2006), article 96.

[6] J. Haluška, O. Hutník, Some inequalities involving integral means, Tatra Mt. Math. Publ. 35 (2007), 131–146.

[7] O. Hutník, Some integral inequalities of Hölder and Minkowski type, Colloq. Math. 108(2) (2007), 247–261.

[8] I. Pawlikowska, An extension of a theorem of Flett, Demonstratio Math. 32 (1999) 281-286.

[9] R. C. Powers, T. Riedel, P. K. Sahoo, Flett's mean value theorem on topological vector spaces, Internat. J. Math. Math. Sci. 27 (2001) 689–694.

[10] T. L. Rădulescu, V. D. Rădulescu, T. Andreescu, Problems in Real Analysis: Advanced Calculus on the Real Axis, Springer Verlag, 2009.

[11] P. K. Sahoo, T. Riedel, Mean Value Theorems and Functional Equations, World Scientific, River Edge, NJ, 1998.

[12] J. Tong, On Flett's mean value theorem, Internat. J. Math. Ed. Sci. Tech. 35 (2004) 936-941.

[13] D. H. Trahan, A new type of mean value theorem, Math. Mag. 39 (1966) 264-268.