Global Uniqueness for an Inverse Stochastic Hyperbolic Equation with Three Unknowns[*]

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Abstract

This paper is addressed to an inverse stochastic hyperbolic equation with three unknowns, i.e., a source term, an initial displacement and an initial velocity. The global uniqueness is proved by a new global Carleman estimate for the stochastic hyperbolic equation. It is found that both the formulation of stochastic inverse problems and the tools to solve them differ considerably from their deterministic counterpart.

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1 Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^2 boundary Γ, and let Γ_0 be a suitable chosen nonempty subset (to be given later) of Γ. Put $Q \stackrel{\triangle}{=} (0, T) \times G$ and $\Sigma \stackrel{\triangle}{=} (0, T) \times \Gamma$. Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined. For any Banach space H, denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded processes, and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$. All of these spaces are endowed with the canonical norm (Similarly, one can define $L^2_{\mathcal{F}}(\Omega; C^k([0, T]; H))$ for any positive integer k).

Throughout this paper, we assume that the functions $b^{ij} \in C^1(\overline{G})$ ($i, j = 1, 2, \cdots, n$) satisfy $b^{ij} = b^{ji}$ and, for some constant $s_0 > 0$,

$$\sum_{i,j=1}^n b^{ij} \xi^i \xi^j \geq s_0 |\xi|^2, \qquad \forall \, (x, \xi) \in G \times \mathbb{R}^n. \tag{1.1}$$

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Consider the following stochastic hyperbolic equation:

$$\begin{cases} dz_t - \sum_{i,j=1}^{n} (b^{ij} z_{x_i})_{x_j} dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z)\, dt + (b_4 z + g) dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0,\ z_t(0) = z_1 & \text{in } G. \end{cases} \tag{1.2}$$

Here, $z_t = \frac{\partial z}{\partial t}$, $z_{x_i} = \frac{\partial z}{\partial x_i}$, and b_i ($1 \le i \le 4$) are some suitable (known) functions to be given later; while $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$ and $g \in L_\mathcal{F}^2(0, T; L^2(G))$ are unknown. Physically, g stands for the intensity of a random force of the white noise type. Put

$$H_T \stackrel{\triangle}{=} L_\mathcal{F}^2(\Omega; C([0,T]; H_0^1(G))) \bigcap L_\mathcal{F}^2(\Omega; C^1([0,T]; L^2(G))). \tag{1.3}$$

It is clear that H_T is a Banach space with the canonical norm. Under suitable assumptions (the assumptions in this paper are enough), for any given (z_0, z_1) and g, one can show that the equation (1.2) admits one and only one solution $z = z(z_0, z_1, g)(t, x, \omega) \in H_T$ (see [12]). We will also denote by $z(z_0, z_1, g)$ or $z(z_0, z_1, g)(t)$ the solution of (1.2).

In this article, the random force $\int_0^t g dB$ is assumed to cause the random vibration starting from some initial state (z_0, z_1). Roughly speaking, our aim is to determine the unknown random force intensity g and the unknown initial displacement z_0 and initial velocity z_1 from the (partial) boundary observation $\frac{\partial z}{\partial \nu}\big|_{(0,T) \times \Gamma_0}$ and the measurement on the terminal displacement $z(T)$, where $\nu = \nu(x)$ denotes the unit outer normal vector of G at $x \in \Gamma$, and Γ_0 is a suitable open subset (to be specified later) of Γ. More precisely, we are concerned with the following global uniqueness problem: *Do $\frac{\partial z}{\partial \nu}(z_0, z_1, g)\big|_{(0,T) \times \Gamma_0} = 0$ and $z(z_0, z_1, g)(T) = 0$ in G, P-a.s. imply that $g = 0$ in Q and $z_0 = z_1 = 0$ in G, P-a.s.?*

In the deterministic setting, there exist numerous literatures addressing the inverse problem of PDEs (See [5, 6] and the rich references cited therein). A typical (deterministic) inverse problem close to the above one is as follows: Fix suitable (known) functions $a(\cdot, \cdot)$ and $f_1(\cdot, \cdot)$ satisfying $\min_{(t,x) \in Q} |f_1(t,x)| > 0$, and consider the following hyperbolic equation:

$$\begin{cases} z_{tt} - \Delta z = a(t,x)z + f_1(t,x)f_2(x) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0,\ z_t(0) = z_1 & \text{in } G. \end{cases} \tag{1.4}$$

In (1.4), both z_1 and f_2 are unknown and one expects to determine them through the boundary observation $\frac{\partial z}{\partial \nu}\big|_{(0,T) \times \Gamma_0}$. As shown in [11], by assuming suitable regularity on functions $a(\cdot, \cdot)$, $f_i(\cdot, \cdot)$ ($i = 1, 2$) and $z_1(\cdot)$, and using the transformation

$$y = y(t,x) = \frac{d}{dt}\left(\frac{z(t,x)}{f_1(t,x)}\right), \tag{1.5}$$

this inverse problem can be reduced to deriving the so-called observability for the following wave

equation with memory

$$
\begin{cases}
y_{tt} - \Delta y = a_1 y_t + a_2 \cdot \nabla y + a_3 y \\
\qquad\qquad + \displaystyle\int_0^t [c_1(t,s,x)y(s,x) + c_2(t,s,x) \cdot \nabla y(s,x)]\, ds & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0,x) = \dfrac{z_1(x)}{f_1(0,x)}, \quad y_t(0,x) = f_2(x) - \dfrac{2\partial_t f_1(0,x)}{|f_1(0,x)|^2} z_1(x) & \text{in } G,
\end{cases}
$$

where $a_i(\cdot,\cdot)$ $(i = 1,2,3)$ and $c_i(\cdot,\cdot,\cdot)$ $(i = 1,2)$ are suitable functions. Concerning this problem, if z_1 is known and both functions $a(\cdot,\cdot)$ and $f_1(\cdot,\cdot)$ are independent of the space variable x, i.e., there is only one unknown in (1.4), then the corresponding inverse problem is now well-understood (e.g. [9, 10] and the references therein). The main tool in the later case is to use the Duhamel principle, instead the transform (1.5), to reduce the problem to the observability estimate for some wave equation (without memory).

Stochastic partial differential equations (PDEs for short) are used to describe a lot of random phenomena appeared in physics, chemistry, biology, control theory and so on. In many situations, stochastic PDEs are more realistic mathematical models than the deterministic ones. Nevertheless, compared to the deterministic setting, there exist a very limited works addressing inverse problems for stochastic PDEs (See [1, 2, 4], etc.). Especially, as far as we know, there is no paper considering the inverse problem for stochastic hyperbolic equations.

One will meet substantially new difficulties in the study of inverse problems for stochastic PDEs. For instance, unlike the deterministic PDEs, the solution of a stochastic PDE is usually non-differentiable with respect to the variable with noise (say, the time variable considered in this paper). Also, the usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs. These new phenomenons lead that some useful methods for solving inverse problems for deterministic PDEs (see [5, 9] for example) cannot be used to solve the corresponding inverse problems in the stochastic setting. Especially, one can see that none of the methods for solving the above inverse problem for the equation (1.4) can be easily adopted to solve our inverse problem for the stochastic hyperbolic equation (1.2), even if g is assumed to be of the form

$$
g(t,x,\omega) = g_1(t,\omega)g_2(x), \qquad \forall\, (t,x,\omega) \in Q \times \Omega, \tag{1.6}
$$

with a known stochastic process $g_1(\cdot,\cdot) \in L^2_{\mathcal{F}}(0,T)$ and an unknown function $g_2(\cdot) \in L^2(G)$. For these reasons, it is necessary to develop new methodology for treating inverse problems for stochastic PDEs.

In this paper, we will use a suitable Carleman estimate to solve the above formulated inverse problem for the equation (1.2). To the best of our knowledge, [12] is the only published reference addressing the Carleman estimate for stochastic hyperbolic equations. In [12], under suitable assumptions, the following estimate was proved for the solution z of (1.2):

$$
|(z(T), z_t(T))|_{L^2(\Omega, \mathcal{F}_T, P; H_0^1(G) \times L^2(G))} \le C \left[\left| \frac{\partial z}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0,T; L^2(\Gamma_0))} + |g|_{L^2_{\mathcal{F}}(0,T; L^2(G))} \right]. \tag{1.7}
$$

(Here and henceforth, C is a generic positive constant, depending only on T, G, Γ_0 and s_0, which may be different from one place to another). Noting that g appears in the right hand side of (1.7), and therefore, the estimate obtained in [12] does not apply to the inverse problem in this paper. In order to solve our stochastic inverse problem, we need to establish a new Carleman

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estimate for (1.2) so that the source term g can be bounded above by the observed data. Hence, we need to avoid employing the usual *energy estimate* because, when applying this estimate to (1.2), the source term g would appear as a bad term. Meanwhile, noting that we are also expected to identity the initial data, hence we need to bound above the initial data by the observed data, too. Because of this, we need to obtain the estimate on the initial data and source term in the Carleman inequality simultaneously. Therefore we cannot use the usual "Carleman estimate" + "energy estimate" method (which works well for the deterministic wave equation, see [3]) to derive the desired estimates. This is the main difficulty that we need to overcome in this paper.

The rest of this paper is organized as follows. In Section 2, we state the main result of this paper. Some preliminary results are collected in Section 3. Finally, Section 4 is addressed to proving the main result.

2 Statement of the main result

To begin with, we introduce the following conditions:

Condition 2.1 *There exists a positive function $d(\cdot) \in C^2(\overline{G})$ satisfying the following:*

1) For some constant $\mu_0 > 0$, it holds that

$$\sum_{i,j=1}^{n} \left\{ \sum_{i',j'=1}^{n} \left[2b^{ij'}(b^{i'j}d_{x_{i'}})_{x_{j'}} - b^{ij}_{x_{j'}} b^{i'j'} d_{x_{i'}} \right] \right\} \xi^i \xi^j \geq \mu_0 \sum_{i,j=1}^{n} b^{ij} \xi^i \xi^j, \tag{2.1}$$

$$\forall (x, \xi^1, \cdots, \xi^n) \in \overline{G} \times \mathbb{R}^n;$$

2) There is no critical point of $d(\cdot)$ in $\overline{G}$, i.e.,

$$\min_{x \in \overline{G}} |\nabla d(x)| > 0. \tag{2.2}$$

Remark 2.1 *If $(b^{ij})_{1 \leq i,j \leq n}$ is the identity matrix, then, by taking $d(x) = |x - x_0|^2$ with $x_0 \notin \overline{G}$, one sees that Condition 2.1 is satisfied. Condition 2.1 was introduced in [3] to show the observability estimate for hyperbolic equations. We refer to [3] for more explanation on Condition 2.1 and further nontrivial examples.*

It is easy to see that if $d(\cdot) \in C^2(\overline{G})$ satisfies Condition 2.1, then for any given constants $a \geq 1$ and $b \in \mathbb{R}$, the function $\tilde{d} = ad + b$ still satisfies Condition 2.1 with μ_0 replaced by $a\mu_0$. Therefore we may choose μ_0 as large as we need in Condition 2.1. Now we choose $0 < c_0 < c_1 < 1$, $\mu_0 > 4$ and T satisfying the following condition:

Condition 2.2

$$\begin{cases} 1) & \mu_0 - 4c_1 - c_0 > 0, \\ 2) & \dfrac{\mu_0}{(8c_1 + c_0)} \sum_{i,j=1}^{n} b^{ij} d_{x_i} d_{x_j} > 4c_1^2 T^2 > \sum_{i,j=1}^{n} b^{ij} d_{x_i} d_{x_j}. \end{cases}$$

Remark 2.2 *Since $\displaystyle\sum_{i,j=1}^{n} b^{ij} d_{x_i} d_{x_j} > 0$, it is easy to see that one can always choose μ_0 in Condition 2.1 large enough so that Condition 2.2 holds true. We put it here simply to emphasize the relationship among $0 < c_0 < c_1 < 1$, $\mu_0 > 4$ and T.*

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In the sequel, we choose

$$\Gamma_0 \overset{\triangle}{=} \left\{ x \in \Gamma : \sum_{i,j=1}^{n} b^{ij} d_{x_i}(x) \nu^j(x) > 0 \right\}. \tag{2.3}$$

Also, we assume that

$$b_1 \in L_{\mathcal{F}}^{\infty}(0,T; L^{\infty}(G)), \quad b_2 \in L_{\mathcal{F}}^{\infty}(0,T; L^{\infty}(G;\mathbb{R}^n)),$$
$$b_3 \in L_{\mathcal{F}}^{\infty}(0,T; L^n(G)), \quad b_4 \in L_{\mathcal{F}}^{\infty}(0,T; L^{\infty}(G)). \tag{2.4}$$

In what follows, we use the notation:

$$\mathcal{A} \overset{\triangle}{=} |b_1|^2_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))} + |b_2|^2_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G;\mathbb{R}^n))} + |b_3|^2_{L_{\mathcal{F}}^{\infty}(0,T;L^n(G))} + |b_4|^2_{L_{\mathcal{F}}^{\infty}(0,T;L^{\infty}(G))} + 1. \tag{2.5}$$

The main result of this paper can be stated as follows.

Theorem 2.1 *Let b_i $(1 \leq i \leq 4)$ satisfy (2.4), and let Condition 2.1 and Condition 2.2 hold. Assume that the solution $z \in H_T$ of (1.2) satisfies that $\frac{\partial z}{\partial \nu} = 0$ on $(0,T) \times \Gamma_0$ and $z(T) = 0$ in G, P-a.s. Then $g = 0$ in Q and $z_0 = z_1 = 0$ in G, P-a.s.*

Several remarks are in order.

Remark 2.3 *Similar to the inverse problem for (1.4), and stimulated by Theorem 2.1, it seems natural to expect a similar uniqueness result for the following equation*

$$\begin{cases} dz_t - \displaystyle\sum_{i,j=1}^{n} (b^{ij} z_{x_i})_{x_j} dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + f) \, dt + b_4 z dB(t) & in \ Q, \\ z = 0 & on \ \Sigma, \\ z(0) = z_0, \ z_t(0) = z_1 & in \ G, \end{cases} \tag{2.6}$$

in which z_0, z_1 and f are unknown and one expects to determine them through the boundary observation $\frac{\partial z}{\partial \nu}\big|_{(0,T) \times \Gamma_0}$ and the terminal measurement $z(T)$. However the same conclusion as that in Theorem 2.1 does not hold true even for the deterministic wave equation. Indeed, we choose any $y \in C_0^{\infty}(Q)$ so that it does not vanish in some subdomain of Q. Putting $f = u_{tt} - \Delta u$, it is obvious that y solves the following wave equation

$$\begin{cases} y_{tt} - \Delta y = f & in \ Q, \\ y = 0, & on \ \Sigma, \\ y(0) = 0, \ y_t(0) = 0 & in \ G. \end{cases}$$

One can show that $y(T) = 0$ in G and $\frac{\partial y}{\partial \nu} = 0$ on Σ. However, it is clear that f does not vanish in Q. This counterexample shows that the formulation of the stochastic inverse problem may differ considerably from its deterministic counterpart.

Remark 2.4 *From the computational point of view, it is quite interesting to study the following stability problem (for the inverse stochastic hyperbolic equation (1.2)): Is the map*

$$\frac{\partial z}{\partial \nu}(z_0, z_1, g)\Big|_{(0,T) \times \Gamma_0} \times z(z_0, z_1, g)(T) \longrightarrow (z_0, z_1, g)$$

continuous in some suitable Hilbert spaces? Unfortunately, we are not able to prove this stability result at this moment. Instead, from the proof of Theorem 2.1 (See Theorem 4.1 in Section 4), it is easy to show the following partial stability result, i.e., for any solution $z \in H_T$ of the equation (1.2) satisfying $z(T) = 0$ in G, P-a.s., it holds that

$$|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} + |\sqrt{T-t}g|_{L_{\mathcal{F}}^2(0,T; L^2(G))} \le C \left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0,T; L^2(\Gamma_0))}.$$

Especially, if g is of the form (1.6) (with $g_1(\cdot, \cdot) \in L_{\mathcal{F}}^2(0,T)$ and $g_2(\cdot) \in L^2(G)$), then the following estimate holds

$$|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} + |g_2|_{L^2(G)} \le C \left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0,T; L^2(\Gamma_0))}.$$

Remark 2.5 *The inverse problem considered in this work is a sort of inverse source problem, which is a linear inverse problem. It would be quite interesting to study some nonlinear inverse problem, say inverse coefficient problem for stochastic PDEs but this remains to be done, and it seems to be a very difficult problem.*

Remark 2.6 *It is also interesting to study the same inverse problems but for other stochastic PDEs, say the stochastic parabolic equation, the stochastic Schödinger equation, the stochastic plate equation and so on. However, it seems that the technique developed in this paper cannot be applied to these equations.*

3 Some preliminaries

In this section, we collect some preliminaries which will be useful later.

First, we show the following hidden regularity result for the solution z to the equation (1.2) (This result means that the observation of the normal derivative of z makes sense, i.e., $|\frac{\partial z}{\partial \nu}|_{L_{\mathcal{F}}^2(0,T; L^2(\Gamma_0))} < +\infty$).

Proposition 3.1 *Let b_i $(1 \le i \le 4)$ satisfy (2.4). Then, for any solution of equation (1.2), it holds that*

$$\left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0,T; L^2(\Gamma))} \le e^{C\mathcal{A}} \left[|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} + |g|_{L_{\mathcal{F}}^2(0,T; L^2(G))} \right]. \tag{3.1}$$

Remark 3.1 *In [12], the author proved Proposition 3.1 when $(b^{ij})_{1 \le i,j \le n}$ is an identity matrix. The proof of Proposition 3.1 for the general coefficient matrix $(b^{ij})_{1 \le i,j \le n}$ is similar, and therefore we give below only a sketch of the proof.*

Proof of Proposition 3.1: Since $\Gamma \in C^2$, one can find a vector field $h = (h^1, \cdots, h^n) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $h = \nu$ on Γ (*see* [7]). A direct computation shows that

$$- \sum_{i=1}^n \left[2(h \cdot \nabla z) \sum_{j=1}^n b^{ij} z_{x_j} + h^i \left(z_t^2 - \sum_{i,j=1}^n b^{ij} z_{x_i} z_{x_j} \right) \right]_{x_i} dt$$

$$= 2 \left\{ \left[dz_t - \sum_{i,j=1}^n (b^{ij} z_{x_i})_{x_j} \right] h \cdot \nabla z - d(z_t h \cdot \nabla z) - \sum_{i,j,k=1}^n b^{ij} z_{x_i} z_{x_k} h^k_{x_j} \right\} dt \tag{3.2}$$

$$- z_t^2 \operatorname{div} h\, dt + \sum_{i,j=1}^n z_{x_j} z_{x_i} \operatorname{div} (b^{ij} h) dt.$$

Integrating the identity (3.2) in Q, taking expectation in Ω and using integration by parts, we obtain the inequality (3.1) immediately. $\qquad\blacksquare$

Next, we recall the following known result, which will play an important role in the proof of Theorem 2.1.

Lemma 3.1 *([12, Theorem 4.1]) Let $p^{ij} \in C^1((0,T) \times \mathbb{R}^n)$ satisfy*

$$p^{ij} = p^{ji}, \qquad i,j = 1,2,\cdots,n, \tag{3.3}$$

l, f, $\Psi \in C^2((0,T) \times \mathbb{R}^n)$. Assume that u is an $H^2_{loc}(\mathbb{R}^n)$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that u_t is an $L^2(\mathbb{R}^n)$-valued semimartingale. Set $\theta = e^l$ and $v = \theta u$. Then, for a.e. $x \in \mathbb{R}^n$ and P-a.s. $\omega \in \Omega$,

$$
\begin{aligned}
&\theta\Big(-2l_t v_t + 2\sum_{i,j=1}^n p^{ij} l_{x_i} v_{x_j} + \Psi v\Big)\Big[du_t - \sum_{i,j=1}^n (p^{ij} u_{x_i})_{x_j} dt\Big] \\
&\quad + \sum_{i,j=1}^n \Big[\sum_{i',j'=1}^n \big(2p^{ij} p^{i'j'} l_{x_{i'}} v_{x_i} v_{x_{j'}} - p^{ij} p^{i'j'} l_{x_i} v_{x_{i'}} v_{x_{j'}}\big) - 2p^{ij} l_t v_{x_i} v_t + p^{ij} l_{x_i} v_t^2 \\
&\qquad\qquad + \Psi p^{ij} v_{x_i} v - \Big(Al_{x_i} + \frac{\Psi_{x_i}}{2}\Big) p^{ij} v^2\Big]_j dt \\
&\quad + d\Big[\sum_{i,j=1}^n p^{ij} l_t v_{x_i} v_{x_j} - 2\sum_{i,j=1}^n p^{ij} l_{x_i} v_{x_j} v_t + l_t v_t^2 - \Psi v_t v + \Big(Al_t + \frac{\Psi_t}{2}\Big) v^2\Big] \\
&= \Big\{\Big[l_{tt} + \sum_{i,j=1}^n (p^{ij} l_{x_i})_j - \Psi\Big] v_t^2 - 2\sum_{i,j=1}^n [(p^{ij} l_{x_j})_t + p^{ij} l_{tj}] v_{x_i} v_t \\
&\qquad + \sum_{i,j=1}^n \Big[(p^{ij} l_t)_t + \sum_{i',j'=1}^n \big(2p^{ij'} (p^{i'j} l_{x_{i'}})_{j'} - (p^{ij} p^{i'j'} l_{x_{i'}})_{j'}\big) + \Psi p^{ij}\Big] v_{x_i} v_{x_j} \\
&\qquad + Bv^2 + \Big(-2l_t v_t + 2\sum_{i,j=1}^n p^{ij} l_{x_i} v_{x_j} + \Psi v\Big)^2\Big\} dt + \theta^2 l_t (du_t)^2,
\end{aligned}
\tag{3.4}
$$

where $(du_t)^2$ denotes the quadratic variation process of u_t, A and B are stated as follows:

$$
\begin{cases}
A \triangleq (l_t^2 - l_{tt}) - \displaystyle\sum_{i,j=1}^n (p^{ij} l_{x_i} l_{x_j} - p^{ij}_{x_j} l_{x_i} - p^{ij} l_{x_i x_j}) - \Psi, \\[4mm]
B \triangleq A\Psi + (Al_t)_t - \displaystyle\sum_{i,j=1}^n (Ap^{ij} l_{x_i})_{x_j} + \frac{1}{2}\Big[\Psi_{tt} - \sum_{i,j=1}^n (p^{ij} \Psi_{x_i})_{x_j}\Big].
\end{cases}
\tag{3.5}
$$

4 Proof of the main result

This section is devoted to proving Theorem 2.1. As mentioned before, we will prove Theorem 2.1 by establishing a new Carleman estimate for the equation (1.2).

The desired Carleman estimate for (1.2) is as follows.

Theorem 4.1 *There exists a constant $\tilde{\lambda} > 0$ such that for any $\lambda \geq \tilde{\lambda}$ and any solution $z \in H_T$ of the equation (1.2) satisfying $z(T) = 0$ in G, P-a.s., it holds that*

$$
\mathbb{E} \int_G \theta^2(\lambda|z_1|^2 + \lambda|\nabla z_0|^2 + \lambda^3|z_0|^2)dx + \lambda\mathbb{E}\int_Q (T-t)\theta^2 g^2 dxdt
$$

$$
\leq C\lambda\mathbb{E}\int_0^T \int_{\Gamma_0} \theta^2 \left|\frac{\partial z}{\partial \nu}\right|^2 d\Gamma dt.
$$

(4.1)

Proof of Theorem 4.1: We borrow some idea from [8] and [12]. In what follows, we shall apply Lemma 3.1 to the equation (1.2) with

$$
u = z, \quad p^{ij} = b^{ij}, \quad l = \lambda\big[d(x) - c_1(t-T)^2\big], \quad \Psi = l_{tt} + \sum_{i,j=1}^{n} (b^{ij}l_{x_i})_{x_j} - \lambda c_0,
$$

and then estimate the terms in (3.4) one by one.

In the sequel, for $\lambda \in \mathbb{R}$, we use $O(\lambda^r)$ to denote a function of order λ^r for large λ. The proof is divided into three steps.

Step 1. In this step, we analyze the terms which stand for the "energy" of the solution of (1.2). As the Carleman estimate for deterministic partial differential equation, the point is to compute the order of λ in the coefficients of $|v_t|^2$, $|\nabla v|^2$ and $|v|^2$. Since the computation is very close to that in the proof of [8, Theorem 1.2.1], we give here only a sketch.

First, it is clear that the coefficient of $|v_t|^2$ reads:

$$
l_{tt} + \sum_{i,j=1}^{n} (b^{ij}l_{x_i})_{x_j} - \Psi = \lambda c_0.
$$

(4.2)

Further, noting that b^{ij} $(1 \leq i,j \leq n)$ are independent of t and $l_{tx_j} = l_{x_j t} = 0$, we find that

$$
\sum_{i,j=1}^{n} (b^{ij}l_{x_j})_t + b^{ij}l_{tx_j}v_{x_i}v_t = 0.
$$

(4.3)

Further, by Condition 2.1, we see that

$$
\sum_{i,j=1}^{n} \left\{ (b^{ij}l_t)_t + \sum_{i',j'=1}^{n} \left[2b^{ij'}(b^{i'j}l_{x_{i'}})_{x_{j'}} - (b^{ij}b^{i'j'}l_{x_{i'}})_{x_{j'}} \right] + \Psi b^{ij} \right\} v_{x_i}v_{x_j}
$$

$$
\geq \lambda(\mu_0 - 4c_1 - c_0)\sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j}.
$$

(4.4)

Further, in order to compute the coefficient B of $|v|^2$, recalling (3.5), we find that

$$
A = \lambda^2\left[4c_1^2(t-T)^2 - \sum_{i,j=1}^{n} b^{ij}d_{x_i}d_{x_j}\right] + O(\lambda).
$$

(4.5)

Hence, by the definition of B (in (3.5)), we conclude that

$$
\begin{aligned}
B &= (4c_1 + c_0)\sum_{i,j=1}^{n} b^{ij}d_{x_i}d_{x_j}\lambda^3 + \sum_{i,j}\sum_{i',j'=1}^{n} b^{ij}d_{x_i}(b^{i'j'}d_{x_{i'}}d_{x_{j'}})_{x_j}\lambda^3 \\
&\quad -4(8c_1^3 + c_0c_1^2)(t-T)^2\lambda^3 + O(\lambda^2).
\end{aligned}
$$

(4.6)

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Recall the following estimate in [8]:

$$\mu_0 \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j} \le \sum_{i,j=1}^n \sum_{i',j'=1}^n b^{ij} d_{x_i} (b^{i'j'} d_{x_{i'}} d_{x_{j'}})_{x_j}. \tag{4.7}$$

Therefore, by Condition 2.2, we obtain that

$$\begin{aligned}
B &\ge (4c_1 + c_0) \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j} \lambda^3 + \mu_0 \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j} \lambda^3 \\
&\quad -4(8c_1 + c_0)c_1^2(t-T)^2 \lambda^3 + O(\lambda^2) \\
&= (4c_1 + c_0) \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j} \lambda^3 + O(\lambda^2).
\end{aligned} \tag{4.8}$$

Hence, there exists a $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, it holds that

$$Bv^2 \ge C\lambda^3 v^2. \tag{4.9}$$

Step 2. In this step, we analyze the terms corresponding to $t = 0$ and $t = T$. For the time $t = 0$, we have

$$\begin{aligned}
&\sum_{i,j=1}^n b^{ij} l_t v_{x_i} v_{x_j} - 2 \sum_{i,j=1}^n b^{ij} l_{x_i} v_{x_j} v_t + l_t v_t^2 - \Psi v_t v + \left(Al_t + \frac{1}{2}\Psi_t\right)v^2 \\
&= 2c_1 T\lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - 2\lambda \sum_{i,j=1}^n b^{ij} d_{x_i} v_{x_j} v_t - \lambda\left(-2c_1 + \sum_{i,j=1}^n (b^{ij} d_{x_i})_{x_j} - c_0\right)v_t v \\
&\quad + 2c_1 T\lambda v_t^2 + \left[2c_1 T\left(4c_1^2 T^2 - \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j}\right)\lambda^3 + O(\lambda^2)\right]v^2 \\
&\ge 2c_1 T\lambda \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - \lambda\left(\sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j}\right)^{\frac{1}{2}} \sum_{i,j=1}^n b^{ij} v_{x_i} v_{x_j} - \lambda\left(\sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j}\right)^{\frac{1}{2}} v_t^2 \\
&\quad + 2c_1 T\lambda v_t^2 - v_t^2 + \left[2c_1 T\left(4c_1^2 T^2 - \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j}\right)\lambda^3 + O(\lambda^2)\right]v^2.
\end{aligned} \tag{4.10}$$

By Condition (2.2), it follows that

$$4c_1^2 T^2 - \sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j} > 0$$

and that

$$2c_1 T - \left(\sum_{i,j=1}^n b^{ij} d_{x_i} d_{x_j}\right)^{\frac{1}{2}} > 0.$$

Hence there exists a $\lambda_1 > 0$ such that for any $\lambda \ge \lambda_1$ and when $t = 0$, it holds that

$$\begin{aligned}
&\sum_{i,j=1}^n b^{ij} l_t v_{x_i} v_{x_j} - 2 \sum_{i,j=1}^n b^{ij} l_{x_i} v_{x_j} v_t + l_t v_t^2 - \Psi v_t v + \left(Al_t + \frac{1}{2}\Psi_t\right)v^2 \\
&\ge C\left[\lambda(v_t^2 + |\nabla v|^2) + \lambda^3 v^2\right].
\end{aligned} \tag{4.11}$$

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On the other hand, since $l_t(T) = 0$, for $t = T$, it holds that

$$\sum_{i,j=1}^{n} b^{ij} l_t v_{x_i} v_{x_j} - 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} v_t + l_t v_t^2 - \Psi v_t v + \left(A l_t + \frac{1}{2} \Psi_t \right) v^2$$

$$= -2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} v_t - \Psi v_t v. \tag{4.12}$$

Noting that $z(T) = 0$ in G, P-a.s., we have $v(T) = 0$ and $v_{x_j}(T) = 0$ in G ($j = 1, 2, \cdots, n$), P-a.s. Thus, from the equality (4.12), we end up with

$$\left\{ \sum_{i,j=1}^{n} b^{ij} l_t v_{x_i} v_{x_j} - 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} v_t + l_t v_t^2 - \Psi v_t v + \left(A l_t + \frac{1}{2} \Psi_t \right) v^2 \right\} \Bigg|_{t=T} = 0, \ P\text{-a.s.} \tag{4.13}$$

Step 3. Integrating (3.4) in Q, taking expectation in Ω and by the argument above, for $\lambda \geq \max\{\lambda_0, \lambda_1\}$, we obtain that

$$\mathbb{E} \int_Q \theta \left\{ \left(-2 l_t v_t + 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} + \Psi v \right) \left[dz_t - \sum_{i,j=1}^{n} (b^{ij} z_{x_i})_{x_j} dt \right] \right\} dx$$

$$+ \lambda \mathbb{E} \int_\Sigma \sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} \left(2 b^{ij} b^{i'j'} d_{x_{i'}} v_{x_i} v_{x_{j'}} - b^{ij} b^{i'j'} d_{x_i} v_{x_{i'}} v_{x_{j'}} \right) \nu_j d\Sigma$$

$$\geq C \left\{ \mathbb{E} \int_Q \left[\theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) + \left(-2 l_t v_t + 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} + \Psi v \right)^2 \right] dx dt \right. \tag{4.14}$$

$$+ \mathbb{E} \int_G \theta^2 \left[\lambda (|\nabla z_0|^2 + |z_1|^2) + \lambda^3 |z_0|^2 \right] dx + \mathbb{E} \int_Q \theta^2 l_t (dz_t)^2 \bigg\}.$$

For the boundary term, noting that $z = 0$ on Σ, it is easy to show that

$$\mathbb{E} \int_\Sigma \sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} \left(2 b^{ij} b^{i'j'} d_{x_{i'}} v_{x_i} v_{x_{j'}} - b^{ij} b^{i'j'} d_{x_i} v_{x_{i'}} v_{x_{j'}} \right) \nu_j d\Sigma$$

$$= \mathbb{E} \int_\Sigma \left(\sum_{i,j=1}^{n} b^{ij} \nu_{x_i} \nu_j \right) \left(\sum_{i',j'=1}^{n} b^{i'j'} d_{x_{i'}} \nu_{j'} \right) \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma. \tag{4.15}$$

From inequality (4.14) and equality (4.15), we obtain that

$$\mathbb{E} \int_Q \theta \left\{ \left(-2 l_t v_t + 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} + \Psi v \right) \left[du_t - \sum_{i,j=1}^{n} (b^{ij} u_{x_i})_{x_j} dt \right] \right\} dx$$

$$+ \lambda \mathbb{E} \int_\Sigma \left(\sum_{i,j=1}^{n} b^{ij} \nu_{x_i} \nu_j \right) \left(\sum_{i',j'=1}^{n} b^{i'j'} d_{x_{i'}} \nu_{j'} \right) \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma$$

$$\geq C \left\{ \mathbb{E} \int_Q \left[\theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) + \left(-2 l_t v_t + 2 \sum_{i,j=1}^{n} b^{ij} l_{x_i} v_{x_j} + \Psi v \right)^2 \right] dx dt \right. \tag{4.16}$$

$$+ \mathbb{E} \int_G \theta^2 \left[\lambda (|\nabla z_0|^2 + |z_1|^2) + \lambda^3 |z_0|^2 \right] dx + \lambda \mathbb{E} \int_Q (T - t) \theta^2 (b_4 z + g)^2 dx dt \bigg\}.$$

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By means of

$$(b_4z + g)^2 \geq \frac{1}{2}g^2 - 2b_4^2z^2,$$

we get

$$\lambda\mathbb{E}\int_Q (T-t)\theta^2(b_4z+g)^2 dxdt \geq \frac{1}{2}\lambda\mathbb{E}\int_Q (T-t)\theta^2 g^2 dxdt - 2\lambda T\mathbb{E}\int_Q \theta^2 b_4^2 z^2 dxdt. \qquad (4.17)$$

On the other hand, by equation (1.2), it is clear that

$$\mathbb{E}\int_Q \theta\Big\{\Big(-2l_tv_t + 2\sum_{i,j=1}^n b^{ij}l_{x_i}v_{x_j} + \Psi v\Big)\Big[dz_t - \sum_{i,j=1}^n (b^{ij}z_{x_i})_{x_j}dt\Big]\Big\}dx$$

$$\leq \mathbb{E}\int_Q \Big(-2l_tv_t + \sum_{i,j=1}^n b^{ij}l_{x_i}v_{x_j} + \Psi v\Big)^2 dxdt + C\Big\{|b_1|_{L_{\mathcal{F}}^\infty(0,T;L^\infty(G))}^2 \mathbb{E}\int_Q \theta^2 z_t^2 dxdt$$

$$+ \Big[|b_2|_{L_{\mathcal{F}}^\infty(0,T;L^\infty(G,\mathbb{R}^n))}^2 + |b_3|_{L_{\mathcal{F}}^\infty(0,T;L^n(G))}^2\Big]\mathbb{E}\int_Q \theta^2 |\nabla z|^2 dxdt$$

$$+ \lambda^2 |b_3|_{L_{\mathcal{F}}^\infty(0,T;L^n(G))}^2 \mathbb{E}\int_Q \theta^2 z^2 dxdt\Big\}. \qquad (4.18)$$

Finally, taking $\tilde{\lambda} = \max\{C\mathcal{A}, \lambda_0, \lambda_1\}$, combining (2.3), (4.16), (4.17) and (4.18), for any $\lambda \geq \tilde{\lambda}$, we conclude the desired estimate (4.1). $\qquad \square$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1: Since $\frac{\partial z}{\partial \nu} = 0$ on Σ_0, P-a.s., we know the right hand side of inequality (4.1) is zero. Therefore, it follows that

$$\mathbb{E}\int_G \theta^2(\lambda|z_1|^2 + \lambda|\nabla z_0|^2 + \lambda^3|z_0|^2)dx = 0 \qquad (4.19)$$

and that

$$\mathbb{E}\int_Q (T-t)\theta^2 g^2 dxdt = 0. \qquad (4.20)$$

From the equality (4.19), we find $z_0 = z_1 = 0$ in G, P-a.s. By means of the equality (4.20), we see $g = 0$ in Q, P-a.s. $\qquad \square$

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