Numerical Functional Analysis and Optimization, 30(5–6):478–485, 2009

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560902987279

REARRANGEMENTS IN REAL ESTATE INVESTMENTS

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□ *In this note we consider an investment problem in real estate. We show that it can be formulated in terms of a constrained optimization problem, and this leads to a linear rearrangement optimization problem. We address existence, uniqueness, and symmetry of the optimal solution.*

Keywords Optimality condition; Optimization; Rearrangements; Rearrangement inequalities.

AMS Subject Classification 49K30.

1. INTRODUCTION

A rearrangement optimization problem is an optimization problem where the admissible set is a rearrangement class; that is, a set comprising functions that are rearrangements of a prescribed function. In recent years, such problems have proved to be interesting and challenging. Rearrangement optimization problems have applications particularly in fluid and solid mechanics, see, for example, [1–3]. A basic rearrangement optimization problem has the following:

$$\sup_{f\in\mathcal{R}} \Phi(f), \tag{1.1}$$

where Φ is a linear functional defined on a suitable function space, and $\mathcal{R}$ is a rearrangement class. In this paper, we introduce an investment problem in real estate and show that it can be formulated as an optimization problem similar to (1.1). We then investigate existence and symmetry questions.

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2. DESCRIPTION AND FORMULATION OF THE PROBLEM

Suppose $\Omega \subset \mathbb{R}^2$, a bounded open set, designates a neighborhood in a city. This neighborhood consists of two jurisdictions denoted U and D, which are separated by a boundary Γ. Here U and D are open sets and $\Omega = U \cup D$. The price of land in U is P_U dollars per square feet and in D is P_D. An investor intends to buy exactly A square feet of land in this neighborhood, where by law no one can buy the whole neighborhood, so $A < \text{area}(\Omega)$. Each jurisdiction has a "risk factor" that is designated by a constant. Let α and β be the risk factors for U and D, respectively. These factors depend on time. So, because the price of land, for example, in U is P_U, then after t units, say years, of time it will be αP_U. We assume that α and β are independent of t, as their dependence on time is irrelevant to our purpose.

If the investor buys a piece of land L, then $L = E \cup F$, where $E \subset U$ and $F \subset D$. Note that E or F could be empty. At the time of sale the investor gains:

$$S(L) = \alpha P_U \text{ area}(E) + \beta P_D \text{ area}(F) \tag{2.1}$$

dollars. Clearly the investor would like to maximize his gain. Hence the following maximization problem is naturally of interest:

$$\max_{L=E\cup F, \text{area}(L)=A} S(L), \tag{2.2}$$

where S is the function defined by (2.1). It is more convenient to write S as an integral, this can be achieved by introducing $p(x)$ as follows:

$$p(x) = \begin{cases} \alpha P_U, & x \in U \\ \beta P_D, & x \in D. \end{cases}$$

Then

$$p(x)\chi_L(x) = p(x)\chi_{E\cup F}(x) = \begin{cases} \alpha P_U, & x \in U \\ \beta P_D, & x \in D \\ 0, & \text{otherwise,} \end{cases}$$

where χ_K stands for the characteristic function of the set K; that is, it takes the value one on K and zero elsewhere. Therefore we derive

$$S(L) = \int_\Omega p(x)\chi_L(x)\,dx.$$

The maximization problem (2.2) is easy to solve. Indeed, let us assume $\alpha P_U > \beta P_D$. Then it is clear that the investor must buy as much land as possible in U. If $A \leq \text{area}(U)$, then he does not need to buy in D at all. However, if $A > \text{area}(U)$, then he must buy all of U and the remaining in D. In conclusion, the investor should exhaust himself primarily in U before buying in D.

The case of finitely many jurisdictions is treated similarly. In this case Ω consists of N jurisdictions $J_1, J_2, J_3, \ldots, J_N$. Assume the price of land in J_i is P_i. At the time of sale the price in J_i will be $\alpha_i P_i$, where α_i is the risk factor in J_i. Without loss of generality we may assume the following ordering:

$$\alpha_{i+1} P_{i+1} > \alpha_i P_i, \quad i = 1, 2, \ldots, N.$$

Setting

$$p(x) = \sum_{i=1}^{N} \alpha_i P_i \chi_{J_i}(x)$$

and

$$S(L) = \int_{\Omega} p(x) \chi_L(x)\, dx,$$

we derive the following maximization problem

$$\max_{\text{area}(L)=A} S(L). \tag{2.3}$$

Following the same line of argument as in the case of two jurisdictions, it is to the benefit of the investor to buy as much land as possible in J_1, and then in J_2 and so on. Observe that any optimal land $\widehat{L}$; that is, any solution of (2.3) satisfies the following optimality condition:

$$\min_{x \in \widehat{L}} p(x) \geq \max_{x \in \Omega \setminus \widehat{L}} p(x). \tag{2.4}$$

Moreover, it is easily verified that any $\widehat{L}$ satisfying (4.4) is an optimal land.

Our interest in this paper is in the case of *infinitesimal* jurisdictions; in the sense that every point $x \in \Omega$ is considered as a jurisdiction. We suppose $p(x)$, which incorporates both the price function and the risk factor, satisfies the following conditions:

($\mathbf{P_1}$) $p \in C(\overline{\Omega})$.
($\mathbf{P_2}$) p is positive in Ω.
($\mathbf{P_3}$) The graph of p has no flat sections; that is, for every non-negative δ, p satisfies

$$|\{x \in \Omega : p(x) = \delta\}| = 0,$$

where $|K|$ stands for the Lebesgue measure of K in $\mathbb{R}^2$. We can now state the maximization problem that we wish to investigate. First we set

$$\widehat{\Psi}(L) = \int_\Omega p(x)\chi_L(x)\,dx.$$

We are interested in the following problem:

$$\sup_{|L|=A} \widehat{\Psi}(L). \tag{2.5}$$

3. REFORMULATING (2.5) INTO A REARRANGEMENT OPTIMIZATION PROBLEM

In this section, we reformulate (2.5) into a rearrangement optimization problem. Let us begin with noting that every set K can be identified with the characteristic function χ_K. On the other hand, from the distribution function of χ_K, defined by

$$\lambda_{\chi_K}(\alpha) = \{x \in \Omega : \chi_K(x) \geq \alpha\}, \quad \alpha \geq 0,$$

we infer $\lambda_{\chi_K}(\alpha) = |K|\chi_{[0,1]}(\alpha)$. Let us recall that two functions defined on Ω are said to be rearrangements of each other provided their respective distribution functions are equal. Thus if K_1 and K_2 have equal measures, then their characteristic functions are rearrangements of each other. Let us now fix a measurable set $D_0 \subset \Omega$, such that $|D_0| = A$, and define

$$\mathcal{R} = \{\chi_D : \chi_D \text{ is a rearrangement of } \chi_{D_0}\}.$$

It is clear that there is a one-to-one correspondence between $\mathcal{R}$ and $\{D \subseteq \Omega : |D| = A\}$. Therefore, by defining $\Psi : L^\infty(\Omega) \to \mathbb{R}$ as

$$\Psi(f) = \int_\Omega p(x)f(x)\,dx,$$

it follows that (2.5) is equivalent to

$$\sup_{f\in\mathcal{R}} \Psi(f). \tag{3.1}$$

We end this section with the following existence result.

Theorem 3.1. *The maximization problem* (3.1) *has a solution; that is, there is* $\widehat{D} \subseteq \Omega$ *such that* $\chi_{\widehat{D}} \in \mathcal{R}$ *and* $\Psi(\chi_{\widehat{D}}) = \sup_{f\in\mathcal{R}}\Psi(f)$.

Proof. We first relax (3.1) by extending $\mathcal{R}$ to $\overline{\mathcal{R}}$, the w^*-closure of $\mathcal{R}$ in $L^\infty(\Omega)$. So the relaxed problem reads

$$\sup_{f\in\overline{\mathcal{R}}} \Psi(f),$$

and as Ψ is w^*-continuous, it has a solution, say $\hat{f}$. Now we claim there exists $f \in \mathcal{R}$ such that $\Psi(f) = \Psi(\hat{f})$; this clearly completes the proof. We prove the claim by way of contradiction. To this end we assume $\Psi(f) < \Psi(\hat{f})$, for every $f \in \mathcal{R}$. It is well known that $\text{ext}(\overline{\mathcal{R}}) = \mathcal{R}$, where $\text{ext}(\overline{\mathcal{R}})$ denotes the extreme points of $\overline{\mathcal{R}}$ in $L^\infty(\Omega)$, see for example [4]. It then follows that there exists t in $(0, 1)$ such that $\hat{f} = tf_1 + (1 - t)f_2$, for some f_1 and f_2 in $\mathcal{R}$. Thus

$$\Psi(\hat{f}) = t\Psi(f_1) + (1 - t)\Psi(f_2) < t\Psi(\hat{f}) + (1 - t)\Psi(\hat{f}) = \Psi(\hat{f}),$$

which is a contradiction, as desired. □

4. THE OPTIMALITY CONDITION

In this section, we derive an optimality condition from the Euler–Lagrange equation satisfied by solutions of (3.1). We have the following result:

Lemma 4.1. *Let $M \subseteq \Omega$. Then*

$$\int_M f(x)\,dx \le \int_0^{|M|} f^\Delta(t)\,dt, \tag{4.1}$$

where f^Δ is the essentially unique decreasing rearrangement of f. Equality in (4.1) holds if and only if M is a cut of f; that is, $M = \{x \in \Omega : f(x) \ge \gamma\}$, for some $\gamma \in \mathbb{R}$.

Proof. The inequality (4.1) is a consequence of the famous Hardy–Littlewood inequality, see for example [5]. We prove the second part of the assertion. Let us first assume M is a cut of f, so $M = \{x \in \Omega : f(x) \ge \gamma\}$, for some $\gamma \in \mathbb{R}$. Then

$$\int_M f(x)\,dx = \int_\Omega \chi_M(x) f(x)\,dx = \int_0^{|\Omega|} (\chi_M f)^\Delta(t)\,dt.$$

Because $(\chi_M f)^\Delta = f^\Delta \chi_{(0,|M|)}$, it follows that

$$\int_M f(x)\,dx = \int_0^{|\Omega|} f^\Delta(t)\chi_{(0,|M|)}(t)\,dt = \int_0^{|M|} f^\Delta(t)\,dt.$$

Now we suppose equality holds in (4.1). To derive a contradiction, let us suppose M is not a cut of f. In this case, if $\delta = \inf_M f$, then M must be a proper subset of the cut $\{x \in \Omega : f(x) \geq \delta\}$. Hence, the measure of the set $\{x \in \Omega : f(x) \geq \delta\} \backslash M$ is non-zero. Thus, there exist sets E and F in M and M^c, respectively, such that $|E| = |F|$, and $\inf_F f > \sup_E f$. Whence,

$$\int_M f(x)\,dx = \int_\Omega \chi_M(x) f(x)\,dx = \int_0^{|\Omega|} (\chi_M f)^\Delta(t)\,dt < \int_0^{|\Omega|} (\chi_W f)^\Delta(t)\,dt,$$

where $W = (M \backslash E) \cup F$. Because $(\chi_M f)^\Delta \leq f^\Delta \chi_{(0,|W|)}$, we then find

$$\int_M f(x)\,dx < \int_0^{|\Omega|} f^\Delta(t)\chi_{(0,|M|)}(t)\,dt = \int_0^{|M|} f^\Delta(t)\,dt,$$

which is a contradiction. □

Theorem 4.2. *Let $\chi_{\widehat{D}}$ be a solution of* (3.1). *Then*

$$\widehat{D} = \{x \in \Omega : p(x) \geq p^\Delta(A)\}, \tag{4.2}$$

which is referred to as the Euler–Lagrange equation for $\chi_{\widehat{D}}$.

Remark 4.3. Clearly the assertion of Theorem 4.2 implies that (3.1) has a unique solution. This, in turn, implies that $\widehat{D}$ is the unique solution of (2.5).

Proof of Theorem 4.2. We begin with the following observation:

$$\int_D p(x)\,dx \leq \int_{\widehat{D}} p(x)\,dx, \tag{4.3}$$

for every $D \subseteq \Omega$, satisfying $|D| = A$. From $(\mathbf{P_1})$–$(\mathbf{P_3})$, it follows that the distribution function of p, denoted λ_p, is continuous and strictly decreasing on $[0, \max_\Omega p]$. Thus there exists a unique $\hat{\alpha} \in (0, \max_\Omega p)$, such that $\lambda_p(\hat{\alpha}) = A$, hence $\hat{\alpha} = p^\Delta(A)$. From Lemma 4.1, we deduce

$$\int_{\{x\in\Omega : p(x) \geq p^\Delta(A)\}} p(x)\,dx = \int_0^A p^\Delta(t)\,dt.$$

Also we have

$$\int_D p(x)\,dx \leq \int_{\{x\in\Omega : p(x) \geq p^\Delta(A)\}} p(x)\,dx,$$

for every $D \subseteq \Omega$ with $|D| = A$. In particular, we obtain

$$\int_{\widehat{D}} p(x)\,dx \le \int_{\{x\in\Omega: p(x)\ge p^{\Delta}(A)\}} p(x)\,dx.$$

However, from (4.3), we have

$$\int_{\{x\in\Omega: p(x)\ge p^{\Delta}(A)\}} p(x)\,dx \le \int_{\widehat{D}} p(x)\,dx,$$

so we derive

$$\int_{\widehat{D}} p(x)\,dx = \int_{\{x\in\Omega: p(x)\ge p^{\Delta}(A)\}} p(x)\,dx.$$

Another application of Lemma 4.1, implies that

$$\int_{\{x\in\Omega: p(x)\ge p^{\Delta}(A)\}} p(x)\,dx = \int_0^A p^{\Delta}(t)\,dt = \int_0^{|\widehat{D}|} p^{\Delta}(t)\,dt.$$

Thus

$$\int_{\widehat{D}} p(x)\,dx = \int_0^{|\widehat{D}|} p^{\Delta}(t)\,dt.$$

So, from Lemma 4.1, it follows that $\widehat{D}$ must be a cut of p. Clearly it must be that $\widehat{D} = \{x \in \Omega : p(x) \ge p^{\Delta}(A)\}$, as in (4.2). □

Corollary 4.4. *D is the solution of* (2.5)*, if and only if the following optimality condition is satisfied:*

$$\inf_D p \ge \sup_{\Omega\setminus D} p. \tag{4.4}$$

Proof. First we suppose D is a solution of (2.5). Then, χ_D is the solution of (3.1). Hence, from (4.2), we deduce $D = \{x \in \Omega : p(x) \ge p^{\Delta}(A)\}$. So (4.4) follows readily.

Conversely, let us suppose D satisfies (4.4). Then, from $(\mathbf{P}_1)$–$(\mathbf{P}_3)$, it follows that $D = \{x \in \Omega : p(x) \ge \inf_D p\}$. So

$$|D| = |\{x \in \Omega : p(x) \ge \inf_D p\}| = \lambda_p(\inf_D p).$$

Because $|D| = A$, we infer that $\lambda_p(\inf_D p) = A$, hence $p^{\Delta}(\lambda_p(\inf_D p)) = p^{\Delta}(A)$. But $p^{\Delta}(\lambda_p) = i$, where i is the identity, hence $\inf_D p = p^{\Delta}(A)$. So we conclude that $D = \{x \in \Omega : p(x) \ge p^{\Delta}(A)\}$. Thus, D is the solution of (2.5). □

5. SYMMETRY

In this section, we show that $\widehat{D}$, the solution of (2.5), inherits the symmetry properties that the domain Ω, and p share. In particular, we address Steiner and radial symmetry.

Recall that we call Ω Steiner symmetric with respect to the line l in $\mathbb{R}^2$, if:

(a) For every $x \in \Omega$, $x_l \in \Omega$, where x_l is the image of x with respect to l.
(b) For every $x \in \Omega$, the segment connecting x and x_l lies entirely inside Ω.

Now, we assume Ω is Steiner symmetric with respect to some line l. We also assume that p is Steiner symmetric with respect to l; that is:

(c) For every $x \in \Omega$, $p(x) = p(x_l)$.
(d) Every curve induced by the intersection of a plane perpendicular to l and the graph of p is strictly concave.

Theorem 5.1. *Under the conditions* (a)–(d) *mentioned above, the unique solution of* (2.5)*,* $\widehat{D}$*, is Steiner symmetric with respect to* l*.*

Proof. From (4.2), we know $\widehat{D} = \{x \in \Omega : p(x) \geq p^{\Delta}(A)\}$. For $x \in \widehat{D}$, $p(x) \geq p^{\Delta}(A)$. So, as $p(x) = p(x_l)$, we infer $p(x_l) \geq p^{\Delta}(A)$, hence $x_l \in \widehat{D}$. It remains to show convexity of $\widehat{D}$. Let us fix $x \in \widehat{D}$ and consider the segment $tx_l + (1-t)x$, $0 \leq t \leq 1$. Because p is concave, it follows that

$$p(tx_l + (1-t)x) \geq tp(x_l) + (1-t)p(x) \geq tp^{\Delta}(A) + (1-t)p^{\Delta}(A) = p^{\Delta}(A),$$

where in the second inequality we used the fact that x and x_l are both in $\widehat{D}$. Thus, $tx_l + (1-t)x \in \widehat{D}$, as desired. □

A straightforward consequence of Theorem 5.1 is the following

Corollary 5.2. *Suppose* Ω *is a disk centered at the origin. Suppose* p *is a radial function; that is,* $p(x) = p(r)$*, where* $r = \|x\|_{\mathbb{R}^2}$*. Then,* $\widehat{D}$*, the solution of* (2.5)*, is a disk concentric with* Ω*.*

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