Families of stable and metastable solitons in coupled system of scalar fields

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In this paper, we obtain stable and metastable soliton solutions of a coupled system of two real scalar fields with five five discrete points of vacua. These solutions have definite topological charges and rest energies and show classical dynamical stability. From a quantum point of view, however, the V-type solutions are expected to be unstable and decay to D-type solutions. The induced decay of a V-type soliton into two D-type ones is calculated numerically, and shown to be chiral, in the sense that the decay products do not respect left-right symmetry.

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I. INTRODUCTION

Relativistic solitons, including those of the well-known Sine-Gordon (SG) equation, exhibit remarkable similarities with classical particles. They exert short range forces on each other and collide, without losing their identities [1–5]. They are localized and do not disperse while propagating in the medium. Because of their field nature, they do tunnel a barrier in certain cases, although this tunnelling is different from the well-known quantum version [2, 6].Topological solitons are stable, due to the boundary conditions at spatial infinity. Their existence, therefore, is essentially dependent on the presence of degenerate vacua [2, 7].

Topology provides an elegant way of classifying solitons in various sectors according to the mappings between the degenerate vacua of the field and the points at spatial infinity. For the Sine-Gordon system in $1 + 1$ dimensions, these mappings are between $\phi = 2n\pi$, $n \in \mathbb{Z}$ and $x = \pm\infty$, which correspond to kinks and antikinks of the SG system. More complicated mappings occur in solitons in higher dimensions [3]. Coupled systems of scalar fields with soliton solutions have found interesting applications in double-strand, long molecules like the DNA molecules [8]-[11] and bi-dimensional QCD [12].

Bazeia et al. [13] considered a system of two coupled real scalar fields with a particular self-

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interaction potential such that the static solutions are derivable from first order coupled differential equations. Riazi et al. [14] employed the same method to investigate the stability of the single-soliton solutions of a particular system of this type. Inspired by the coupled system introduced in [2], we propose a new coupled system of two real scalar fields which shows interesting types of solitons with well-defined topological charges and rest energies (masses).

In this paper, we focus on a system with the self-interaction potential

$$V(\phi,\psi)=\phi^2(\psi^2-\psi_0^2)^2+\psi^2(\phi^2-\phi_0^2)^2, \tag{1}$$

in which ϕ and ψ are real scalar fields, and ϕ_0 and ψ_0 are constants. This potential is plotted in Fig. **??** for $\phi_0=1$ and $\psi_0=2$. Following the common terminology in field theory, this potential has absolute degenerate minima at $\phi=\psi=0$, $\phi=\pm\phi_0$ and $\psi=\pm\psi_0$ known as the true vacua. In our proposed system, a spectrum of solitons with different rest energies exists which are stable or meta-stable, depending on their energies and boundary conditions.

The structure of this paper is as follows: in Section II we review some basic properties of the proposed system. In Section III, some exact solutions together with the corresponding charges and energies are derived. The necessary nomenclature and general behavior of the solutions due to the boundary conditions are also introduced in this section. Numerical solutions corresponding to different boundary conditions are presented, and properties of these solutions like their charges, masses, and stability status are addressed in this section. In order to investigate the stability of the numerical solutions, their evolution is worked out numerically. The classical dynamical stability of the solutions is investigated further in IV. Our conclusions and a summary of the results are given in the last section V.

II. DYNAMICAL EQUATIONS, CONSERVED CURRENTS, AND TOPOLOGICAL CHARGES

Within a relativistic formulation, the Lagrangian density of the system is given by:

$$\mathcal{L}=\frac{1}{2}\partial^\mu\phi\partial_\mu\phi+\frac{1}{2}\partial^\mu\psi\partial_\mu\psi-[\phi^2(\psi^2-\psi_0^2)^2+\psi^2(\phi^2-\phi_0^2)^2]. \tag{2}$$

From this Lagrangian density, we obtain the following equations for ϕ and ψ, respectively:

$$\Box\phi=-2\phi(\psi^2-\psi_0^2)^2-4\phi\psi^2(\phi^2-\phi_0^2); \tag{3}$$

and

$$\Box\psi=-2\psi(\phi^2-\phi_0^2)^2-4\psi\phi^2(\psi^2-\psi_0^2). \tag{4}$$

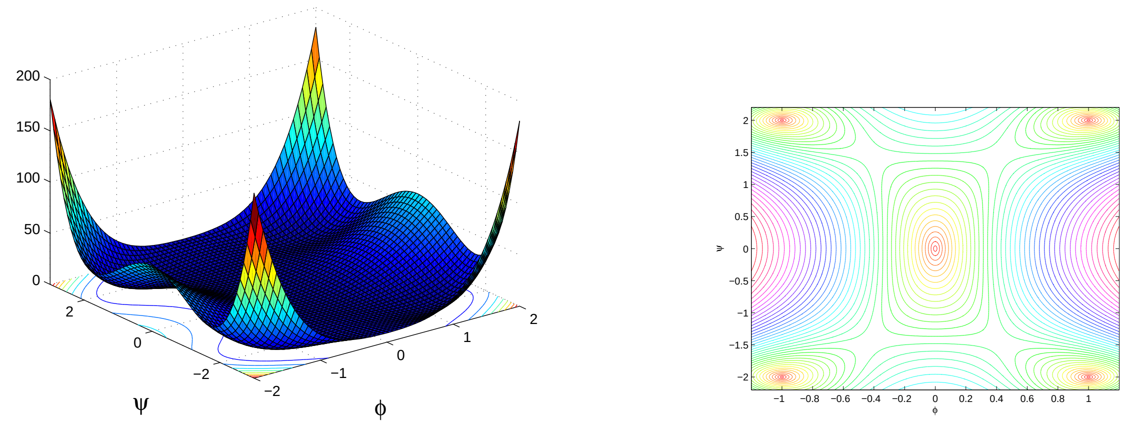

FIG. 1: Left: The self-interaction potential (1) is shown as a height diagram over the (ϕ,ψ) plane. $(\phi_0, \psi_0) = (1, 2)$ is assumed throughout this paper. Right: The self-interaction potential as a contour map over the (ϕ, ψ) plane.

Since the lagrangian density Eq.(2) is Lorentz invariant, the corresponding energy-momentum tensor[15, 16] is:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \partial_\mu \psi \partial_\nu \psi - g_{\mu\nu} \mathcal{L}; \tag{5}$$

which satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0. \tag{6}$$

In Equation (5), $g_{\mu\nu} = diag(1, -1)$ is the metric of the $(1 + 1)$ dimensional Minkowski spacetime. The Hamiltonian (energy) density is obtained from Eq.(5) according to

$$\mathcal{H} = T^{00} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 + V(\phi, \psi). \tag{7}$$

It can be shown that the following topological currents can be defined, which are conserved independently, and lead to quantized charges:

$$\begin{aligned} J^\mu_H &= \frac{1}{2\phi_0} \epsilon^{\mu\nu} \partial_\nu \phi, \\ J^\mu_V &= \frac{1}{2\psi_0} \epsilon^{\mu\nu} \partial_\nu \psi. \end{aligned} \tag{8}$$

The subscripts H, V and D (to be used later), denote “horizontal”, “vertical” and “diagonal” which will be explained later. The currents $J^\mu_{H,V}$ are conserved, independent of each other:

$$\begin{aligned} \partial_\mu J^\mu_H &= 0, \\ \partial_\mu J^\mu_V &= 0. \end{aligned} \tag{9}$$

It is obvious that the total horizontal and vertical charges are conserved separately in any local dynamical evolution of the system. The corresponding topological charges are given by:

$$Q_H = \int_{-\infty}^{+\infty} J_H^0 dx = \frac{1}{2}[\phi(+\infty) - \phi(-\infty)],$$
$$Q_V = \int_{-\infty}^{+\infty} J_V^0 dx = \frac{1}{4}[\psi(+\infty) - \psi(-\infty)]. \tag{10}$$

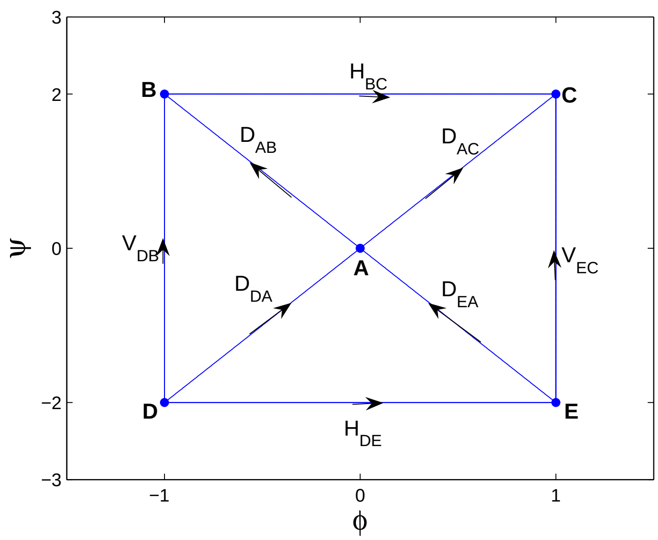

FIG. 2: Nomenclature of horizontal (H), vertical (V) and diagonal (D) solutions according to the boundary conditions on the (ϕ, ψ) plane.

III. SINGLE-SOLITON SOLUTIONS

Static solutions which correspond to transitions between neighboring vacua are symbolically shown in Fig. 2. Accordingly, we call the static solutions H (horizontal), V (vertical) and D (diagonal) types, depending on the beginning and end points of the solution on the (ϕ, ψ) plane. We have called them "horizontal", "vertical" and "Diagonal" simply because of their orientation in the $(\phi\ ,\psi)$ plane. Equations (3) and (4) possess the following exact single-soliton solutions[17, 18], summarized in Table 1.

Table 1. Exact static solutions and the corresponding horizontal and vertical charges.

Type	Solution	Q_H	Q_V
Horizontal	$\phi = \pm\tanh(2\sqrt{2}x), \quad \psi_0 = \pm 2$	± 1	0
Vertical	$\phi_0 = \pm 1, \quad \psi = \pm 2\tanh(2\sqrt{2}x)$	0	± 1
Diagonal	$\phi = \pm(1/2)\psi, \quad \psi = \pm\frac{2}{1+\exp(\pm 2\sqrt{10}x)}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$

Although these static solutions satisfy the field equations (3) and (4), they are not stable, minimum energy solutions. Any static solution has an orbit $f(\phi,\psi)=0$ in the (ϕ,ψ) plane, which begins and ends at one of the true vacua. We therefore have

$$\frac{df}{dx}=\frac{\partial f}{\partial\phi}\phi'+\frac{\partial f}{\partial\psi}\psi'=0, \tag{11}$$

where prime means differentiation with respect to x. The field equations (3-4) can be integrated once to yield

$$\frac{1}{2}(\phi')^2=\int\frac{\partial V}{\partial\phi}d\phi+C_1, \tag{12}$$

and

$$\frac{1}{2}(\psi')^2=\int\frac{\partial V}{\partial\psi}d\psi+C_2, \tag{13}$$

where C_1 and C_2 are integration constants. Combining equations (11)-(13), we obtain the following integro-differential equation for the orbit:

$$\left(\frac{\partial f}{\partial\phi}\right)^2\left(\int\frac{\partial V}{\partial\phi}d\phi+C_1\right)=\left(\frac{\partial f}{\partial\psi}\right)^2\left(\int\frac{\partial V}{\partial\psi}d\psi+C_2\right). \tag{14}$$

Unfortunately, this equation cannot be solved for the orbit $f(\phi,\psi)$ analytically.

In order to find stable static solutions, we have used the analytical solutions depicted in Table 1 as the starting "guesses" in a step-wise variational procedure. In this procedure, the initial guess is repeatedly varied in steps, with the total energy checked at each step. The fields are thus varied toward the minimum energy solution, were small variations no longer lead to a decrease in total energy. The minimum energy, static solutions obtained in this way are plotted in Figure 3. In Figure 4, we have mapped these solutions into the (ϕ,ψ) plane.

The conservation of V and H type charges (Equation 9) does not come from a continuous symmetry of the Lagrangian, but rather it stems from the boundary conditions and topological considerations [21]. The system under consideration respects various types of symmetries: 1) Lorentz symmetry: this symmetry comes from the invariance of the Lagrangian (2) under Lorentz transformations. Since ϕ and ψ are scalar fields, we conclude that if $(\phi(x),\psi(x))$ is a static solution, then $(\phi(\gamma(x-vt),\psi(\gamma(x-vt))$ where v is a constant (soliton velocity) and $\gamma=(1-v^2)^{-1/2}$, is also solution. 2) If $(\phi(x,t),\psi(x,t))$ is a solution, then $(\pm\phi(x,t),\pm\psi(x,t))$ are solutions, also. 3) For the special case $\phi_0=\psi_0$, we have dual solutions $\phi\leftrightarrow\psi$. 4) Parity and time reversal are symmetries of the system. Therefore if $(\phi(x,t),\psi(x,t))$ is a solution, then $(\phi(\pm x,\pm t),\psi(\pm x,\pm t))$ are solutions, too.

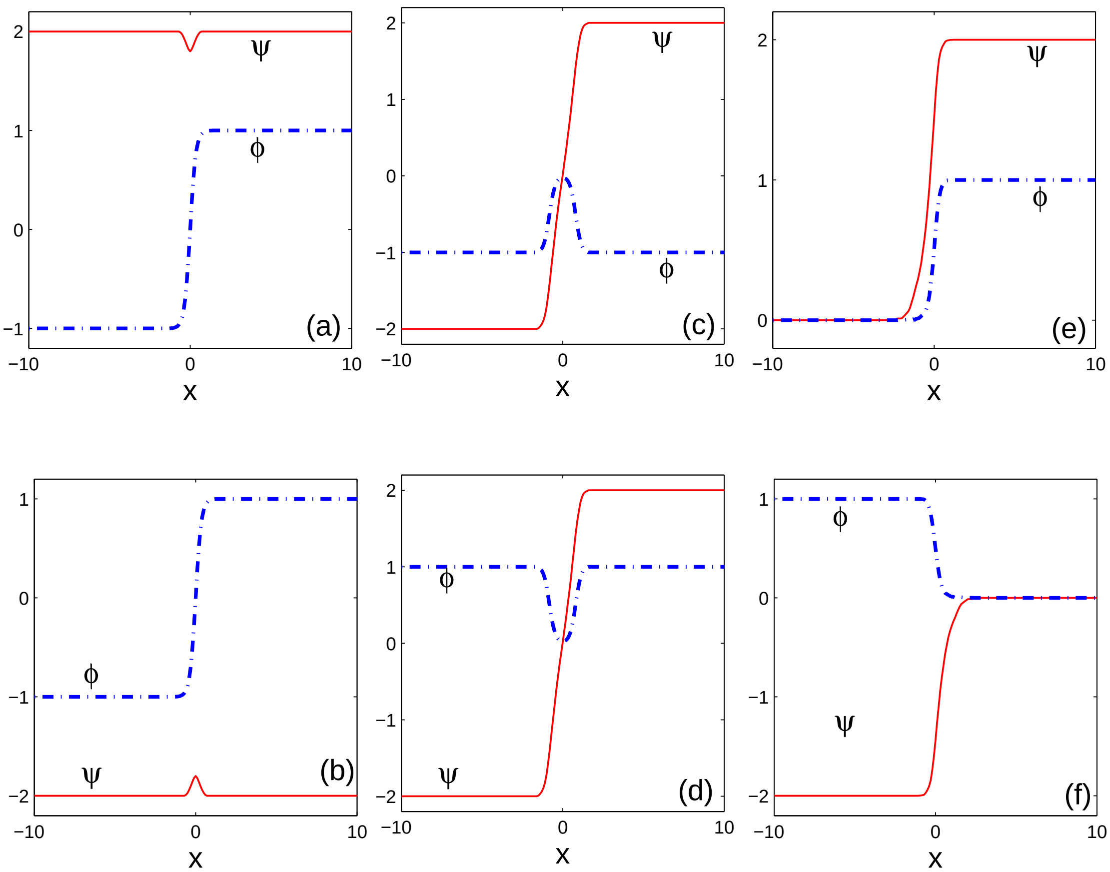

FIG. 3: Minimu m energy, static solutions.(a) H_{BC}, (b) H_{DE}, (c) V_{DB}, (d) V_{EC}, (e) D_{AC} and (f) D_{EA}. The solid curves represent ψ and the dash-dotted curves are for ψ.

IV. SOLITON STABILITY AND CHIRAL DECAY OF V-TYPE SOLITONS

Basic properties of the H, V, and D solitons are summarized in Table 1. It can be seen that all D solitons are degenerate (i.e. have the same rest energy). This degeneracy results from the $(\phi, \psi) \leftrightarrow (\pm\phi, \pm\psi)$ symmetry of the Lagrangian density (2).

Another interesting property of these solitons is that the mass (rest energy) of the V_{EC} soliton is larger than the sum of the D_{EA} and D_{AC} solitons, while the topological charge of V_{EC} is equal to the sum of the D_{EA} and D_{AC} charges. The same is for V_{DB} which is more massive than the sum of D_{DA} plus D_{AB}. The decay of V_{EC} into $D_{EA} + D_{AC}$ (or V_{DB} into $D_{DA} + D_{AB}$) is therefore possible with regard to energy and charge conservations. These decays were not observed to occur spontaneously in our dynamical simulations, implying that V_{EC} and V_{DB} are classically

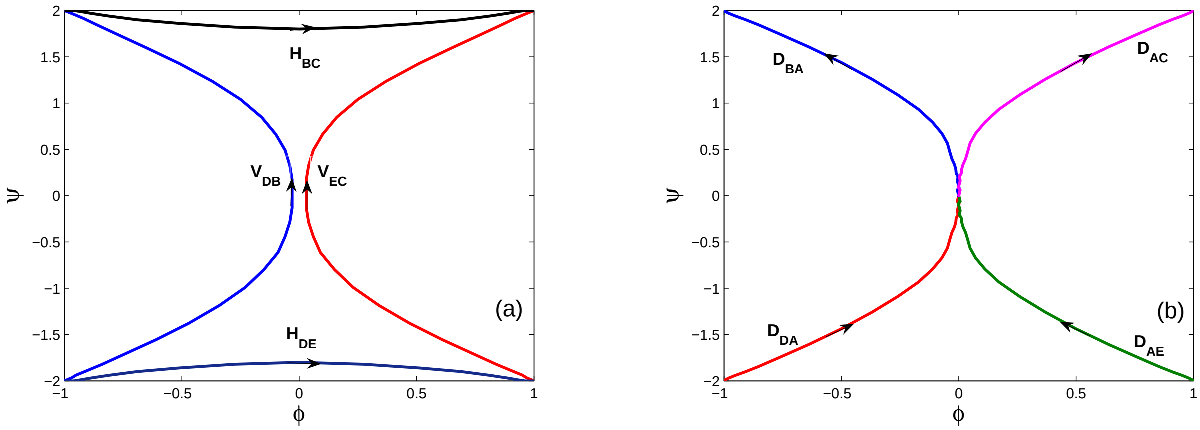

FIG. 4: The minimum energy, static solutions mapped on the (ϕ,ψ) plane: (a) H_{DE}, V_{DB}, H_{BC} and V_{EC} and (b) D_{DA}, D_{AC}, D_{EA} and D_{AB}.

metastable configurations. Quantum tunnelling or external perturbations can -in principle- cause such a decay (see [2]). In order to show that stimulated decay is in fact possible in the system under consideration, we have performed the following simulation: The static vertical soliton V_{DB} is used as the initial condition in a program which calculates the dynamics of the system, by solving the coupled PDEs 3 and 4. This soliton is excited, by pumping some energy into the ϕ field, via an increase in its amplitude. The system is then observed to become unstable and decay into D_{DA} and D_{AB}. This simulation is shown in Figure 5. Note that the excess energy is transferred into both the kinetic energy of the decay products and also emission of some low amplitude waves.

It is interesting to note that the decay of V-type solitons is chiral, in the sense that when V_{EC} decays into D_{EA} and D_{AC}, the former always moves to the left, while the latter always move to the right. The fact that the reverse never happens is due to the boundary conditions and topological reasons. Similarly, for the decay $V_{DB} \rightarrow D_{DA} + D_{AB}$, the decay product D_{DA} always moves to the left and D_{AB} always moves to the right. This property can be extended to an arbitrary array of solitons in a multi-soliton system. The consistency of boundary conditions between successive solitons permits certain sequences and forbids certain ones. Examples of allowed and forbidden arrays are the followings:

$$\text{Allowed arrays}: \quad ...D_{DA}D_{AB}H_{BC}V_{CE}H_{ED}...$$

$$\text{Forbidden arrays}: \quad ...D_{DA}D_{BA}H_{BC}V_{EC}H_{EC}...$$

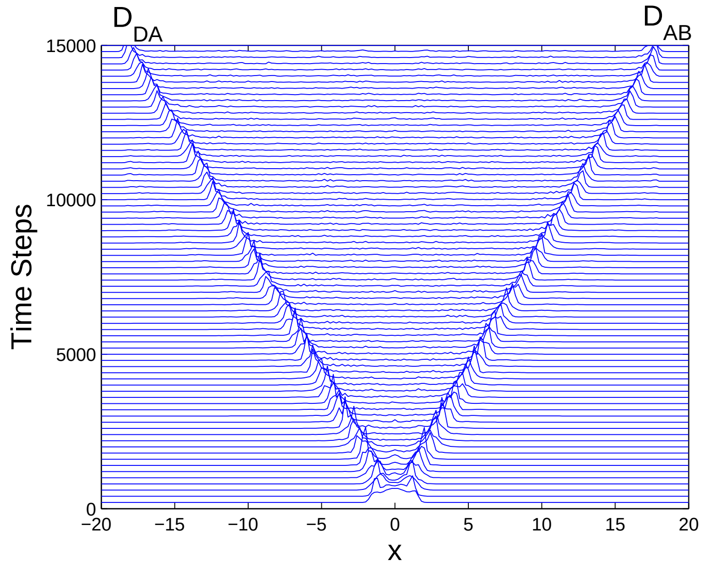

FIG. 5: Stimulated decay of a V_{DB} soliton into $D_{DA} + D_{AB}$. Note that the excess energy is transferred into the kinetic energy of daughter solitons and also emission of low amplitude waves. Also note that the decay is chiral, in the sense that D_{DA} always moves to the left.

Table 2. General properties of different soliton families. Note that the soliton masses and stability status depend on the choice of ϕ_0 and ψ_0. Here, we have assumed $\phi_0 = 1$ and $\psi_0 = 2$.

Symbol	Mass	Q_H	Q_V	Stability	Decay Mode
D_{AB}	2.858	-1/2	+1/2	stable	–
D_{AC}	2.858	+1/2	+1/2	stable	–
D_{DA}	2.858	+1/2	+1/2	stable	–
D_{EA}	2.858	-1/2	+1/2	stable	–
H_{BC}	3.594	+1	0	stable	–
V_{EC}	6.383	0	+1	metastable	$V_{EC} \to D_{EA} + D_{AC}$
H_{DE}	3.594	+1	0	stable	–
V_{DB}	6.383	0	+1	metastable	$V_{DB} \to D_{DA} + D_{AB}$

V. CONCLUDING REMARKS

We studied a nonlinear system of coupled scalar fields which had three (H, V, and D) types of stable and metastable soliton solutions. We started by deriving analytical, static solutions with definite topological charges. These static solutions, however, were shown to be unstable and the stable solutions were obtained numerically, by minimizing the total energy, using a step-wise variational method. The full, dynamical equations were used to ensure the stability of these minimum energy solutions. According to energy and topological charge conservations, the V-type solutions are expected to decay into D-type ones. This, however, does not happen spontaneously at the classical level. Either quantum tunnelling or external triggers are needed for decays like $V_{EC} \rightarrow D_{EA} + DAC$. Conversely, the formation of V_{EC} from $D_{EA} + D_{AC}$ collision is endothermic, requiring -at least- the incoming kinetic energy $\Delta E = 6.38 - 2 \times 2.86 = 0.66$ in dimensionless units. The stimulated decay of a V-type soliton were calculated numerically, by pumping some energy into the ϕ-field. This decay is exothermic and the excess energy was observed to be transferred into the kinetic energy of decay products and emission of low amplitude waves.

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[1] G.L. Lamb, Jr., *Elements of Soliton Theory*, John Wiley and Sons, NewYork(1980).

[2] N. Riazi, A. Azizi and S. M. Zebarjad, Phys. Rev. **D 66**, 065003 (2002).

[3] T. Dauxois and M. Peyrard, *Physics of Solitons*, Cambridge University Press (2006).

[4] M. Peyravi, A. Montakhab, N. Riazi and A. Gharaati, Eur. Phys. J. **B 72**, 269-277 (2009).

[5] M. Peyravi, N. Riazi and A. Montakhab, Eur. Phys. J. **B 76**, 547555 (2010).

[6] N. Riazi, Int. J. Theor. Phys. **35**, 101 (1996).

[7] T.D. Lee, *Particle Physics and Introduction to field Theory*, Harwood, Chur, Switzerland (1981).

[8] L.V. Yakushevich, *Nonlinear Physics of DNA*, Wiley,(2004).

[9] L. V. Yakushevich, A. V. Savin and L. I. Manevitch, Phys. Rev. **E 66**, 016614 (2002).

[10] S. Cuenda, A. Sanchez, and N.R. Quintero, Physica **D 223**, 214221 (2006).

[11] M. Peyrard, Nonlinearity **17**, R1-R40, (2004).

[12] H. Blas, JHEP, **0703**, 055, (2007).

[13] D. Bazeia, J. R. S. Nascimento, R. F. Ribeiro, and D. Toledo, J. Phys. **A 30**, 8157 (1997).

[14] N. Riazi, M. M. Golshan, and K. Mansuri, Int. J. Theor. Phys. Group Theor. Non. Opt. **7**, 91 ((2001).

[15] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press (1985).

[16] M. Guidray, *Gauge Field Theories, An Introduction with Application*, John Wiley and Sons, NewYork(1991).

[17] R. H. Goodman and R. Haberman, Siam J. Applied Dynamical Systems **4**, No. 4, 1195, (2005).

[18] S. Hoseinmardy and N. Riazi, IJMPA. **25**, 3261, (2010).

[19] R. Cordero and R. D. Mota, arXiv:0709.2822v1 [hep-th] (2007).

[20] K.D. Asit, A. Harindranath, J. Maiti and T. Sinha, arXiv:hep-lat/0506003v2 (2005).

[21] R. Rajaraman, Soltions and Instantons, Elsevier, B.V. (1989).