Several inequalities regarding sdepth

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Abstract

We give several bounds for $\operatorname{sdepth}_S(I+J)$, $\operatorname{sdepth}_S(I\cap J)$, $\operatorname{sdepth}_S(S/(I+J))$, $\operatorname{sdepth}_S(S/(I\cap J))$, $\operatorname{sdepth}_S(I:J)$ and $\operatorname{sdepth}_S(S/(I:J))$ where $I,J\subset S = K[x_1,\ldots,x_n]$ are monomial ideals. Also, we give several equivalent forms of Stanley Conjecture for I and S/I, where $I\subset S$ is a monomial ideal.

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Introduction

Let K be a field and $S=K[x_1,\ldots,x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D}: M=\bigoplus_{i=1}^r m_iK[Z_i]$ as K-vector space, where $m_i\in M$, $Z_i\subset\{x_1,\ldots,x_n\}$ such that $m_iK[Z_i]$ is a free $K[Z_i]$-module. We define $\operatorname{sdepth}(\mathcal{D})=\min_{i=1}^r|Z_i|$ and $\operatorname{sdepth}_S(M)=\max\{\operatorname{sdepth}(\mathcal{D})|\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M. Herzog, Vladoiu and Zheng show in [4] that this invariant can be computed in a finite number of steps if $M=I/J$, where $J\subset I\subset S$ are monomial ideals. There are two important particular cases. If $I\subset S$ is a monomial ideal, we are interested in computing $\operatorname{sdepth}_S(S/I)$ and $\operatorname{sdepth}_S(I)$ and to find some relation between them.

Let $I\subset S'=K[x_1,\ldots,x_r]$, $J\subset S''=K[x_{r+1},\ldots,x_n]$ two monomial ideals, and consider $S=K[x_1,\ldots,x_n]$. In Theorem 1.2, we give some lower and upper bounds for $\operatorname{sdepth}_S(IS+JS)$ and $\operatorname{sdepth}_S(S/(IS\cap JS))$. Some lower bounds for $\operatorname{sdepth}_S(IS\cap JS)$ and $\operatorname{sdepth}_S(S/(IS+JS))$ were given in [7], respective in [10]. An important fact, which will use implicitly in our paper, is that $\operatorname{sdepth}_S(IS)=\operatorname{sdepth}_{S'}(I)+n-r$, see [4]. Also, obviously, $\operatorname{depth}_S(IS)=\operatorname{depth}_{S'}(I)+n-r$. In [10], A. Rauf conjectured that $\operatorname{sdepth}_S(I)\geq \operatorname{sdepth}_S(S/I)+1$. We prove that this inequality holds, if $\operatorname{sdepth}_S(I)=\operatorname{sdepth}_{S[y_1]}(I,y_1)$, see Remark 1.4. In the first section we also give some corollaries of Theorem 1.1.

In section 2, we consider the more general case, when $I,J\subset S$ are two monomial ideals. In Theorem 2.2, we give lower bounds for $\operatorname{sdepth}_S(I+J)$, $\operatorname{sdepth}_S(I\cap J)$, $\operatorname{sdepth}_S(S/(I+J))$ and $\operatorname{sdepth}_S(S/(I\cap J))$, where $I,J\subset S$ are two monomial ideals. In section 3, we prove that if $I\subset S$ is a monomial ideal, and $v\in S$ a monomial, then $\operatorname{sdepth}_S S/(I:v)\geq \operatorname{sdepth}_S(S/I)$, see Proposition 2.7. As a consequence, we give lower bounds for $\operatorname{sdepth}_S(I:J)$ and $\operatorname{sdepth}_S(S/(I:J))$, where $I,J\subset S$ are monomial ideals, see Corollary 2.12. Also, if $I\subset S$ is a monomial ideal, we give some bounds for $\operatorname{sdepth}_S(I)$ and $\operatorname{sdepth}_S(S/I)$, in terms of the irreducible irredundant decomposition of I, see Corollary 2.13, and in terms of the primary irredundant decomposition of I, see Corollary 2.14.

In section 3, we give several equivalent forms of Stanley Conjecture for I and S/I, where $I\subset S$ is a monomial ideal. See Propositions 3.1, 3.3, 3.4 and 3.8.

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1 Case of ideals with disjoint support

We denote $S = K[x_1,\ldots,x_n]$ the ring of polynomials in n variables, where $n \geq 2$. For a monomial $u \in S$, we denote $\operatorname{supp}(u) = \{x_i : x_i|u\}$. We begin this section with the following lemma.

Lemma 1.1. *Let $u, v \in S$ be two monomials and $Z, W \subset \{x_1,\ldots,x_n\}$, such that $\operatorname{supp}(u) \subset W$ and $\operatorname{supp}(v) \subset Z$. Then $uK[Z] \cap vK[W] = \operatorname{lcm}(u,v)K[Z \cap W]$.*

Proof. "$\supseteq$": Since $\operatorname{lcm}(u,v) = u \cdot (v/gcd(u,v))$ and $\operatorname{supp}(v) \in K[Z]$, it follows that $\operatorname{lcm}(u,v) \in uK[Z]$. Analogously, $\operatorname{lcm}(u,v) \in vK[W]$ and therefore, it follows that $\operatorname{lcm}(u,v) \in uK[Z] \cap vK[W]$.

"$\subseteq$": Let $w \in uK[Z] \cap vK[W]$ be a monomial. It follows that $w = u \cdot a = v \cdot b$, where $a \in K[Z]$ and $b \in K[W]$ are some monomial. Thus $\operatorname{lcm}(u,v)|w$ and $w = \operatorname{lcm}(u,v) \cdot c$, where $c = w/\operatorname{lcm}(u,v) = a/(\operatorname{lcm}(u,v)/u) = b/(\operatorname{lcm}(u,v)/v)$. Therefore, $c \in K[Z] \cap K[W] = K[Z \cap W]$. ☐

Theorem 1.2. *Let $I \subset S' = K[x_1,\ldots,x_r]$, $J \subset S'' = K[x_{r+1},\ldots,x_n]$ be monomial ideals, where $1 \leq r < n$. Then, we have the following inequalities:*

(1) $\operatorname{sdepth}_S(IS) \geq \operatorname{sdepth}_S(IS+JS) \geq \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_{S''}(J)+\operatorname{sdepth}_{S'}(S'/I)\}$.

(2) $\operatorname{sdepth}_S(IS \cap JS) \geq \operatorname{sdepth}_{S'}(I) + \operatorname{sdepth}_{S''}(J)$.

(3) $\operatorname{sdepth}_S(S/IS) \geq \operatorname{sdepth}_S(S/(IS \cap JS)) \geq \min\{\operatorname{sdepth}_S(S/IS), \operatorname{sdepth}_{S''}(S''/J) + \operatorname{sdepth}_{S'}(I)\}$.

(4) $\operatorname{sdepth}_S(S/(IS + JS)) \geq \operatorname{sdepth}_{S'}(S'/I) + \operatorname{sdepth}_{S''}(S''/J)$.

(5) $\operatorname{depth}_S(S/(IS \cap JS)) - 1 = \operatorname{depth}_S(S/(IS+JS)) = \operatorname{depth}_{S'}(S'/I) + \operatorname{depth}_{S''}(S''/J)$.

(6) $\operatorname{depth}_S(IS \cap JS) = \operatorname{depth}_S(IS + JS) + 1 = \operatorname{depth}_{S'}(I) + \operatorname{depth}_{S''}(J)$ *and* $\operatorname{depth}_S((IS + JS)/IS) = \operatorname{depth}_S(IS + JS)$.

Proof. (1) For the first inequality, let $IS+JS = \bigoplus_{i=1}^r w_iK[W_i]$ be a Stanley decomposition of the ideal $IS + JS \subset S$. Note that $(IS + JS) \cap S' = IS \cap S' = I$, since $JS \cap S' = (0)$. Therefore, $I = \bigoplus_{i=1}^r (w_iK[W_i] \cap S')$. If $w_i \in S'$, we have $w_iK[W_i] \cap S' = w_iK[W_i \cap \{x_1,\ldots,x_r\}]$, by Lemma 1.1. On the other hand, if $w_i \notin S'$, we have $w_iK[W_i] \cap S' = (0)$. Thus, $I = \bigoplus_{w_i \in S'} w_iK[W_i \cap \{x_1,\ldots,x_r\}]$. It follows that $IS = \bigoplus_{w_i \in S'} w_iK[W_i \cup \{x_{r+1},\ldots,x_n\}]$. Therefore, $\operatorname{sdepth}_S(IS + JS) \leq \operatorname{sdepth}_S(IS)$.

In order to prove the second inequality, we consider the Stanley decompositions $S'/I = \bigoplus_{i=1}^r u_iK[U_i]$ and $J = \bigoplus_{j=1}^s v_jK[V_j]$. It follows that $S/IS = \bigoplus_{i=1}^r u_iK[U_i \cup \{x_{r+1},\ldots,x_n\}]$ and $JS = \bigoplus_{j=1}^s v_jK[V_j \cup \{x_1,\ldots,x_r\}]$ are Stanley decompositions for S/IS, respectively for JS. We consider the decomposition:

$$(*) \quad IS + JS = ((IS + JS) \cap IS) \oplus ((IS + JS) \cap S/IS) = IS \oplus (JS \cap S/IS).$$

We have $JS \cap S/IS = \bigoplus_{i=1}^r \bigoplus_{j=1}^s u_iK[U_i \cup \{x_{r+1},\ldots,x_n\}] \cap v_jK[V_j \cup \{x_{r+1},\ldots,x_n\}]$. Since $u_i \in S'$ and $v_j \in S''$ for all $(i,j)'s$, by Lemma 1.1, it follows that $JS \cap S/IS = \bigoplus_{i=1}^r \bigoplus_{j=1}^s u_iv_jK[U_i \cup V_j]$ and therefore $\operatorname{sdepth}_S(JS \cap S/IS) \geq \operatorname{sdepth}''_S(J)$. Thus, by $(*)$, we get the required conclusion.

(2) It was proved in [7, Lemma 1.1].

(3) For the first inequality, let $S/(IS+JS) = \bigoplus_{i=1}^r w_iK[W_i]$ be a Stanley decomposition of $S/(IS+JS)$. As in the proof of (1), we get $S/IS = \bigoplus_{w_i\in S'} w_iK[W_i \cup \{x_{r+1},\ldots x_n\}]$ and thus we get $\operatorname{sdepth}_S(S/IS) \geq \operatorname{sdepth}_S(S/(IS\cap JS))$. In order to prove the second inequality, we consider the decomposition:

$$S/(IS\cap JS) = (S/(IS\cap JS)\cap S/IS) \oplus (S/(IS\cap JS)\cap IS) = S/IS \oplus ((S/JS)\cap IS)$$

and, as in the proof of (1), we get $\operatorname{sdepth}_S((S/JS)\cap IS) \geq \operatorname{sdepth}_{S'}(I) + \operatorname{sdepth}_{S''}(S''/J)$ and thus we obtain the required conclusion.

(4) It was proved in [10, Theorem 3.1].

(5) It is a consequence of Depth's Lemma for the short exact sequence of S-modules

$$0 \to S/(IS\cap JS) \to S/IS \oplus S/JS \to S/(IS+JS) \to 0.$$

See also [7, Lemma 1.1] for more details.

(6) The first equality is a direct consequence of (5). The second follows by Depth Lemma for the short exact sequence $0 \to I \to I+J \to (I+J)/I \to 0$. □

Remark 1.3. *If $I \subset S$ is a monomial ideal, we define* the support of I *to be the set* $\operatorname{supp}(I) = \bigcup_{u\in G(I)} \operatorname{supp}(u)$, *where $G(I)$ is the set on minimal monomial generators of I. With this notation, we can reformulate Theorem* 1.2 *in terms of two monomial ideals $I, J \subset S$ with* $\operatorname{supp}(I)\cap\operatorname{supp}(J) = \emptyset$. *The conclusions should be also modified, as follows. If $I, J \subset S$ are two monomial ideals with disjoint supports, then* $\operatorname{sdepth}_S(I\cap J) \geq \operatorname{sdepth}_S(I) + \operatorname{sdepth}_S(J) - n$ *etc.*

With the above notations, we may consider the short exact sequences $0 \to I \to I+J \to (I+J)/I \to 0$ *and* $0 \to I/(I\cap J) \cong (I+J)/J \to S/(I\cap J) \to S/J \to 0$. *It follows that* $\operatorname{sdepth}_S(I+J) \geq \min\{\operatorname{sdepth}_S(I), \operatorname{sdepth}_S((I+J)/I)\}$ *and* $\operatorname{sdepth}_S(S/(I\cap J)) \geq \min\{\operatorname{sdepth}_S(S/I), \operatorname{sdepth}_S((I+J)/J)\}$. *Note that $(I+J)/I = J\cap(S/I)$ and $(I+J)/J = I\cap(S/J)$. From the proof of Theorem* 1.2(1), *we get* $\operatorname{sdepth}_S((I+J)/I) \geq \operatorname{sdepth}_S(J) + \operatorname{sdepth}_S(S/I) - n$, *if* $\operatorname{supp}(I)\cap\operatorname{supp}(J) = \emptyset$.

We recall the facts that if $I = (u_1,\ldots,u_m) \subset S$ is a monomial complete intersection, then $\operatorname{sdepth}_S(I) = n - \lfloor m/2\rfloor$, see [12, Theorem 2.4] and $\operatorname{sdepth}_S(S/I) = n-m$, see [11, Theorem 1.1]. On the other hand, if $I = (u_1,\ldots,u_m) \subset S$ is an arbitrary monomial ideal, then, according to [6, Theorem 2.1], $\operatorname{sdepth}_S(I) \geq n - \lfloor m/2\rfloor$ and according to [2, Proposition 1.2], $\operatorname{sdepth}_S(S/I) \geq n-m$. Using these results, we proved the following:

Corollary 1.4. *Let $I \subset S' = K[x_1,\ldots,x_r]$ be a monomial ideal and $J = (u_1,\ldots,u_m) \subset S'' = K[x_{r+1},\ldots,x_n]$ be a monomial ideal. Then:*

(1) $\operatorname{sdepth}_S(IS) \geq \operatorname{sdepth}_S(IS+JS) \geq \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_S(S/SI) - \lfloor m/2\rfloor\}$.

(2) $\operatorname{sdepth}_S(IS\cap JS) \geq \operatorname{sdepth}_S(IS) - \lfloor m/2\rfloor$.

(3) $\operatorname{sdepth}_S(S/IS) \geq \operatorname{sdepth}_S(S/(IS\cap JS)) \geq \min\{\operatorname{sdepth}_S(S/IS), \operatorname{sdepth}_S(IS) - m\}$.

(4) $\operatorname{sdepth}_S(S/(IS+JS)) \geq \operatorname{sdepth}_S(S/IS) - m$.

(5) In particular, if J is complete intersection, then: $\operatorname{depth}_S(S/(IS\cap JS)) - 1 = \operatorname{depth}_S(S/(IS+JS)) = \operatorname{depth}_S(S/IS) - m$.

Remark 1.5. Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. If we denote $\bar{S} = S[y_1, \ldots, y_m]$, then, by Corollary 1.4(1), we have

$$\operatorname{sdepth}_S(I) + m \geq \operatorname{sdepth}_{\bar{S}}(I, y_1, \ldots, y_m) \geq \min\{\operatorname{sdepth}_S(I) + m, \operatorname{sdepth}_S(S/I) + \lceil m/2 \rceil\}.$$

Assume $\operatorname{sdepth}_S(I) + m > \operatorname{sdepth}_{\bar{S}}(I, y_1, \ldots, y_m)$. It follows that $\operatorname{sdepth}_S(I) + m > \operatorname{sdepth}_S(S/I) + \lceil m/2 \rceil$ and therefore $\operatorname{sdepth}_S(I) \geq \operatorname{sdepth}_S(S/I) + \lfloor m/2 \rfloor + 1$. In particular, if $m = 1$ and $\operatorname{sdepth}_{\bar{S}}(I, y_1) = \operatorname{sdepth}_S(I)$, then $\operatorname{sdepth}_S(I) \geq \operatorname{sdepth}_S(S/I) + 1$ and thus we get a positive answer to the problem put by Asia in [10].

Corollary 1.6. *With the notations of Theorem* 1.2*, we have the followings:*

(1) If the Stanley conjecture hold for I and J, then the Stanley conjecture holds for $IS \cap JS$.

(2) If the Stanley conjecture hold for S'/I and S''/J, then the Stanley conjecture holds for $S/(IS + JS)$.

(3) If the Stanley conjecture hold for J and S'/I or for I and S''/J, then the Stanley conjecture hold for $(IS + JS)$ and $S/(IS \cap JS)$.

Proof. (1) It is a direct consequence of Theorem 1.2(2) and 1.2(6). (2) It is a direct consequence of Theorem 1.2(4) and 1.2(5).

(3) Assume the Stanley conjecture hold for J and S'/I. According to Theorem 1.2(1), we have $\operatorname{sdepth}_S(IS + JS) \geq \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_{S''}(J) + \operatorname{sdepth}_{S'}(S'/I)\}$. If $\operatorname{sdepth}_S(IS+JS) = \operatorname{sdepth}_S(IS)$, then, by 1.2(6), we get $\operatorname{sdepth}_S(IS+JS) \geq \operatorname{depth}_S(IS) = \operatorname{depth}_{S'}(I) + n - r \geq \operatorname{depth}_{S'}(I) + \operatorname{depth}_{S''}(J) > \operatorname{depth}_S(IS + JS)$.

If $\operatorname{sdepth}_S(IS+JS) < \operatorname{sdepth}_S(IS)$, it follows that $\operatorname{sdepth}_S(IS+JS) \geq \operatorname{sdepth}_{S''}(J) + \operatorname{sdepth}_{S'}(S'/I) \geq \operatorname{depth}_{S''}(J) + \operatorname{depth}_{S'}(S'/I) = \operatorname{depth}_S(IS + JS)$. In the both cases, the ideal $IS + JS$ satisfies the Stanley conjecture. The case when I and S''/J satisfy the Stanley conjecture is similar. Also, the proof of the fact that $S/(IS \cap JS)$ satisfies the Stanley conjecture follows in the same way from 1.2(3) and 1.2(5). $\square$

Note that, by the proof of Corollary 1.6(1), if $\operatorname{sdepth}_S(IS + JS) = \operatorname{sdepth}_S(IS)$, then $\operatorname{sdepth}_S(IS + JS) \geq \operatorname{depth}_S(IS + JS) + n - r - \operatorname{depth}_{S''}(S''/J)$. Analogously, if $\operatorname{sdepth}_S(S/(IS \cap JS)) = \operatorname{sdepth}_S(IS)$ then $\operatorname{sdepth}_S(S/(IS \cap JS)) \geq \operatorname{depth}_S(S/(IS \cap JS)) + n - r - \operatorname{depth}_{S''}(S''/J)$.

Corollary 1.7. *Let $I_j \subset S_j := [x_{j1}, \ldots, x_{jn_j}]$ be some monomial ideals, where $k \geq 2, n_j \geq 1$ and $1 \leq j \leq k$. Denote $S = K[x_{ji} : \ 1 \leq j \leq k, \ 1 \leq i \leq n_j]$. Then, the following inequalities hold:*

(1) $\operatorname{sdepth}_S(I_1S \cap \cdots \cap I_kS) \geq \operatorname{sdepth}_{S_1}(I_1) + \cdots + \operatorname{sdepth}_{S_k}(I_k)$.

(2) $\operatorname{sdepth}_S(I_1S + \cdots + I_kS) \geq \min\{\operatorname{sdepth}_{S_1}(I_1) + n_2 + \cdots + n_k, \operatorname{sdepth}_{S_2}(I_2) + \operatorname{sdepth}_{S_1}(S_1/I_1) + n_3 + \cdots + n_k, \ldots, \operatorname{sdepth}_{S_k}(I_k) + \operatorname{sdepth}_{S_{k-1}}(S_{k-1}/I_{k-1}) + \cdots + \operatorname{sdepth}_{S_1}(S_1/I_1)\}$.

$\operatorname{sdepth}_S(I_1S + \cdots + I_kS) \leq \min\{\operatorname{sdepth}_S(I_jS) : \ j = 1, \ldots, k\}$.

(3) $\operatorname{sdepth}_S(S/(I_1S \cap \cdots \cap I_kS)) \geq \min\{\operatorname{sdepth}_{S_1}(S_1/I_1) + n_2 + \cdots + n_k, \operatorname{sdepth}_{S_2}(S_2/I_2) + \operatorname{sdepth}_{S_1}(I_1) + n_3 + \cdots + n_k, \ldots, \operatorname{sdepth}_{S_k}(S_k/I_k) + \operatorname{sdepth}_{S_{k-1}}(I_{k-1}) + \cdots + \operatorname{sdepth}_{S_1}(I_1)\}$

$\operatorname{sdepth}_S(S/(I_1S \cap \cdots \cap I_kS)) \leq \min\{\operatorname{sdepth}_S(S/I_jS) : \ j = 1, \ldots, k\}$.

(4) $\mathrm{sdepth}_S(S/(I_1S+\cdots+I_kS)) \geq \mathrm{sdepth}_{S_1}(I_1S)+\cdots+\mathrm{sdepth}_{S_k}(I_kS)$.

(5) $\mathrm{depth}_S(I_1S\cap\cdots\cap I_kS) = \mathrm{depth}_S(I_1S+\cdots+I_kS)+(k-1) = \mathrm{depth}_{S_1}(I_1)+\cdots+\mathrm{depth}_{S_k}(I_k)$.

Proof. We use induction on $k\geq 2$ and we apply Theorem 1.2. □

Corollary 1.8. *With the notation of the previous Corollary, we have:*

(1) If $I_1,\ldots,I_k$ satisfy the Stanley Conjecture, then $I_1S\cap\cdots\cap I_kS$ satisfies the Stanley Conjecture.

(2) If $1\leq l\leq n$ is an integer and the Stanley conjecture holds for I_l and S/I_j for all $j\neq l$ then, the Stanley Conjecture holds for $I_1S+\cdots+I_kS$.

(3) If $1\leq l\leq n$ is an integer and the Stanley conjecture holds for S_l/I_l and I_j for all $j\neq l$ then, the Stanley Conjecture holds for $S/(I_1S\cap\cdots\cap I_kS)$.

(4) If $S/I_1,\ldots,S/I_k$ satisfy the Stanley Conjecture, then $S/(I_1S+\cdots+I_kS)$ satisfies the Stanley Conjecture.

Proof. (1) We use induction on k and apply Corollary 1.7(1).

(2) We may assume $l=k$. Denote $S'=K[x_{ji}:\ 1\leq j\leq k-1,\ 1\leq i\leq n_j]$ and consider the ideal $I':=I_1S'+\cdots+I_{k-1}S'\subset S$. By (1), it follows that the Stanley Conjecture holds for S'/I'. We denote $I=I_1S+\cdots+I_kS$. According to Corollary 1.7(3), since Stanley conjecture holds for S'/I' and I_k and since $I=I'S+I_kS$, it follows that the Stanley Conjecture holds for I.

(3) The proof is similar to the proof of (2).

(4) We use induction on k and apply Corollary 1.7(4). □

Corollary 1.9. *With the notations of* 1.7*, if all $n_j\leq 5$ and all $I_j's$ are squarefree, then $I_1S\cap\cdots\cap I_kS$, $I_1S+\cdots+I_kS$, $S/(I_1S\cap\cdots\cap I_kS)$ and $S/(I_1S+\cdots+I_kS)$ satisfy the Stanley Conjecture.*

Proof. Indeed, if $I\subset K[x_1,\ldots,x_n]$ is a squarefree monomial ideal with $n\leq 5$, then both I and S/I satisfies the Stanley Conjecture, see [8] and [9]. Therefore, $I_j's$ and $S_j/I_j's$ satisfy the Stanley Conjecture. By Corollary 1.8 we are done. □

Example 1.10. *Let $I=(x_{11},\ldots,x_{1n_1})\cap(x_{21},\ldots,x_{2n_2})\cap\cdots\cap(x_{k1},\ldots,x_{kn_k})\subset S$, where $k\geq 2$,$n_j\geq 1$, $1\leq j\leq k$ and $S=K[x_{ji}:\ 1\leq j\leq k,\ 1\leq i\leq n_j]$. According to Corollary* 1.7(1)*, $\mathrm{sdepth}_S(I)\geq\lceil n_1/2\rceil+\cdots+\lceil n_k/2\rceil$. Note that $\mathrm{sdepth}_S(I)\geq\mathrm{depth}_S(I)=k$. Also, according to Corollary* 3.2 *or [5, Theorem 3.1], $\mathrm{sdepth}_S(I)\leq\min\{n-\lfloor n_j/2\rfloor:\ 1\leq j\leq k\}$.*

Now, we want to estimate $\mathrm{sdepth}_S(S/I)$. According to Corollary 1.7(3)*, we have:*

$$\mathrm{sdepth}_S(S/I)\geq\min\{n_2+\cdots+n_k,\lceil n_1/2\rceil+n_3+\cdots+n_k,\lceil n_1/2\rceil+$$

$$+\lceil n_2/2\rceil+n_4+\cdots+n_k,\ldots,\lceil n_1/2\rceil+\cdots+\lceil n_{k-1}/2\rceil+n_k\}$$

Note that $\mathrm{sdepth}_S(S/I)\geq\mathrm{depth}_S(S/I)=k-1$. Also, according to Corollary 3.2 *or Corollary* 1.7(3)*, we have $\mathrm{sdepth}_S(S/I)\leq\min\{n-n_j:\ 1\leq j\leq k\}$.*

2 The general case

In the following, we consider $1 \leq s \leq r+1 \leq n$ three integers, with $n \geq 2$. We denote $S' := K[x_1,\ldots,x_r]$, $S'' := K[x_s,\ldots,x_n]$ and $S := K[x_1,\ldots,x_n]$. Let $p := r-s+1$.

Lemma 2.1. *Let $u \in S'$ and $v \in S''$ be two monomials, $Z \subset \{x_1,\ldots,x_r\}$ and $W \subset \{x_s,\ldots,x_n\}$ two subsets of variables. We denote $\bar{Z} := Z \cup \{x_{r+1},\ldots,x_n\}$ and $\bar{W} := W \cup \{x_1,\ldots,x_{s-1}\}$. If $L := uK[\bar{Z}] \cap vK[\bar{W}]$, then $L = \{0\}$ or $L = \operatorname{lcm}(u,v)K[(Z \cup W) \setminus Y]$, where $Y \subset \{x_s,\ldots,x_r\}$ and with $|(Z \cup W) \setminus Y| \geq |Z| + |W| - p$.*

Proof. We use induction on $p = r-s+1$. If $p = 0$, it follows that $s = r+1$ and therefore $\operatorname{supp}(u) \subset \{x_1,\ldots,x_{s-1}\}$ and $\operatorname{supp}(v) \subset \{x_{r+1},\ldots,x_n\}$. Thus, by Lemma 1.1, we get

$$L = \operatorname{lcm}(u,v)K[(Z \cup \{x_{r+1},\ldots,x_n\}) \cap (W \cup \{x_1,\ldots,x_r\})] = \operatorname{lcm}(u,v)K[Z \cup W].$$

Now, assume $p > 0$, i.e. $r \geq s$. We must consider several cases. First, suppose $x_s \notin \operatorname{supp}(u)$ and $x_s \notin \operatorname{supp}(v)$. If $x_s \in Z \cap W$, we can write $L = uK[\bar{Z}] \cap vK[\bar{W}] = (uK[\bar{Z} \setminus \{x_s\}] \cap vK[\bar{W} \setminus \{x_s\}])[x_s]$. Using the induction hypothesis, we are done. On the other hand, if $x_s \notin Z \cap W$, then $L = uK[\bar{Z} \setminus \{x_s\}] \cap vK[\bar{W} \setminus \{x_s\}]$. Note that $|\bar{Z} \cap \bar{W}| = |\bar{Z} \setminus \{x_s\} \cap \bar{W} \setminus \{x_s\}| \geq |W \setminus \{x_s\}| + |Z \setminus \{x_s\}| - p + 1 \geq |Z| + |W| - p$, since the variable x_s appear only in one of the sets W and Z. Therefore, by induction, we are done.

Now, assume $x_s \in \operatorname{supp}(u)$, and denote $\alpha = \max\{j :\ x_s^j | u\}$ and $\beta = \max\{j :\ x_s^j | v\}$. We write $u = x_s^{\alpha}\tilde{u}$ and $v = x_s^{\beta}\tilde{v}$. If $x_s \notin Z$ we have two subcases:

a) Assume $x_s \notin W$. If $\alpha \neq \beta$, it follows that $L = \{0\}$. If $\alpha = \beta$, then $L = x_s^{\alpha}(\tilde{u}K[Z] \cap \tilde{v}K[W])$ and we are done by induction, noting that $\operatorname{lcm}(u,v) = x_s^{\alpha}\operatorname{lcm}(\tilde{u},\tilde{v})$.

b) If $x_s \in W$ and $\alpha < \beta$, we have $L = \{0\}$. If $\alpha \geq \beta$, we have $L = x_s^{\alpha}(\tilde{u}K[Z] \cap \tilde{v}K[W])$ and we are done by induction, noting that $\operatorname{lcm}(u,v) = x_s^{\alpha}\operatorname{lcm}(\tilde{u},\tilde{v})$.

If $x_s \in Z$, we must also consider two subcases:

a) If $x_s \notin W$ and $\alpha > \beta$, it follows that $L = \{0\}$. If $\alpha \leq \beta$, we have $L = x_s^{\beta}(\tilde{u}K[Z] \cap \tilde{v}K[W])$ and we are done by induction.

b) If $x_s \in W$, we have $L = x_s^{\max\{\alpha,\beta\}}(\tilde{u}K[\bar{Z} \setminus \{x_s\}] \cap \tilde{v}K[\bar{W} \setminus \{x_s\}])[x_s]$ and, again, we are done by induction. □

Now, we are able to prove the following theorem, which generalize some results of Theorem 1.2.

Theorem 2.2. *Let $I \subset S'$ and $J \subset S''$ be two monomial ideals. Then:*

(1) $\operatorname{sdepth}_S(IS \cap JS) \geq \operatorname{sdepth}_{S'}(I) + \operatorname{sdepth}_{S''}(J) - p = \operatorname{sdepth}_S(IS) + \operatorname{sdepth}_S(JS) - n$.

(2) $\operatorname{sdepth}_S(S/(IS+JS)) \geq \operatorname{sdepth}_{S'}(S'/I) + \operatorname{sdepth}_{S''}(S''/J) - p = \operatorname{sdepth}_S(S/IS) + \operatorname{sdepth}_S(S/JS) - n$.

(3) $\operatorname{sdepth}_S(IS+JS) \geq \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_{S''}(J) + \operatorname{sdepth}_{S'}(S'/I) - p\} = \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_S(JS) + \operatorname{sdepth}_S(S/IS) - n\}$.

(4) $\operatorname{sdepth}_S(S/(IS \cap JS)) \geq \min\{\operatorname{sdepth}_S(S/IS), \operatorname{sdepth}_{S''}(S''/J) + \operatorname{sdepth}_{S'}(I) - p\} = \min\{\operatorname{sdepth}_S(S/IS), \operatorname{sdepth}_S(S/JS) + \operatorname{sdepth}_S(IS) - n\}$.

Proof. (1) We consider $I=\bigoplus_{i=1}^{a}u_iK[Z_i]$ and $J=\bigoplus_{j=1}^{b}v_jK[W_j]$ two Stanley decomposition for I, respective for J. Then $IS=\bigoplus_{i=1}^{a}u_iK[\bar{Z}_i]$, where $\bar{Z}_i=Z_i\cup\{x_{r+1},\ldots,x_n\}$ and $JS=\bigoplus_{j=1}^{b}v_iK[\bar{W}_i]$, where $\bar{W}_j=W_j\cup\{x_1,\ldots,x_{s-1}\}$. We have $IS\cap JS=\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}L_{ij}$ a Stanley decomposition for $IS\cap JS$, where $L_{ij}:=u_iK[\bar{Z}_i]\cap v_j[\bar{W}_j]$. According to Lemma 2.1, $L_{ij}=\{0\}$ or $L_{ij}=\operatorname{lcm}(u_i,v_j)K[(Z_i\cup W_j)\setminus Y_{ij}]$, where $Y_{ij}\subset\{x_s,\ldots,x_r\}$ and $|(Z_i\cup W_j)\setminus Y_{ij}|\geq|Z_i|+|W_j|-p$. Therefore, we are done.

(2) The proof is similar with the proof of (1).

(3) We consider $S'/I=\bigoplus_{i=1}^{a}u_iK[Z_i]$ and $J=\bigoplus_{j=1}^{b}v_jK[W_j]$ two Stanley decomposition for S'/I, respective for J. Then $S/IS=\bigoplus_{i=1}^{a}u_iK[\bar{Z}_i]$, where $\bar{Z}_i=Z_i\cup\{x_{r+1},\ldots,x_n\}$ and $JS=\bigoplus_{j=1}^{b}v_iK[\bar{W}_i]$, where $\bar{W}_j=W_j\cup\{x_1,\ldots,x_{s-1}\}$. We use the decomposition:

$$IS+JS=((IS+JS)\cap IS)\oplus((IS+JS)\cap(S/IS))=IS\oplus(JS\cap(S/IS)).$$

If follows, that $\operatorname{sdepth}_S(IS+JS)\geq\min\{\operatorname{sdepth}_S(IS),\operatorname{sdepth}_S(JS\cap(S/IS))\}$. We have $JS\cap S/IS=\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b}L_ij$ a Stanley decomposition for $IS\cap JS$, where $L_{ij}:=u_iK[\bar{Z}_i]\cap v_j[\bar{W}_j]$. By Lemma 2.1, it follows that $\operatorname{sdepth}_S(JS\cap(S/IS))\geq\operatorname{sdepth}_{S'}(S'/I)+\operatorname{sdepth}_{S''}(J)$ and therefore we are done.

(4) The proof is similar with the proof of (3). □

Remark 2.3. *Note that the results of the previous Theorem do not depend on the numbers r and s. Therefore, we can reformulate the Theorem 2.2 in terms of arbitrary monomial ideals $I,J\subset S$. Also, if $I,J\subset S$ are two monomial ideals, the minimal number p which can be chose, by a reordering of the variables, is $p=|\operatorname{supp}(I)\cap\operatorname{supp}(J)|$.*

Also, as in Remark 1.3, we have $\operatorname{sdepth}_S((I+J)/I)\geq\operatorname{sdepth}_S(J)+\operatorname{sdepth}_S(S/I)-n$. *Therefore, in particular, if $I\subset J$, then* $\operatorname{sdepth}_S(J/I)\geq\operatorname{sdepth}_S(J)+\operatorname{sdepth}_S(S/I)-n$.

Using the previous remark, we have the following Corollary.

Corollary 2.4. *If $I,J\subset S$ are two monomial ideals and $|G(J)|=m$, then:*
(1) $\operatorname{sdepth}_S(I\cap J)\geq\operatorname{sdepth}_S(I)-\lfloor m/2\rfloor$.
(2) $\operatorname{sdepth}_S(I+J)\geq\min\{\operatorname{sdepth}_S(I),\operatorname{sdepth}_S(S/I)-\lfloor m/2\rfloor\}$.
$\operatorname{sdepth}_S(I+J)\geq\operatorname{sdepth}_S(I)-m$.
(3) $\operatorname{sdepth}_S(S/(I+J))\geq\operatorname{sdepth}_S(S/I)-m$.
(4) $\operatorname{sdepth}_S(S/(I\cap J))\geq\min\{\operatorname{sdepth}_S(S/I),\operatorname{sdepth}_S(I)-m\}$.
$\operatorname{sdepth}_S(S/(I\cap J))\geq\min\{n-m,\operatorname{sdepth}_S(S/I)-\lfloor m/2\rfloor\}$.
(5) $\operatorname{sdepth}_S((I+J)/I)\geq\operatorname{sdepth}_S(S/I)-\lfloor m/2\rfloor$.
$\operatorname{sdepth}_S((I+J)/J)\geq\operatorname{sdepth}_S(I)-m$.

Proof. We apply Theorem 2.2 and use the facts that $\operatorname{sdepth}_S(J)\geq n-\lfloor m/2\rfloor$, see [6, Theorem 2.1] and $\operatorname{sdepth}_S(S/J)\geq n-m$, see [2, Proposition 1.2]. □

Corollary 2.5. *If $I\subset S$ is a monomial ideal and $u\in S$ a monomial, then:*
(1) $\operatorname{sdepth}_S(I\cap(u))\geq\operatorname{sdepth}_S(I)$.
(2) $\operatorname{sdepth}_S(I,u)\geq\min\{\operatorname{sdepth}_S(I),\operatorname{sdepth}_S(S/I)\}$.
(3) $\operatorname{sdepth}_S(S/(I,u))\geq\operatorname{sdepth}_S(S/I)-1$.
(4) $\operatorname{sdepth}_S(S/(I\cap(u)))\geq\operatorname{sdepth}_S(S/I)$.

A. Rauf [10] proved that $\operatorname{depth}_S(S/(I : u)) \geq \operatorname{depth}_S(S/I)$, for any monomial ideal $I \subset S$ and any monomial $u \in S$, see [10, Corollary 1.3]. Similar results hold for $\operatorname{sdepth}_S(I : u)$ and $\operatorname{sdepth}_S(S/(I : u))$. In order to show that, we use Corollary 2.5 and the following result from [2].

Theorem 2.6. *[2, Theorem 1.4] Let $I \subset S$ be a monomial ideal such that $I = v(I : v)$, for a monomial $v \in S$. Then $\operatorname{sdepth}_S(I) = \operatorname{sdepth}_S(I : v)$, $\operatorname{sdepth}_S(S/I) = \operatorname{sdepth}_S(S/(I : v))$.*

Proposition 2.7. *If $I \subset S$ is a monomial ideal and $u \in S$ a monomial, then:*
(1) $\operatorname{sdepth}_S(I : u) \geq \operatorname{sdepth}_S(I)$. *([8, Proposition 1.3])*
(2) $\operatorname{sdepth}_S(S/(I : u)) \geq \operatorname{sdepth}_S(S/I)$.

Proof. (1) Note that $I \cap (u) = u(I : u)$. By Theorem 2.6, it follows that $\operatorname{sdepth}_S(I : u) = \operatorname{sdepth}_S(I \cap (u)) \geq \operatorname{sdepth}_S(I)$. See another proof in [8].
(2) By Theorem 2.6 and Corollary 2.5, $\operatorname{sdepth}_S(S/(I : u)) = \operatorname{sdepth}_S(S/(I \cap (u))$. □

Note that if $P \in Ass(S/I)$ is an associated prime, then there exists a monomial $v \in S$ such that $P = (I : v)$. Using the above Proposition, we obtain again the results of Ishaq [5] and Apel [1] .

Corollary 2.8. *If $I \subset S$ is a monomial ideal, with $Ass(S/I) = \{P_1, \ldots, P_r\}$. If we denote $d_i = |P_i|$, we have:*
(1) $\operatorname{sdepth}_S(I) \leq \min\{n - \lfloor d_i/2 \rfloor : i = 1, \ldots r\}$. *(Ishaq)*
(2) $\operatorname{sdepth}_S(S/I) \leq \min\{n - d_i : i = 1, \ldots r\}$. *(Apel)*

Proof. (1) It is enough to notice that $\operatorname{sdepth}_S(P_i) = n - \lfloor d_i/2 \rfloor$. See also [5, Theorem 1.1].
(2) It is enough to notice that $\operatorname{sdepth}_S(P_i) = n - d_i$. See also [1]. □

Corollary 2.9. *Let $I \subset S$ be a monomial ideal minimally generated by m monomials, such that there exists a prime ideal $P \in Ass(S/I)$ with $ht(P) = m$. Then $\operatorname{sdepth}_S(S/I) = n - m$.*

Proof. It is a direct consequence of Theorem 2.6 and Corollary 2.8(2). □

Remark 2.10. Let $I \subset S$ be a monomial ideal. Then $\operatorname{sdepth}_S(S/I) = n - 1$ if and only if I is principal. Indeed, I is principal if and only if all the primes in $Ass(S/I)$ have height 1. Therefore, we are done by Corollary 2.8(2).

Corollary 2.11. *Let $k \geq 2$ be an integer, and let $I_j \subset S$ be some monomial ideals, where $1 \leq j \leq k$. Then:*
(1) $\operatorname{sdepth}_S(I_1 \cap \cdots \cap I_k) \geq \operatorname{sdepth}_S(I_1) + \cdots + \operatorname{sdepth}_S(I_k) - n(k - 1)$.
(2) $\operatorname{sdepth}_S(I_1 + \cdots + I_k) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - n, \ldots, \operatorname{sdepth}_S(I_k) + \operatorname{sdepth}_S(S/I_{k-1}) + \cdots + \operatorname{sdepth}_S(S/I_1) - n(k - 1)\}$.
(3) $\operatorname{sdepth}_S(S/(I_1 \cap \cdots \cap I_k)) \geq \min\{\operatorname{sdepth}_S(S/I_1), \operatorname{sdepth}_S(S/I_2) + \operatorname{sdepth}_S(I_1) - n, \ldots, \operatorname{sdepth}_S(S/I_k) + \operatorname{sdepth}_S(I_{k-1}) + \cdots + \operatorname{sdepth}_S(I_1) - n(k - 1)\}$.
(4) $\operatorname{sdepth}_S(S/(I_1 + \cdots + I_k)) \geq \operatorname{sdepth}_S(S/I_1) + \cdots + \operatorname{sdepth}_S(S/I_k) - n(k - 1)$.

Proof. We use induction on $k \geq 2$ and we apply Theorem 2.2. □

Corollary 2.12. *Let $I, J \subset S$ be two monomial ideals, such that $G(J) = \{u_1, \ldots, u_k\}$ is the set of minimal monomial generators of J. Then:*

(1) $\operatorname{sdepth}_S(I : J) \geq \operatorname{sdepth}_S(I : u_1) + \operatorname{sdepth}_S(I : u_2) + \cdots + \operatorname{sdepth}_S(I : u_k) - n(k-1) \geq k \operatorname{sdepth}_S(I) - n(k-1)$.

(2) $\operatorname{sdepth}_S(S/(I : J)) \geq \min\{\operatorname{sdepth}_S(S/(I : u_1)), \operatorname{sdepth}_S(S/(I : u_2)) + \operatorname{sdepth}_S(I : u_1) - n, \ldots, \operatorname{sdepth}_S(S/(I : u_k)) + \operatorname{sdepth}_S(I : u_{k-1}) + \cdots + \operatorname{sdepth}_S(I : u_1) - n(k-1)\} \geq \operatorname{sdepth}_S(S/I) + (k-1)\operatorname{sdepth}_S(I) - n(k-1)$.

Proof. (1) Note that $(I : J) = (I : u_1) \cap (I : u_2) \cap \cdots \cap (I : u_k)$. Therefore, the first inequality is a direct consequence of 2.11(1). The second inequality is a consequence of Proposition 2.7(1).

(2) Similarly to (1), we use Corollary 2.11(3) and Proposition 2.7(2). □

Now, let $I \subset S$ be a monomial ideal and let $I = C_1 \cap \cdots \cap C_k$, be the irredundant minimal decomposition of I. If we denote $P_j = \sqrt{C_j}$ for $1 \leq j \leq k$, we have $Ass(S/I) = \{P_1, \ldots, P_k\}$. In particular, if I is squarefree, $C_j = P_j$ for all j. Denote $d_j = |P_j|$, where $1 \leq i \leq k$. We may assume that $d_1 \geq d_2 \geq \cdots \geq d_k$. Using [3, Theorem 1.3], Proposition 2.8 and Corollary 2.11, we obtain, by straightforward computations, the following bounds for $\operatorname{sdepth}_S(I)$ and $\operatorname{sdepth}_S(S/I)$.

Corollary 2.13. *(1)* $n - \lfloor d_1/2 \rfloor \geq \operatorname{sdepth}_S(I) \geq n - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_k/2 \rfloor$.

(2) $n - d_1 \geq \operatorname{sdepth}_S(S/I) \geq n - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_{k-1}/2 \rfloor - d_k$.

In a more general case, let $I = Q_1 \cap \cdots \cap Q_k$ be the primary irredundant decomposition of I, $P_i = \sqrt{Q_i}$ and denote $q_j = \operatorname{sdepth}_S(Q_j)$ and $d_j = |P_j|$. We may assume that $d_1 \geq d_2 \geq \cdots \geq d_k$. Note that $q_j \leq n - d_j/2$, since $P_j = (Q_j : u_j)$, where $u_j \in S$ is a monomial, and therefore $\operatorname{sdepth}_S(Q_j) \leq \operatorname{sdepth}_S(P_j)$, by Proposition 2.7(1). On the other hand, we obviously have $\operatorname{sdepth}_S(S/Q_j) = \operatorname{sdepth}_S(S/P_j)$. Using Proposition 2.8 and Corollary 2.11, we obtain, by straightforward computations, the following bounds for $\operatorname{sdepth}_S(I)$ and $\operatorname{sdepth}_S(S/I)$.

Corollary 2.14. *(1)* $n - \lfloor d_1/2 \rfloor \geq \operatorname{sdepth}_S(I) \geq q_1 + \cdots + q_k - n(k-1)$.

(2) $n - d_1 \geq \operatorname{sdepth}_S(S/I) \geq \min\{n - d_1, q_1 - d_2, q_1 + q_2 - d_3 - n, \ldots, q_1 + \cdots + q_{k-1} - d_k - n(k-2)\}$.

Example 2.15. *Let* $I = Q_1 \cap Q_2 \cap Q_3 \subset S := K[x_1, \ldots, x_7]$, *where* $Q_1 = (x_1^2, \ldots, x_5^2)$, $Q_2 = (x_4^3, x_5^3, x_6^3)$ *and* $Q_3 = (x_6^3, x_6x_7, x_7^2)$. *Denote* $P_j = \sqrt{Q_j}$. *Note that* $q_3 = \operatorname{sdepth}_S(Q_3) = \operatorname{sdepth}_{K[x_6,x_7]}(Q_3 \cap K[x_6, x_7]) + 5 = 1 + 5 = 6$. *Also, since* Q_1 *and* Q_2 *are generated by powers of variables, by [3, Theorem 1.3],* $q_1 = 7 - \lfloor 5/2 \rfloor = 5$ *and* $q_2 = 7 - \lfloor 3/2 \rfloor = 6$. *According to Corollary* 2.14, *we have* $5 = 7 - \lfloor d_1/2 \rfloor \geq \operatorname{sdepth}_S(I) \geq q_1 + q_2 + q_3 - 14 = 3$ *and* $2 = 7 - d_1 \geq \operatorname{sdepth}_S(S/I) \geq \min\{7 - d_1, q_1 - d_2, q_1 + q_2 - d_3 - 7\} = \min\{7-5, 5-3, 5+6-2-7\} = 2$. *Thus* $\operatorname{sdepth}_S(I) \in \{3, 4, 5\}$ *and* $\operatorname{sdepth}_S(S/I) = 2$.

On the other hand, $\operatorname{depth}_S(S/I) \leq \min\{n - \operatorname{depth}_S(S/P_j) : j = 1, 2, 3\} = 2$. *In particular, we have* $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$ *and* $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$. *Thus both I and S/I satisfy the Stanley conjecture. In fact, using CoCoA, we get* $\operatorname{depth}_S(S/I) = 2$.

3 Equivalent forms of Stanley conjecture

Proposition 3.1. *The following assertions are equivalent:*

(1) For any integer $n \geq 1$ and any monomial ideal $I \subset S = K[x_1,\ldots,x_n]$, Stanley conjecture holds for I, i.e. $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$.

(2) For any integer $n \geq 1$ and any monomial ideals $I, J \subset S$, if $\operatorname{sdepth}_S(I+J) \geq \operatorname{depth}_S(I+J)$, *then* $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$.

(3) For any integers $n, m \geq 1$, any monomial ideal $I \subset S = K[x_1,\ldots,x_n]$, if $u_1,\ldots,u_m \in S$ is a regular sequence on S/I and $J = (u_1,\ldots,u_m)$, then if:

$$\operatorname{sdepth}_S(I+J) \geq \operatorname{depth}_S(I+J) \Rightarrow \operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I).$$

(4) For any integers $n, m \geq 1$, any monomial ideal $I \subset S = K[x_1,\ldots,x_n]$, if $u_1,\ldots,u_m \in S$ is a regular sequence on S/I and $J = (u_1,\ldots,u_m)$, then if:

$$\operatorname{sdepth}_S(I+J) = \operatorname{depth}_S(I+J) \Rightarrow \operatorname{sdepth}_S(I) = \operatorname{depth}_S(I).$$

(5) For any integer $n \geq 1$, any monomial ideal $I \subset S = K[x_1,\ldots,x_n]$, if $\bar{S} = S[y]$, then: $\operatorname{sdepth}_{\bar{S}}(I,y) = \operatorname{depth}_S(I) \Rightarrow \operatorname{sdepth}_S(I) = \operatorname{depth}_S(I)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Are obvious.

(3) $\Rightarrow$ (4). Assume $\operatorname{sdepth}_S(I+J) = \operatorname{depth}_S(I+J)$. Note that $\operatorname{depth}_S(I+J) = \operatorname{depth}_S(I) - m$, since $u_1,\ldots,u_m \in S$ is a regular sequence on S/I. By Corollary 2.4(2), $\operatorname{sdepth}_S(I+J) \geq \operatorname{sdepth}_S(I) - m$. Since $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$ by (3), we get $\operatorname{sdepth}_S(I) = \operatorname{depth}_S(I)$.

(4) $\Rightarrow$ (5). It is obvious, since y is regular on $\bar{S}/I\bar{S}$ and we apply (4) for $I\bar{S}$.

(5) $\Rightarrow$ (1). Let $I \subset S$ be a monomial ideal. Assume by contradiction that $\operatorname{sdepth}_S(I) < \operatorname{depth}_S(I)$. If $k \geq 1$ is an integer, we denote $I_k = (I, y_1,\ldots,y_k) \subset S_k := S[y_1,\ldots,y_k]$. Note that $y_1,\ldots,y_k$ is a regular sequence on S_k/I_k and therefore $\operatorname{depth}_{S_k}(I_k) = \operatorname{depth}_S(I)$. According to Corollary 1.4(1), we have:

$$\operatorname{sdepth}_{S_k}(I_k) \geq \min\{\operatorname{sdepth}_S(I) + k,\ \operatorname{sdepth}_S(S/I) + \lceil k/2 \rceil\}.$$

It follows that there exists $k_0 \leq 1$, such that $\operatorname{sdepth}_{S_k}(I_k) \geq \operatorname{depth}_S(I)$ for any $k \geq k_0$. If we chose k_0 minimal with this property, we claim that $\operatorname{sdepth}_{S_{k_0}}(I_{k_0}) = \operatorname{depth}_S(I)$. Indeed, it is enough to notice that $\operatorname{sdepth}_{S_k}(I_k) \leq \operatorname{sdepth}_{S_{k-1}}(I_{k-1}) + 1$. Now, by applying (5) inductively, it follows that $\operatorname{sdepth}_S(I) = \operatorname{depth}_S(I)$, a contradiction. ☐

Remark 3.2. Let $I \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal such that $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$. Let $u_1,\ldots,u_m \in S$ be a regular sequence on S/I and $J = (u_1,\ldots,u_m)$. Note that $\operatorname{depth}_S(I \cap J) = \operatorname{depth}_S(I+J) + 1 = \operatorname{depth}_S(I) - m + 1$. Also, by Corollary 2.4(1), we have $\operatorname{sdepth}_S(I \cap J) \geq \operatorname{sdepth}_S(I) - \lfloor m/2 \rfloor$. Assume $\operatorname{sdepth}_S(I \cap J) = \operatorname{depth}_S(I \cap J)$. It follows that $\operatorname{depth}_S(I) - m + 1 \geq \operatorname{sdepth}_S(I) - \lfloor m/2 \rfloor \geq \operatorname{depth}_S(I) - \lfloor m/2 \rfloor \geq \operatorname{depth}_S(I) - m + 1$.

Therefore, $\operatorname{sdepth}_S(I) = \operatorname{depth}_S(I)$ and $\lfloor m/2 \rfloor = m - 1$, and thus $m \leq 2$. In particular, if we could find an ideal $I \subset S$ such that, by denoting $\bar{S} = S[y_1, y_2, y_3]$, if $\operatorname{sdepth}_{\bar{S}}(I\bar{S} \cap (y_1, y_2, y_3)) = \operatorname{depth}_S(I)$, we contradict the Stanley conjecture for I.

Proposition 3.3. *The following assertions are equivalent:*

(1) For any integer $n \geq 1$ and any monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, Stanley conjecture holds for I, i.e. $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$.

(2) For any integer $n \geq 1$ and any monomial ideals $I, J \subset S$, if $\operatorname{sdepth}_S(I \cap J) \geq \operatorname{depth}_S(I \cap J)$ *then* $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$.

(3) For any integers $n, m \geq 1$, any monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, if $u_1, \ldots, u_m \in S$ is a regular sequence on S/I and $J = (u_1, \ldots, u_m)$, then:

$$\operatorname{sdepth}_S(I \cap J) \geq \operatorname{depth}_S(I \cap J) \Rightarrow \operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I).$$

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). There is nothing to prove.

(3) $\Rightarrow$ (1). Let $I \subset S$ be a monomial ideal. Assume by contradiction that $\operatorname{sdepth}_S(I) < \operatorname{depth}_S(I)$. For any integer $k \geq 1$, we define $I_k := (I, y_1, \ldots, y_k) \subset S_k := S[y_1, \ldots, y_k]$. Denote $J = (y_1, \ldots, y_k) \subset S_k$. Note that $y_1, \ldots, y_k$ is a regular sequence on S_k/IS_k. By Corollary 2.4(1), we have $\operatorname{sdepth}_{S_k}(I_k) \geq \operatorname{sdepth}_S(I) + \lceil k/2 \rceil$. On the other hand, by Corollary 1.4(5), $\operatorname{depth}_{S_k}(I_k) = \operatorname{depth}_S(I)$. It follows that there exists a $k_0 \geq 1$, such that $\operatorname{sdepth}_{S_k}(I_k) \geq \operatorname{depth}_{S_k}(I_k)$ for any $k \geq k_0$, and therefore, by (2), we get $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$, as required. $\square$

Proposition 3.4. *The following assertions are equivalent:*

(1) For any integer $n \geq 1$ and any monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, Stanley conjecture holds for S/I, i.e. $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

(2) For any integer $n \geq 1$ and any monomial ideals $I, J \subset S$, if $\operatorname{sdepth}_S(S/(I \cap J)) \geq \operatorname{depth}_S(S/(I \cap J))$ *then* $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

(3) For any integers $n, m \geq 1$, any monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, if $u_1, \ldots, u_m \in S$ is a regular sequence on S/I and $J = (u_1, \ldots, u_m)$, then:

$$\operatorname{sdepth}_S(S/(I \cap J)) \geq \operatorname{depth}_S(S/(I \cap J)) \Rightarrow \operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I).$$

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). There is nothing to prove.

(3) $\Rightarrow$ (1). Let $I \subset S$ be a monomial ideal. Assume by contradiction that $\operatorname{sdepth}_S(I) < \operatorname{depth}_S(I)$. For any integer $k \geq 1$, we define $I_k := (I, y_1, \ldots, y_k) \subset S_k := S[y_1, \ldots, y_k]$. Note that $y_1, \ldots, y_k$ is a regular sequence on S_k/IS_k. By Corollary 2.4(4), $\operatorname{sdepth}_{S_k}(S_k/I_k) \geq \min\{n, \operatorname{sdepth}_S(S/I) + \lceil k/2 \rceil\}$. On the other hand, by Corollary 1.4(5), $\operatorname{depth}_{S_k}(S_j/I_k) = \operatorname{depth}_S(S/I)$. It follows that there exists a $k_0 \geq 1$, such that $\operatorname{sdepth}_{S_k}(I_k) \geq \operatorname{depth}_{S_k}(I_k)$ for any $k \geq k_0$, and therefore, by (2), we get $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$, as required. $\square$

Remark 3.5. Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal such that $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$. Let $u_1, \ldots, u_m \in S$ be a regular sequence on S/I and $J = (u_1, \ldots, u_m)$. Note that $\operatorname{depth}_S(S/(I \cap J)) = \operatorname{depth}_S(S/(I + J)) + 1 = \operatorname{depth}_S(S/I) - m + 1$. Also, by Corollary 2.4(4), we have $\operatorname{sdepth}_S(S/(I \cap J)) \geq \min\{n - m, \operatorname{sdepth}_S(S/I) - \lfloor m/2 \rfloor\}$ Assume $\operatorname{sdepth}_S(S/(I \cap J)) = \operatorname{depth}_S(S/(I \cap J))$.

It follows that $\operatorname{depth}_S(S/I) - m + 1 \geq \min\{n - m, \operatorname{sdepth}_S(S/I) - \lfloor m/2 \rfloor\} \geq \min\{n - m, \operatorname{depth}_S(S/I) - \lfloor m/2 \rfloor\} \geq \min\{n - m, \operatorname{depth}_S(S/I) - m + 1\} = \operatorname{depth}_S(S/I) - m + 1$ and therefore, we have equalities.

If I is principal, then $\operatorname{depth}_S(S/I) = n-1$ and therefore $\min\{n-m, \operatorname{depth}_S(S/I) - \lfloor m/2 \rfloor\} = n-m$. It follows that $\operatorname{depth}_S(S/I) - \lfloor m/2 \rfloor = n-1-\lfloor m/2 \rfloor \geq n-m$ which is true for all m. If I is not principal, then by Remark 2.10, $\operatorname{depth}_S(S/I) \leq n-2$. It follows that $\min\{n-m, \operatorname{sdepth}_S(S/I) - \lfloor m/2 \rfloor\} = \operatorname{sdepth}_S(S/I) - \lfloor m/2 \rfloor = \operatorname{depth}_S(S/I) - m + 1$. Therefore, $\operatorname{sdepth}_S(S/I) = \operatorname{depth}_S(S/I)$ and $m \leq 2$.

In particular, if we could find an ideal $I \subset S$ which is not principal, such that, denoting $\bar{S} = S[y_1, y_2, y_3]$, if $\operatorname{sdepth}_{\bar{S}}(\bar{S}/(I\bar{S} \cap (y_1, y_2, y_3))) = \operatorname{depth}_S(I)$, we contradict the Stanley conjecture for S/I.

Lemma 3.6. *Let $I \subset J \subset S = K[x_1, \ldots, x_n]$ be two monomial ideals and denote $\bar{S} := S[y]$. Then:*

$$\operatorname{sdepth}_S(J/I) + 1 \geq \operatorname{sdepth}_{\bar{S}}((J\bar{S} + (y))/I\bar{S}) \geq \min\{\operatorname{sdepth}_S(J/I), \operatorname{sdepth}_S(S/I) + 1\}.$$

Proof. In order to prove the first inequality, we consider $\bigoplus_{i=1}^{r} u_i K[Z_i]$, a Stanley decomposition of $(J\bar{S} + (y))/I\bar{S}$. Note that $((J\bar{S} + (y))/I\bar{S}) \cap S = J/I$ and therefore, $J/I = \bigoplus_{y \nmid u_i} u_i K[Z_i \setminus \{y\}]$ is a Stanley decomposition.

The second inequality follows from the fact that $(J\bar{S} + (y))/I\bar{S} = J/I \oplus y(S/I)[y]$. □

As a particular case of Example 1.10, we consider the following Lemma.

Lemma 3.7. *Let $J = (x_1, \ldots, x_n) \cap (y_1, \ldots, y_m) \subset S' = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with $n \geq m$. Then:*

(1) $m \geq \operatorname{sdepth}_{S'}(S'/J) \geq \min\{m, \lceil n/2 \rceil\}$.

(2) $\operatorname{depth}_{S'}(S'/J) = 1$.

In particular, if $n \geq 2m-1$, then $\operatorname{sdepth}_{S'}(S'/J) = m$.

Proposition 3.8. *The following assertions are equivalent:*

(1) For any integer $n \geq 1$ and any monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, Stanley conjecture holds for S/I and I.

(2) For any integer $n \geq 1$ and any monomial ideals $I, J \subset S$ with $\operatorname{supp}(I) \cap \operatorname{supp}(J) = \emptyset$, we have: If $\operatorname{sdepth}_S((I+J)/I) = \operatorname{depth}_S((I+J)/I)$, then $\operatorname{sdepth}_S(S/I) = \operatorname{depth}_S(S/I)$ and $\operatorname{sdepth}_S(J) \geq \operatorname{depth}_S(J)$.

Proof. (1) $\Rightarrow$ (2). Let $I, J \subset S$ be two monomial ideals, with $\operatorname{supp}(I) \cap \operatorname{supp}(J) = \emptyset$, and assume $\operatorname{sdepth}_S((I+J)/I) = \operatorname{depth}_S((I+J)/I)$. According to Theorem 1.2(6), we have $\operatorname{depth}_S((I+J)/I) = \operatorname{depth}_S(I+J) = \operatorname{depth}_S(S/I) + \operatorname{depth}_S(J) - n$. On the other hand, by Remark 1.3, $\operatorname{sdepth}_S((I+J)/I) \geq \operatorname{sdepth}_S(S/I) + \operatorname{sdepth}_S(J) - n$. By (1), it follows that $\operatorname{sdepth}_S(S/I) = \operatorname{depth}_S(S/I)$ and $\operatorname{sdepth}_S(J) = \operatorname{depth}_S(J)$. In particular, $\operatorname{sdepth}_S(J) \geq \operatorname{depth}_S(J)$.

(2) $\Rightarrow$ (1). Let $I \subset S$ be a monomial ideal. For any positive integer k, we denote $S_k = S[y_1, \ldots, y_k]$ and $I_k = (I, y_1, \ldots, y_k) \subset S_k$. Assume $\operatorname{sdepth}_S(S/I) < \operatorname{depth}_S(S/I)$. Since $\operatorname{sdepth}_{S_k}(I_k/IS_k) \geq \operatorname{sdepth}_S(S/I) + \lfloor k/2 \rfloor$, it follows that there exists a positive integer k_0 such that $\operatorname{sdepth}_{S_k}(I_k/IS_k) \geq \operatorname{depth}_{S_k}(I_k/IS_k) = \operatorname{depth}_S(S/I)$, $(\forall) k \geq k_0$ $(*)$. If we apply Lemma 3.6 for $I_k \subset S_k$ and y_{k+1}, we obtain $\operatorname{sdepth}_{S_{k+1}}(I_{k+1}/IS_{k+1}) \leq \operatorname{sdepth}_{S_k}(I_k/IS_k) + 1$.

Thus, if we chose the minimal k_0 with the property $(*)$, we have in fact $\operatorname{sdepth}_{S_{k_0}}(I_{k_0}/IS_{k_0}) = \operatorname{depth}_S(S/I)$. By (2), it follows that $\operatorname{sdepth}_S(S/I) = \operatorname{depth}_S(S/I)$, a contradiction.

Now, assume $\operatorname{sdepth}_S(I) < \operatorname{depth}_S(I)$, and denote $J_k = (y_1, \ldots, y_{2k-1}) \cap (y_{2k}, \ldots, y_{3k-1}) \subset S_k := S[y_1, \ldots, y_{3k-1}]$. According to Lemma 3.7, we have $\operatorname{sdepth}_{S_k}(S_k/J_k) = n + k$ and $\operatorname{depth}_{S_k}(S_k/J_k) = 1$. Let $I_k := IS_k + J_k$. By Remark 1.3, $\operatorname{sdepth}_{S_k}(I_k/J_k) \geq \operatorname{sdepth}_S(I) + k$. On the other hand $\operatorname{depth}_{S_k}(I_k/J_k) = \operatorname{depth}_S(I) + \operatorname{depth}_{S_k}(S_k/J_k) - n = \operatorname{depth}_S(I) + 1$.

Therefore, there exists a positive integer k_0, such that $\operatorname{sdepth}_{S_k}(I_k/J_k) \geq \operatorname{sdepth}_{S_k}(I_k/J_k)$ for any $k \geq k_0$. It follows, by (2), that $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$, a contradiction. $\square$

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