On Toeplitz localization operators

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Abstract. We present a unified approach to study properties of Toeplitz localization operators based on the Calderón and Gabor reproducing formula. We show that these operators with functional symbols on a plane domain may be viewed as certain pseudodifferential operators (with symbols on a line, or certain compound symbols).

1 Introduction and preliminaries

A starting point for the construction of time-frequency localization or filter operators are the famous reproducing formulas of Calderón (in wavelet analysis) and of Gabor (in time-frequency analysis). In this paper we will work with both the reproducing formulas and therefore we introduce the following unified notation. Write

$$f = \int_G \langle f, \Psi_\zeta \rangle \Psi_\zeta \, \mathrm{d}\zeta,$$

where G denotes either the half-plane $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}$, or the whole plane $\mathbb{R}^2$. In the first (wavelet) case $\zeta = (u, v)$, $u > 0$, $v \in \mathbb{R}$, and

$$\psi_{u,v}(x) = \frac{1}{\sqrt{u}} \, \psi\left(\frac{x - v}{u}\right)$$

is the action of the group $\mathbb{R}^2_+$ on $L_2(\mathbb{R})$, where $\mathbb{R}^2_+$ is equipped with the hyperbolic measure $\mathrm{d}\zeta = u^{-2} \, \mathrm{d}u\mathrm{d}v$. Here (the real-valued) *admissible wavelet* is the function $\psi \in L_2(\mathbb{R})$ satisfying the condition

$$\int_{\mathbb{R}_+} |\hat{\psi}(t\xi)|^2 \frac{\mathrm{d}t}{t} = 1$$

for almost every $\xi \in \mathbb{R}$, where $\hat{\psi}$ stands for the Fourier transform (in the time-frequency convention) $\mathscr{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ given by

$$\mathscr{F}\{f\}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) \, \mathrm{e}^{-2\pi \mathrm{i} x\xi} \, \mathrm{d}x.$$

In the second (time-frequency) case $\zeta = (q, p)$, $q, p \in \mathbb{R}$ with

$$\phi_{q,p}(x) = \mathrm{e}^{2\pi \mathrm{i} p x} \phi(x - q)$$

being the action of $\mathbb{R}^2$ on $L_2(\mathbb{R})$, and $\mathrm{d}\zeta = \mathrm{d}q\mathrm{d}p$ being the measure on $\mathbb{R}^2$. The *admissible window* is the function $\phi \in L_2(\mathbb{R})$ satisfying $\|\phi\|_{L_2(\mathbb{R})} = 1$. In what

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follows the symbol Ψ always means either an admissible wavelet $\psi \in L_2(\mathbb{R})$, or an admissible window $\phi \in L_2(\mathbb{R})$.

If $L_2(G, \mathrm{d}\zeta)$ denotes the Hilbert space of all square-integrable complex-valued functions on G, then for a fixed $\Psi \in L_2(\mathbb{R})$ the functions $W_\Psi f$ on G of the form

$$(W_\Psi f)(\zeta) = \langle f, \Psi_\zeta \rangle, \quad f \in L_2(\mathbb{R}),$$

form a reproducing kernel Hilbert space $W_\Psi(L_2(\mathbb{R}))$. Then the transform $W_\Psi : L_2(\mathbb{R}) \to L_2(G, \mathrm{d}\zeta)$ is an isometry, and the integral operator $P_\Psi : L_2(G, \mathrm{d}\zeta) \to L_2(G, \mathrm{d}\zeta)$ given by

$$(P_\Psi F)(\eta) = \int_G F(\zeta) \langle \Psi_\eta, \Psi_\zeta \rangle \, \mathrm{d}\zeta, \quad F \in L_2(G, \mathrm{d}\zeta),$$

is the orthogonal projection onto $W_\Psi(L_2(\mathbb{R}))$, where $\langle \Psi_\eta, \Psi_\zeta \rangle$ is the reproducing kernel in $W_\Psi(L_2(\mathbb{R}))$. For a given bounded function a on G define the *Toeplitz localization operator* T_a^Ψ with symbol a as follows

$$T_a^\Psi : f \in W_\Psi(L_2(\mathbb{R})) \longmapsto P_\Psi(af) \in W_\Psi(L_2(\mathbb{R})).$$

In wavelet case the Toeplitz operator T_a^ψ is usually called the *Calderón-Toeplitz operator*, whereas in the case of time-frequency analysis the operator T_a^ϕ is called the *Gabor-Toeplitz operator*.

In this paper we underline some interesting features of spaces of transforms $W_\Psi(L_2(\mathbb{R}))$ and Toeplitz localization operators acting on them. In fact, with the above notation we provide a unified approach to both cases and give a more natural construction of unitary operators which does not use the decomposition of $L_2(G, \mathrm{d}\zeta)$ onto spaces $W_\Psi(L_2(\mathbb{R}))$ as it was done in [4] and [5]. Indeed, in Section 2 according to the general scheme presented in [7] we give the construction of the unitary operator R_Ψ which is an exact analog of the Bargmann transform mapping the Fock space $F_2(\mathbb{C}^n)$ of Gaussian square-integrable entire functions on $\mathbb{C}^n$ onto $L_2(\mathbb{R}^n)$, see [1]. Then, via R_Ψ, the Toeplitz localization operators $T_a^\Psi : W_\Psi(L_2(\mathbb{R})) \to W_\Psi(L_2(\mathbb{R}))$ can be identified with certain pseudodifferential operators

$$\mathfrak{C}_a^\Psi := R_\Psi T_a^\Psi R_\Psi^* : L_2(\mathbb{R}) \to L_2(\mathbb{R}).$$

This passing from T_a^Ψ to $\mathfrak{C}_a^\Psi$ is nothing but an analog of the Berezin reducing of (Toeplitz) operators with anti-Wick symbols on the Fock space $F_2(\mathbb{C}^n)$ to Weyl pseudodifferential operators on $L_2(\mathbb{R}^n)$, see [2] for further details.

In particular, in Section 3.1 the above mentioned observation is applied to operator symbols $a(r, s) : G \to \mathbb{C}$ that are only depending on the variable r. In this case (cf. Theorem 3.1) the operator $\mathfrak{C}_a^\Psi$ is simply a multiplication operator with explicitly computable symbol γ_a^Ψ. As a consequence the boundedness (also for the case where a is unbounded!) and the spectrum of T_a^Ψ is precisely characterized in terms of function γ_a^Ψ. Moreover, for a fixed $\Psi \in L_2(\mathbb{R})$ the space of Toeplitz localization operators T_a^Ψ with bounded symbols a depending only on r generates a commutative C^*-sub-algebra of the Toeplitz C^*-algebra. In Theorem 3.6 the commutative C^*-algebra generated by operators T_a^Ψ with symbols $a(r, s) = \alpha(r)$ and α in a certain space of piecewise constant functions is shown to be isometrically isomorphic to an explicitly given algebra of continuous functions.

In Section 3.2 the case of an operator symbol $a(r,s) = \beta(s)$ depending only on the second variable s of G is studied. In this case $\mathfrak{C}_a^\Psi$ has the form of a certain integral operator on $L_2(\mathbb{R})$. Finally, Section 3.3 treats the mixed case $a(r,s) = \alpha(r)\beta(s)$ which leads to pseudodifferential operators $\mathfrak{C}_a^\Psi$ with an explicitly computable compound (double) symbol. These results provide an interesting tool for further study of Toeplitz localization operators via investigating their unitary equivalent images in the classes of pseudodifferential operators.

2 Bargmann-type transform

In what follows let $\Psi \in L_2(\mathbb{R})$ be fixed. In order to construct unitary operators which will be used to study Toeplitz localization operators, we represent the Hilbert space $L_2(G, \mathrm{d}\zeta)$ as a tensor product in the form

$$L_2(G, \mathrm{d}\zeta) = L_2(G_1, \mathrm{d}\zeta_1) \otimes L_2(G_2, \mathrm{d}\zeta_2),$$

where $G_1 = \mathbb{R}_+$, $G_2 = \mathbb{R}$ with $\mathrm{d}\zeta_1 = u^{-2}\,\mathrm{d}u$, $\mathrm{d}\zeta_2 = \mathrm{d}v$ in the first (wavelet) case, and $G_1 = G_2 = \mathbb{R}$ with $\mathrm{d}\zeta_1 = \mathrm{d}q$, $\mathrm{d}\zeta_2 = \mathrm{d}p$ in the second (time-frequency) case, respectively. Introduce the unitary operator

$$U_\Psi : L_2(G, \mathrm{d}\zeta) = L_2(G_1, \mathrm{d}\zeta_1) \otimes L_2(G_2, \mathrm{d}\zeta_2) \to L_2(G_1, \mathrm{d}\zeta_1) \otimes L_2(G_2, \mathrm{d}\zeta_2)$$

given by $U_\Psi = (I \otimes \mathscr{F}^{\pm 1})$, where the Fourier transform $\mathscr{F} = \mathscr{F}^{+1}$ corresponds to wavelet case, and the inverse Fourier transform $\mathscr{F}^{-1}$ corresponds to time-frequency case. The image Δ_Ψ of the space $W_\Psi(L_2(\mathbb{R}))$ under the mapping U_Ψ consists of all functions $F(z,\omega) = f(\omega)\ell_\Psi(z,\omega)$, where $f \in L_2(\mathbb{R})$ and

$$\ell_\psi(u,\omega) = \sqrt{u}\,\overline{\hat{\psi}(u\omega)} \quad \text{and} \quad \ell_\phi(q,\omega) = \overline{\phi(\omega - q)},$$

respectively. Clearly, for each $\omega \in G_2$ holds

$$\ell_\Psi(\cdot,\omega) \in L_2(G_1, \mathrm{d}\zeta_1) \ \text{ with } \ \|\ell_\Psi(\cdot,\omega)\|_{L_2(G_1,\,\mathrm{d}\zeta_1)} = 1,$$

and thus we obviously have

$$\|F(z,\omega)\|_{\Delta_\Psi} = \|f(\omega)\|_{L_2(G_2,\,\mathrm{d}\zeta_2)}.$$

Then the operator $\Lambda_\Psi : L_2(G, \mathrm{d}\zeta) \to \Delta_\Psi$ given by $\Lambda_\Psi = U_\Psi P_\Psi U_\Psi^*$ has the explicit form

$$(\Lambda_\Psi F)(z,\omega) = \ell_\Psi(z,\omega) \int_{G_1} F(t,\omega)\,\overline{\ell_\Psi(t,\omega)}\,\mathrm{d}\zeta_1(t).$$

Thus, $\operatorname{Im}\Lambda_\Psi = \Delta_\Psi$. Moreover, $\Lambda_\Psi^2 = \Lambda_\Psi$, and Λ_Ψ is obviously self-adjoint. Introduce the isometric imbedding $Q_\Psi : L_2(G_2, \mathrm{d}\zeta_2) \to \Delta_\Psi$ by the rule

$$(Q_\Psi f)(z,\omega) = f(\omega)\ell_\Psi(z,\omega).$$

Then the adjoint operator $Q_\Psi^* : L_2(G, \mathrm{d}\zeta) \to L_2(G_2, \mathrm{d}\zeta_2)$ is given by

$$(Q_\Psi^* F)(\xi) = \int_{G_1} F(t,\xi)\,\overline{\ell_\Psi(t,\xi)}\,\mathrm{d}\zeta_1(t),$$

$$\begin{tikzcd}
L_2(G_2, \mathrm{d}\zeta_2) \arrow[r, "Q_\Psi"] \arrow[d, "I"] & L_2(G, \mathrm{d}\zeta) \arrow[r, "U_\Psi^*"] \arrow[d, "\Lambda_\Psi"] & L_2(G, \mathrm{d}\zeta) \arrow[d, "P_\Psi"] \\
L_2(G_2, \mathrm{d}\zeta_2) & \Delta_\Psi \arrow[l, "Q_\Psi^*"'] & W_\Psi(L_2(\mathbb{R})) \arrow[l, "U_\Psi"']
\end{tikzcd}$$

Figure 1: Relationships among the constructed unitary operators

and it is easy to verify that the operators Q_Ψ and Q_Ψ^* provide the following decomposition of identity on $L_2(G_2, \mathrm{d}\zeta_2)$ and of orthogonal projection Λ_Ψ, i.e.,

$$\begin{aligned} Q_\Psi^* Q_\Psi &= I \quad : L_2(G_2, \mathrm{d}\zeta_2) \to L_2(G_2, \mathrm{d}\zeta_2), \\ Q_\Psi Q_\Psi^* &= \Lambda_\Psi : L_2(G, \mathrm{d}\zeta) \to \Delta_\Psi. \end{aligned}$$

The whole situation of constructed operators is described on Figure 1.

Theorem 2.1 *The operator $R_\Psi = Q_\Psi^* U_\Psi$ maps the space $L_2(G, \mathrm{d}\zeta)$ onto $L_2(G_2, \mathrm{d}\zeta_2)$, and the restriction*

$$R_\Psi \,|_{W_\Psi(L_2(\mathbb{R}))} \colon W_\Psi(L_2(\mathbb{R})) \to L_2(G_2, \mathrm{d}\zeta_2)$$

is an isometrical isomorphism. The adjoint

$$R_\Psi^* = U_\Psi^* Q_\Psi : L_2(G_2, \mathrm{d}\zeta_2) \to W_\Psi(L_2(\mathbb{R})) \subset L_2(G, \mathrm{d}\zeta)$$

is an isometrical isomorphism of $L_2(G_2, \mathrm{d}\zeta_2)$ onto the subspace $W_\Psi(L_2(\mathbb{R}))$ of the space $L_2(G, \mathrm{d}\zeta)$. Moreover,

$$\begin{aligned} R_\Psi R_\Psi^* &= I \quad : L_2(G_2, \mathrm{d}\zeta_2) \to L_2(G_2, \mathrm{d}\zeta_2), \\ R_\Psi^* R_\Psi &= P_\Psi : L_2(G, \mathrm{d}\zeta) \to W_\Psi(L_2(\mathbb{R})). \end{aligned}$$

In what follows we show that the Bargmann-type transform R_Ψ essentially simplifies the previous computations made in [4] and [5], and enables to obtain many interesting results for the Toeplitz localization operators which both cases share in common in a more transparent way.

3 Toeplitz localization operators

For each function $a(r, s) \in L_\infty(G, \mathrm{d}\zeta)$ consider the Toeplitz localization operator (TLO, for short)

$$T_a^\Psi : f \in W_\Psi(L_2(\mathbb{R})) \longmapsto P_\Psi(af) \in W_\Psi(L_2(\mathbb{R})).$$

In our original publications [4] and [5] the operators R_Ψ and R_Ψ^* were defined after applying the second unitary operator, let us say

$$V_\Psi : L_2(G_1, \mathrm{d}\zeta_1) \otimes L_2(G_2, \mathrm{d}\zeta_2) \to L_2(G_1, \mathrm{d}x) \otimes L_2(G_2, \mathrm{d}y)$$

given by

$$V_\psi : F(u, \omega) \mapsto \frac{\sqrt{|y|}}{x} F\left(\frac{x}{|y|}, y\right) \quad \text{and} \quad V_\phi : F(q, \omega) \mapsto F(y - x, y).$$

Under the operator $V_\Psi U_\Psi : L_2(G, \mathrm{d}\zeta) \to L_2(G_1, \mathrm{d}x) \otimes L_2(G_2, \mathrm{d}y)$ we have obtained the structural result saying "how much space occupies the subspace $W_\Psi(L_2(\mathbb{R}))$ inside $L_2(G, \mathrm{d}\zeta)$", see [4, Theorem 3.3] and [5, Theorem 1] for more details. Now the trick is that in comparison with our previous approach the second operator V_Ψ in both cases *is not needed* to study the TLO's T_a^Ψ, thus providing a much easier way to the properties of T_a^Ψ. Of course, the previous approach has its own advantages in connection with understanding the structure of $W_\Psi(L_2(\mathbb{R}))$ inside $L_2(G, \mathrm{d}\zeta)$, as well as with study of certain algebras of operators, e.g., algebras generated by operators of the form $A_\Psi = aI + bP_\Psi$ acting on $L_2(G, \mathrm{d}\zeta)$ with a, b the bounded functions on G depending only on the first coordinate. In what follows we gradually apply the Bargmann-type transform R_Ψ to the TLO T_a^Ψ with a symbol as a function of individual coordinates of G.

3.1 TLO's with symbols depending on the first variable

In what follows we consider the case where the symbol of the TLO depends only on horizontal variable of G. This case is very important because it gives rise to commutative operator algebras with some interesting features.

Theorem 3.1 *If a measurable function $a(r,s) = \alpha(r)$ on G does not depend on s, then the TLO T_α^Ψ acting on $W_\Psi(L_2(\mathbb{R}))$ is unitarily equivalent to the multiplication operator $\gamma_\alpha^\Psi I$ acting on $L_2(G_2, \mathrm{d}\zeta_2)$. The function $\gamma_\alpha^\Psi : G_2 \to \mathbb{C}$ is given by*

$$\gamma_\alpha^\Psi(\xi) = \int_{G_1} \alpha(r) |\ell_\Psi(r,\xi)|^2 \, \mathrm{d}\zeta_1(r), \quad \xi \in G_2.$$

Proof. The operator T_α^Ψ is obviously unitarily equivalent to the following operator

$$\begin{aligned} R_\Psi T_\alpha^\Psi R_\Psi^* &= R_\Psi P_\Psi \alpha(r) P_\Psi R_\Psi^* = R_\Psi (R_\Psi^* R_\Psi) \alpha(r) (R_\Psi^* R_\Psi) R_\Psi^* \\ &= (R_\Psi R_\Psi^*) R_\Psi \alpha(r) R_\Psi^* (R_\Psi R_\Psi^*) = R_\Psi \alpha(r) R_\Psi^* \\ &= Q_\Psi^* \alpha(r) Q_\Psi, \end{aligned}$$

where the result of Theorem 2.1 has been used. Finally, for $f \in L_2(G_2, \mathrm{d}\zeta_2)$ we have

$$(Q_\Psi^* \alpha(r) Q_\Psi f)(\xi) = \int_{G_1} \alpha(r) f(\xi) |\ell_\Psi(r,\xi)|^2 \, \mathrm{d}\zeta_1(r) = f(\xi) \cdot \gamma_a^\Psi(\xi), \quad \xi \in G_2,$$

where $\gamma_\alpha^\Psi(\xi) = \int_{G_1} \alpha(r) |\ell_\Psi(r,\xi)|^2 \, \mathrm{d}\zeta_1(r)$ for $\xi \in G_2$.

Remark 3.2 The explicit form of the corresponding function γ_a^Ψ for both cases is as follows

$$\gamma_\alpha^\psi(\xi) = \int_{\mathbb{R}_+} \alpha(u) |\hat{\psi}(u\xi)|^2 \frac{\mathrm{d}u}{u}, \qquad \gamma_\alpha^\phi(\xi) = \int_{\mathbb{R}} \alpha(q) |\phi(\xi - q)|^2 \, \mathrm{d}q,$$

where $\xi \in \mathbb{R}$. In fact, the functions γ_α^ψ and γ_α^ϕ are constructed by putting a multiplier in admissibility conditions for the wavelet ψ and the window ϕ, respectively.

Clearly, the result of Theorem 3.1 opens an easy and direct way to properties of TLO's with symbols depending only on first variable. Since T_α^Ψ is unitarily equivalent to a multiplication operator, then it is never compact. If $a(r,s)=\alpha(r)$ is a bounded symbol, then the operator T_α^Ψ is obviously bounded on $W_\Psi(L_2(\mathbb{R}))$, and for its operator norm holds

$$\|T_\alpha^\Psi\| \leq \text{ess-sup}\,|\alpha(r)|.$$

Therefore, the spaces $W_\Psi(L_2(\mathbb{R}))$ are natural and appropriate for TLO's with bounded symbols. However, we may observe that the result of Theorem 3.1 suggests considering not only bounded symbols, but also unbounded ones. In this case we obviously have

Theorem 3.3 *Let $a(r,s)=\alpha(r)$ be a measurable symbol on G. Then the TLO T_α^Ψ is bounded on $W_\Psi(L_2(\mathbb{R}))$ if and only if the corresponding function $\gamma_\alpha^\Psi(\xi)$ is bounded on G_2, and*

$$\|T_\alpha^\Psi\| = \sup_{\xi\in G_2} |\gamma_\alpha^\Psi(\xi)|.$$

In this case we may also easily describe the spectrum of TLO as follows.

Theorem 3.4 *The spectrum of a bounded TLO T_α^Ψ acting on $W_\Psi(L_2(\mathbb{R}))$ with a measurable symbol $a(r,s)=\alpha(r)$ coincides with its essential spectrum, and is given by*

$$\operatorname{sp} T_\alpha^\Psi = \operatorname{clos}\left\{\gamma_\alpha^\Psi(\xi);\ \xi\in G_2\right\}.$$

Moreover, for a real-valued function $a(r,s)=\alpha(r)$ we have

$$\operatorname{sp} T_\alpha^\Psi = \left[\inf_{\xi\in G_2}\gamma_\alpha^\Psi(\xi), \sup_{\xi\in G_2}\gamma_\alpha^\Psi(\xi)\right].$$

Introduce the C^*-algebra $\mathscr{A}_\infty$ of all $L_\infty(G,d\zeta)$-functions depending on r only, where $(r,s)\in G$. Then as a consequence of Theorem 3.1 we have

Theorem 3.5 *The C^*-algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$ of TLO's T_a^Ψ with symbols $a\in\mathscr{A}_\infty$ is commutative and is isometrically imbedded to the algebra $C_b(G_2)$ of bounded continuous functions on G_2. The isomorphic imbedding*

$$\tau_\infty^\Psi : \mathscr{T}_\Psi(\mathscr{A}_\infty) \longrightarrow C_b(G_2)$$

is generated by the following mapping

$$\tau_\infty^\Psi : T_a^\Psi \longmapsto \gamma_a^\Psi(\xi)$$

of generators of the algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$.

Commutativity of the algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$ is a rather interesting feature, see the book [8] devoted to this phenomena for Toeplitz operators on the Bergman spaces. For two symbols $a,b\in\mathscr{A}_\infty$ we have obviously that, in general,

$$\gamma_a^\Psi(\xi)\gamma_b^\Psi(\xi) - \gamma_{ab}^\Psi(\xi) \neq 0,$$

which means that the C^*-algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$ generated by TLO's with such symbols is a further example of algebra with the property that for each pair $a,b\in$

$\mathscr{A}_\infty$ the commutator $[T_a^\Psi, T_b^\Psi] = 0$, while the semi-commutator $[T_a^\Psi, T_b^\Psi)$ is not compact.

Fix a number $m \in \mathbb{N}$. Let Y_k^Ψ, $k = 1, \ldots, m$, be disjoint measurable sets in G_1 with a positive measure, such that $\bigcup_{k=1}^m Y_k^\Psi = G_1$. Let $\Pi_k^\Psi = G_2 + \mathrm{i}\, Y_k^\Psi$, $k = 1, \ldots, m$, be the corresponding sets in $G = G_2 + \mathrm{i}\, G_1$. Denote by $\chi_{Y_k^\Psi}$ the characteristic function of the set Y_k^Ψ, and by $\chi_{\Pi_k^\Psi}$ the characteristic function of the set Π_k^Ψ, $k = 1, \ldots, m$, respectively. For the algebra

$$\mathscr{A}_m^\Psi = \left\{ a_1 \chi_{\Pi_1^\Psi} + \cdots + a_m \chi_{\Pi_m^\Psi};\ a_k \in \mathbb{C}, k = 1, \ldots, m \right\}$$

we immediately have the following result.

Theorem 3.6 *The algebra $\mathscr{T}_\Psi(\mathscr{A}_m^\Psi)$ is isometric and isomorphic to the algebra $C(\nabla_\Psi)$, where*

$$\nabla_\Psi = \nabla_\Psi \left(Y_1^\Psi, \ldots, Y_m^\Psi \right) = \mathrm{clos} \left\{ \left(\gamma^\Psi_{\chi_{Y_1^\Psi}}(\xi), \ldots, \gamma^\Psi_{\chi_{Y_m^\Psi}}(\xi) \right);\ \xi \in G_2 \right\},$$

and the functions $\gamma^\Psi_{\chi_{Y_k^\Psi}}(\xi)$, $k = 1, \ldots, m$, are given by

$$\gamma^\Psi_{\chi_{Y_k^\Psi}}(\xi) = \int_{G_1} \chi_{Y_k^\Psi}(z) |\ell_\Psi(z, \xi)|^2 \,\mathrm{d}\zeta_1(z) = \int_{Y_k^\Psi} |\ell_\Psi(z, \xi)|^2 \,\mathrm{d}\zeta_1(z), \quad \xi \in G_2.$$

The isomorphism

$$\tau_m^\Psi : \mathscr{T}_\Psi(\mathscr{A}_m^\Psi) \longrightarrow C(\nabla_\Psi)$$

is generated by the following mapping of generators T_a^Ψ of the algebra $\mathscr{T}_\Psi(\mathscr{A}_m^\Psi)$

$$\tau_m^\Psi : T_a^\Psi \longmapsto a_1 z_1^\Psi + \cdots + a_m z_m^\Psi, \quad z^\Psi = \left(z_1^\Psi, \ldots, z_m^\Psi \right) \in \nabla_\Psi,$$

where

$$a = a_1 \chi_{\Pi_1^\Psi} + \cdots + a_m \chi_{\Pi_m^\Psi} \in \mathscr{A}_m^\Psi.$$

Property to be unitarily equivalent to a multiplication operator permits us to describe easily a sufficiently rich structure of invariant subspaces of C^*-algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$.

Theorem 3.7 *The commutative C^*-algebra $\mathscr{T}_\Psi(\mathscr{A}_\infty)$ is reducible and every invariant subspace $\mathscr{S}_\Psi$ of $\mathscr{T}_\Psi(\mathscr{A}_\infty)$ is defined by a measurable subset $S_\Psi \subset G_2$ and has the form*

$$\mathscr{S}_\Psi = (R_\Psi^* \chi_{S_\Psi} I) L_2(G_2, d\zeta_2)$$

with χ_{S_Ψ} being the characteristic function of S_Ψ.

3.2 TLO's with symbols depending on the second variable

Now we are interested in symbols depending only on second (vertical) variable of G. In this case the TLO is no more unitarily equivalent to a multiplication operator, but certain class of integral operators appears.

Theorem 3.8 *If a measurable function $a(r,s) = \beta(s)$ does not depend on r, then the TLO T_β^Ψ acting on $W_\Psi(L_2(\mathbb{R}))$ is unitarily equivalent to the following integral operator*

$$\left(\mathfrak{B}_\beta^\Psi f\right)(\xi) = \int_{G_2} \mathfrak{b}_\Psi(\xi,\omega)\,\hat{\beta}(\pm(\xi-\omega))f(\omega)\,\mathrm{d}\zeta_2(\omega), \quad \xi \in G_2,$$

acting on $L_2(G_2, \mathrm{d}\zeta_2)$. The function $\mathfrak{b}_\Psi : G_2 \times G_2 \to \mathbb{C}$ is given by

$$\mathfrak{b}_\Psi(\xi,\omega) = \int_{G_1} \ell_\Psi(r,\omega)\overline{\ell_\Psi(r,\xi)}\,\mathrm{d}\zeta_1(r).$$

Proof. Similarly as in the proof of Theorem 3.1 we get

$$\mathfrak{B}_\beta^\Psi = R_\Psi T_{\beta(s)}^\Psi R_\Psi^* = R_\Psi \beta(s) R_\Psi^* = Q_\Psi^* U_\Psi \beta(s) U_\Psi^* Q_\Psi.$$

Using the convolution theorem for Fourier transform we have

$$\begin{aligned}
\left(\mathfrak{B}_\beta^\Psi f\right)(\xi) &= Q_\Psi^* \left(\int_{G_2} \hat{\beta}(\pm(s-\omega))(Q_\Psi f)(r,\omega)\,\mathrm{d}\zeta_2(\omega)\right)(\xi) \\
&= \int_{G_1} \overline{\ell_\Psi(r,\xi)}\,\mathrm{d}\zeta_1(r) \int_{G_2} \hat{\beta}(\pm(\xi-\omega))f(\omega)\,\ell_\Psi(r,\omega)\,\mathrm{d}\zeta_2(\omega) \\
&= \int_{G_2} \hat{\beta}(\pm(\xi-\omega))f(\omega)\,\mathrm{d}\zeta_2(\omega) \int_{G_1} \ell_\Psi(r,\omega)\overline{\ell_\Psi(r,\xi)}\,\mathrm{d}\zeta_1(r) \\
&= \int_{G_2} \mathfrak{b}_\Psi(\xi,\omega)\hat{\beta}(\pm(\xi-\omega))f(\omega)\,\mathrm{d}\zeta_2(\omega), \quad \xi \in G_2,
\end{aligned}$$

where $f \in L_2(G_2, \mathrm{d}\zeta_2)$, and

$$\mathfrak{b}_\Psi(\xi,\omega) = \int_{G_1} \ell_\Psi(r,\omega)\overline{\ell_\Psi(r,\xi)}\,\mathrm{d}\zeta_1(r),$$

which completes the proof.

Remark 3.9 Observe that the class of integral operators

$$\left(\mathfrak{B}_\beta^\Psi f\right)(\xi) = \int_{G_2} \mathfrak{b}_\Psi(\xi,\omega)\hat{\beta}(\pm(\xi-\omega))f(\omega)\,\mathrm{d}\omega, \quad \xi \in G_2,$$

is interesting itself, and in some sense extends and generalizes the class of operators considered in [3] to the whole line. Note that for each $\Psi \in L_2(\mathbb{R})$ the function $\mathfrak{b}_\Psi$ has the following properties:

$$\mathfrak{b}_\Psi(\xi,\omega) = \overline{\mathfrak{b}_\Psi(\omega,\xi)} \text{ for all } \xi,\omega \in G_2, \qquad \mathfrak{b}_\Psi(\xi,\xi) = 1 \text{ for a.e. } \xi \in G_2.$$

Further remarkable properties may be obtained when considering some special cases of wavelets, or windows, respectively.

3.3 General case of symbols

The above mentioned construction of unitary operators may be used to study more general symbols $a(r,s)$ for which the TLO T_a^Ψ is no longer unitarily equivalent to a multiplication operator, because the operator $R_\Psi T_a^\Psi R_\Psi^*$ might have a more complicated structure as we have demonstrated above for the case of symbols depending on the second variable. As we will prove now the TLO T_a^Ψ with symbols which depend on both variables $(r,s) \in G$ is unitarily equivalent to a pseudodifferential operator with certain compound (double) symbol. We clarify this statement for the case of symbol a in the product form $a(r,s) = \alpha(r)\beta(s)$.

Theorem 3.10 *Let $a(r,s) = \alpha(r)\beta(s)$ be a measurable symbol on G. Then the TLO T_a^Ψ acting on $W_\Psi(L_2(\mathbb{R}))$ is unitarily equivalent to the pseudodifferential operator $\mathfrak{A}_\mathfrak{a}^\Psi$ acting on $L_2(G_2, \mathrm{d}\zeta_2)$. The operator $\mathfrak{A}_\mathfrak{a}^\Psi$ is given by the iterated integral*

$$\left(\mathfrak{A}_\mathfrak{a}^\Psi f\right)(x) = \int_{G_2} \mathrm{d}\zeta_2(\xi) \int_{G_2} \mathfrak{a}_\Psi(x,y,\xi) f(y)\, \mathrm{e}^{\mp 2\pi \mathrm{i}(x-y)\xi}\, \mathrm{d}\zeta_2(y), \tag{1}$$

for $x \in G_2$, where the compound (double) symbol $\mathfrak{a}_\Psi : G_2 \times G_2 \times G_2 \to \mathbb{C}$ has the form $\mathfrak{a}_\Psi(x,y,\xi) = \Gamma_\alpha^\Psi(x,y)\beta(\xi)$ with

$$\Gamma_\alpha^\Psi(x,y) = \int_{G_1} \alpha(r)\overline{\ell_\Psi(r,x)}\ell_\Psi(r,y)\, \mathrm{d}\zeta_1(r).$$

Similarly as above, since we consider the operator $U_\Psi = \left(I \otimes \mathscr{F}^{\pm 1}\right)$ to describe the both cases of wavelet and time-frequency analysis, also here the signs $\mp$ in (1) correspond to $\mathfrak{A}_\mathfrak{a}^\psi$ (for wavelet case) and $\mathfrak{A}_\mathfrak{a}^\phi$ (for time-frequency case), respectively.

Proof. Let $f \in L_2(G_2, \mathrm{d}\zeta_2)$. Then T_a^Ψ is unitarily equivalent to the operator

$$\mathfrak{A}_\mathfrak{a}^\Psi = R_\Psi T_{\alpha(r)\beta(s)}^\Psi R_\Psi^* = R_\Psi \alpha(r)\beta(s) R_\Psi^* = Q_\Psi^* \alpha(r) U_\Psi \beta(s) U_\Psi^* Q_\Psi.$$

Then

$$\begin{aligned}
(\mathfrak{A}_\mathfrak{a}^\Psi f)(\lambda) &= \int_{G_1} \alpha(r)\overline{\ell_\Psi(r,\lambda)}\, \mathrm{d}\zeta_1(r) \int_{G_2} \beta(s)\, \mathrm{e}^{\mp 2\pi \mathrm{i} s\lambda}\, \mathrm{d}\zeta_2(s) \\
&\quad \times \int_{G_2} f(\omega)\ell_\Psi(r,\omega)\, \mathrm{e}^{\pm 2\pi \mathrm{i}\omega s}\, \mathrm{d}\zeta_2(\omega) \\
&= \int_{G_2} \beta(s)\, \mathrm{e}^{\mp 2\pi \mathrm{i} s\lambda}\, \mathrm{d}\zeta_2(s) \int_{G_2} f(\omega)\, \mathrm{e}^{\pm 2\pi \mathrm{i}\omega s}\, \mathrm{d}\zeta_2(\omega) \\
&\quad \times \int_{G_1} \alpha(r)\overline{\ell_\Psi(r,\lambda)}\ell_\Psi(r,\omega)\, \mathrm{d}\zeta_1(r).
\end{aligned}$$

If the last integral is denoted by $\Gamma_\alpha^\Psi(\lambda,\omega)$, then we finally have

$$\begin{aligned}
\left(\mathfrak{A}_\mathfrak{a}^\Psi f\right)(\lambda) &= \int_{G_2} \mathrm{d}\zeta_2(s) \int_{G_2} \Gamma_\alpha^\Psi(\lambda,\omega)\beta(s) f(\omega)\, \mathrm{e}^{\mp 2\pi \mathrm{i}(\lambda-\omega)s}\, \mathrm{d}\zeta_2(\omega) \\
&= \int_{G_2} \mathrm{d}\zeta_2(s) \int_{G_2} \mathfrak{a}_\Psi(\lambda,\omega,s) f(\omega)\, \mathrm{e}^{\mp 2\pi \mathrm{i}(\lambda-\omega)s}\, \mathrm{d}\zeta_2(\omega).
\end{aligned}$$

Changing the variables $\lambda = x$, $\omega = y$ and $s = \xi$ we finally get the standard notation (1) for the pseudodifferential operator with a compound symbol, see e.g. [6].

Remark 3.11 Observe that $\Gamma_\alpha^\Psi(x,x) = \gamma_\alpha^\Psi(x)$ is the function from Theorem 3.1 which is responsible for properties of the TLO T_α^Ψ whose symbol $a(r,s) = \alpha(r)$ depends on the first variable only. Also, for $\alpha(r) \equiv 1$ on G_1 we get $\Gamma_1^\Psi(x,y) = \mathfrak{b}_\Psi(x,y)$, the function appearing in Theorem 3.8. Note that in the proof of Theorem 3.10 we did not use the convolution theorem as in Theorem 3.8 to get the desired form (1) of pseudodifferential operator. Thus, the operator $\mathfrak{B}_\beta^\Psi$ may also be viewed as a pseudodifferential operator of the form (1). Further research on TLO's T_a^Ψ using the deeper connection with pseudodifferential operators will be considered elsewhere.

Concluding remarks

In the end let us note that the presented technique and the obtained results are interesting from various viewpoints, because (inter alia)

(i) they represent a unified approach to study both Calderón-Toeplitz and Gabor-Toeplitz operators and properties which they share;

(ii) the presented technique is purely analytic based only on operator theory and does not use neither the specifics of groups behind the construction of localization operators (affine, or Weyl-Heisenberg group, respectively), nor time-scale, or time-frequency methods;

(iii) they give rise to commutative algebras of TLO's which are practically unknown;

(iv) they enable to study the TLO's using their unitarily equivalent images in the class of pseudodifferential operators with compound symbols as an analog of the Berezin approach known for Toeplitz and Weyl pseudodifferential operators.

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