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Krausz dimension and its generalizations in special graph classes

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A *krausz* (k, m)*-partition* of a graph G is the partition of G into cliques, such that any vertex belongs to at most k cliques and any two cliques have at most m vertices in common. The m*-krausz* dimension $kdim_m(G)$ of the graph G is the minimum number k such that G has a krausz (k, m)-partition. 1-krausz dimension is known and studied krausz dimension of graph $kdim(G)$.

In this paper we prove, that the problem "$kdim(G) \leq 3$" is polynomially solvable for chordal graphs, thus partially solving the problem of P. Hlineny and J. Kratochvil. We show, that the problem of finding m-krausz dimension is NP-hard for every $m \geq 1$, even if restricted to (1,2)-colorable graphs, but the problem "$kdim_m(G) \leq k$" is polynomially solvable for $(\infty, 1)$-polar graphs for every fixed $k, m \geq 1$.

Keywords: Krausz dimension, intersection graphs, linear k-uniform hypergraphs, chordal graphs, polar graphs

1 Introduction

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph (hypergraph) G are denoted by $V(G)$ and $E(G)$ respectively. $N(v) = N_G(v)$ is the neighborhood of a vertex v in G and $deg(v)$ is the degree of v. Let $G(X)$ denote the subgraph of G induced by a set $X \subseteq V(G)$ and $ecc_G(v)$ is the eccentricity of a vertex $v \in V(G)$.

A *krausz partition* of a graph G is the partition of G into cliques (called *clusters* of the partition), such that every edge of G belongs to exactly one cluster. If every vertex of G belongs to at most k clusters then the partition is called *krausz* k*-partition*. The *krausz dimension* $kdim(G)$ of the graph G is a minimal k such that G has krausz k-partition.

Krausz k-partitions are closely connected with the representation of a graph as an intersection graph of a hypergraph. The *intersection graph* $L(H)$ of a hypergraph $H = (V(H), E(H))$ is defined as follows:

1) the vertices of $L(H)$ are in a bijective correspondence with the edges of H;

2) two vertices are adjacent in $L(H)$ if and only if the corresponding edges have a nonempty intersection.

Hypergraph H is called *linear*, if any two of its edges have at most one common vertex; k*-uniform*, if every edge contains k vertices.

The *multiplicity* of the pair of vertices u, v of the hypergraph H is the number $m(u, v) = |\{\mathcal{E} \in E(H) : u, v \in \mathcal{E}\}|$; the *multiplicity* $m(H)$ of the hypergraph H is the maximum multiplicity of the pairs of its vertices. So, linear hypergraphs are the hypergraphs with the multiplicity 1.

Denote by H^* the dual hypergraph of H and by $H_{[2]}$ the 2-section of H (i.e. the simple graph obtained from H by transformation each edge into a clique). It follows immediately from the definition that

$$L(H) = (H^*)_{[2]} \tag{1}$$

(first this relation was implicitly formulated by C. Berge in [2]). This relation implies that a graph G has krausz k-partition if and only if it is intersection graph of linear k-uniform hypergraph.

A graph is called (p, q)*-colorable* [4], if its vertex set could be partitioned into p cliques and q stable sets. In this terms $(1, 1)$-colorable graphs are well-known split graphs.

Another generalization of split graphs are *polar graphs* (see [6],[20]). A graph G is called *polar* if there exists a partition of its vertex set

$$V(G) = A \cup B,\ A \cap B = \emptyset \tag{2}$$

(*bipartition* (A, B)) such that all connected components of the graphs $\overline{G}(A)$ and $G(B)$ are complete graphs. If, in addition, α and β are fixed integers, and the orders of connected components of the graphs $\overline{G}(A)$ and $G(B)$ are at most α and β respectively, then the polar graph G with bipartition (2) is called (α, β)*-polar*. Given a polar graph G with bipartition (2), if the order of connected components of the graph $\overline{G}(A)$ (the graph $G(B)$) is not restricted above, then the parameter α (respectively β) is supposed to be equal ∞. Thus an arbitrary polar graph is (∞, ∞)-polar, and a split graph is $(1, 1)$-polar.

Denote by $KDIM(k)$ the problem of determining whether $kdim(G) \leq k$ and by $KDIM$ the problem of finding the krausz dimension.

The class of line graphs (intersection graphs of linear 2-uniform hypergraphs, i.e graphs with krausz dimension at most 2) have been studied for a long time. It is characterized by a finite list of forbidden induced subgraphs [1], efficient algorithms for recognizing it (i.e. solving the problem $KDIM(2)$) and constructing the corresponding krausz 2-partition are also known (see for example [5], [11], [17], [18]).

The situation changes radically if one takes $k = 3$ instead of $k = 2$: the problem $KDIM(k)$ is NP-complete for every fixed $k \geq 3$ [8]. The case $k = 3$ was studied in the different papers (see [9],[14],[15],[16],[19]), and several graph classes, where the problem $KDIM(3)$ is polynomially solvable or remains NP-complete, were found.

In [8] P. Hlineny and J. Kratochvil studied the computational complexity of the krausz dimension in detail. Besides another results, the following results were obtained in their paper:

1) The problem $KDIM$ is polynomially solvable for graphs with bounded treewidth. In particular, it is polynomially solvable for chordal graphs with bounded clique size.

2) For the whole class of chordal graphs the problem $KDIM(k)$ is NP-complete for every $k \geq 6$.

So, the problem of deciding the complexity of $KDIM(k)$ restricted to chordal graphs for $k = 3, 4, 5$ was posed by P. Hlineny and J. Kratochvil. As a partial answer to it, in the Section 2 we prove that the problem $KDIM(3)$ is polynomially solvable in the class of chordal graphs.

In the Section 3 we consider the natural generalization of the krausz dimension. The *krausz* (k,m)-*partition* of a graph G is the partition of G into cliques (called *clusters* of the partition), such that any vertex belongs to at most k clusters of the partition, and any two clusters have at most m vertices in common. As above, the relation (1) implies the following statement:

Proposition 1 *A graph G has krausz (k,m)-partition if and only if it is the intersection graph of a k-uniform hypergraph with the multiplicity at most m.*

The m-*krausz* dimension $kdim_m(G)$ of the graph G is the minimum k such that G has a krausz (k,m)-partition. The krausz dimension in this terms is the 1-krausz dimension.

Denote by $KDIM_m$ the problem of determining the m-krausz dimension of graph, by $KDIM_m(k)$ the problem of determining whether $kdim_m(G) \leq k$ and by L_k^m the class of graphs with a krausz (k,m)-partition. It was proved in [10] that the class L_3^m could not be characterized by a finite set of forbidden induced subgraphs for every $m \geq 2$, but the complexity of the problem $KDIM_m$ for an arbitrary m was not established yet. We prove that the problem $KDIM_m$ is NP-hard for every $m \geq 1$, even if restricted to the class of $(1,2)$-colorable graphs.

The class L_k^m is hereditary (i.e. closed with respect to deleting the vertices) and therefore can be characterized in terms of forbidden induced subgraphs. We prove that for every fixed integers m, k such finite characterization of the class exists when restricted to $(\infty,1)$-polar graphs. In particular, it follows that the problem $KDIM_m(k)$ is polynomially solvable for $(\infty,1)$-polar graphs for every fixed m and k. In particular, it generalizes the result of [8] and [12], that for every fixed k the problem $KDIM(k)$ is polynomially solvable for split graphs.

2 Krausz 3-partitions of chordal graphs

Let F be a family of cliques of graph G. The cliques from F are called *clusters of* F. Denote by $l_F(v)$ the number of clusters from F covering the vertex v.

A maximal clique with at least $k^2 - k + 2$ vertices is called a k-*large* clique. For such cliques the following statement holds:

Lemma 2 *[8, 9, 16] Any k-large clique of a graph G belongs to every krausz k-partition of G.*

Further in this section 3-large clique will be called simply *large clique*.

Let G be a graph with $kdim(G) \leq 3$ and Q be some its krausz 3-partition. Any subset $F \subseteq Q$ is called *a fragment of the krausz 3-partition* Q (or simply *a fragment*).

Let F be some fragment of krausz 3-partition Q and H be the subgraph of G obtained by deleting edges covered by F (F could be empty). Fix some vertex $a \in V(H)$ and positive integer k. Denote by $B_k[a]$ the kth neighborhood of a in H, i.e. the set of vertices at distance at most k from a. A family of cliques $F_k(a)$ in H is called (a,k)-*local fragment* (or simply a *local fragment*), if

(1) every edge with at least one end in $B_k[a]$ is covered by some cluster of $F_k(a)$;

(2) every vertex $v \in B_k[a]$ belongs to at most $3 - l_F(v)$ clusters of $F_k(a)$.

(3) every two clusters of $F_k(a)$ have at most one common vertex.

A clique C is called *special*, if C is a cluster of every (a,k)-local fragment for some a and k. In particular, by Lemma 2 large cliques are special.

The following statements are evident.

Lemma 3 *1) If $deg(v) \geq 19$ for some vertex $v \in V(G)$, then v is contained in some large clique.*

2) If $l_F(v) = 2$, then $C = N_H(v) \cup \{v\}$ is a special clique.

3) If $v \in B_k[a]$ is adjacent to at least $4 - l_F(v)$ vertices of the cluster C of some local fragment $F_k(a)$, then $v \in C$.

4) for every $a \in V(H)$ and every k there exists at least one (a, k)-local fragment;

5) If the clique C is special, then $F \cup \{C\}$ is a fragment.

Proof: Let's illustrate, for example, 3) and 5). If $v \in B_k[a]$ is adjacent to vertices $v_1, ..., v_{4-l_F(v)} \in C \in F_k(a)$, but $v \notin C$, then the edges $vv_1, ...vv_{4-l_F(v)}$ should be covered by different clusters of $F_k(a)$. It contradicts (2).

The family of cliques $X = \{C \in Q \setminus F : C \cap B_k[a] \neq \emptyset\}$ is a local fragment. Since C is special, $C \in X$ and therefore $C \in Q \setminus F$. □

Denote by $lc(H)$ the length of a longest induced cycle of the graph H.

Lemma 4 *Let G be a chordal graph with $kdim(G) \leq 3$. Let further there are no special cliques in H. Then $lc(H) \leq 6$.*

Proof: Suppose contrary, i.e. let $a_1, \ldots, a_k$ form the induced cycle $S \cong C_k$ in H, $k \geq 7$, $a_i a_{i+1} \in E(H)$, indices are taken modulo k.

Since for every a_i there are two nonadjacent neighbors in H, then in every local fragment with center in a_i it is covered by at least 2 clusters. It implies $l_F(a_i) \leq 1$ for every $i = 1, \ldots, k$.

As G is a chordal graph, there exist chords of the cycle S covered by the fragment F. It is easy to see, that for every two consecutive vertices a_i, a_{i+1} of S at least one of them belongs to some chord of S (indices are taken modulo k). Indeed, let without lost of generality $a_i = a_k$, $a_{i+1} = a_1$. If our statement is not true, then one can choose the chord $a_p a_q$, $1 < p < q < k$ such, that $(p-1) + (k-q)$ is minimal. But then $G(a_1, \ldots, a_p, a_q, \ldots, a_k)$ is a chordless cycle.

Assume without lost of generality, that one of chords of S contains a_1. As $l_F(a_i) \leq 1$, for every vertex a_i chords incident to this vertex are covered by exactly one cluster of F. It implies that there are no pairs of chords of the form $\{a_i a_j, a_i a_{j+1}\}$, since in this case the vertices a_i, a_j, a_{j+1} are covered by one cluster of F and thus the edge $a_j a_{j+1}$ should be covered by F.

Let us show, that all chords of S are covered by the cluster $C_{chord} \supseteq \{a_1, a_3, \ldots, a_{k-1}\}$ (and thus k is even). Indeed, suppose that some chords of S are covered by the cluster $C \supseteq \{a_{i_1}, \ldots, a_{i_r}\}$, $i_1 < i_2 < \ldots < i_r$, $i_1 = 1$, $C \neq C_{chord}$. Then there exist $1 \leq p < q \leq r$ such, that $q - p \geq 3$. So, $G(a_{i_p}, a_{i_p+1}, \ldots, a_{i_q-1}, a_{i_q})$ is a cycle of length at least 4, where without lost of generality a_{i_p} belongs to some chord. That chord should be covered by a cluster $C' \in F$, $C' \neq C$. So, we have $l_F(a_{i_p}) \geq 2$, the contradiction.

In particular, this proposition implies that for any odd i and even j such that $a_i a_j$ is not the edge of S, the vertices a_i and a_j are nonadjacent in G (otherwise $l_F(a_i) \geq 2$).

Let us denote by $C(v_1, \ldots, v_r)$ the clique containing vertices $v_1, \ldots, v_r$.

Let $e = min\{ecc_H(a_1), 5\}$. Since there is no special cliques in H there exists two different local fragments $F_e(a_1) \supseteq \{C(a_1, a_2), C(a_1, a_k)\}$ and $F'_e(a_1) \supseteq \{C'(a_1, a_2), C'(a_1, a_k)\}$, such that without lost of generality $C(a_1, a_2) \setminus C'(a_1, a_2) \neq \emptyset$.

Let $v \in C(a_1, a_2) \setminus C'(a_1, a_2)$. Then $v \in C'(a_1, a_k)$ and therefore $a_1 v, a_2 v, a_k v \in E(H)$.

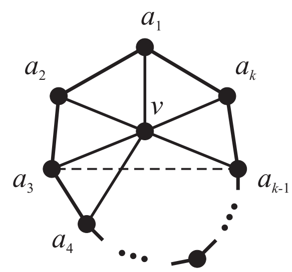

Figure 1:

The vertices $a_2, v, a_k, a_{k-1}, a_3$ form a cycle in G. It should have at least 2 chords. Since $a_2a_{k-1}, a_3a_k \notin E(G)$, there are edges $va_3, va_{k-1} \in E(G)$. The edges va_3, va_{k-1} are not covered by F (otherwise $v \in C_{chord}$ and thus $\{a_1, v\} \in C_{chord} \cap C(a_1, a_2)$) and hence $va_3, va_{k-1} \in E(H)$. It implies, that $v \in C(a_3, a_4) \in F_e(a_1)$. So, $va_4 \in E(H)$ (see Figure 1). Note, that since $k \geq 7$ we have $a_4a_{k-1} \notin E(H)$.

Let us remind, that in the local fragment $F'_e(a_1)$ the vertex v is covered by the cluster $C'(v, a_1, a_k)$. So, all other edges of H incident to v, should be covered by at most two clusters of $F_e(a_1)$. But it is impossible, since the vertices a_2, a_4, a_{k-1} are pairwise nonadjacent. This contradiction proves Lemma 4. □

The considerations above suggest the following algorithm which reduces the problem of recognition chordal graphs with krausz dimension at most 3 to the same problem for graphs with bounded maximum degree and maximum induced cycle length.

Algorithm 1
Input: chordal graph G.
Output: One of the following:
1) graph H with $\Delta(H) \leq 18$ and $lc(H) \leq 6$ such that $kdim(G) \leq 3$ if and only if $kdim(H) \leq 3$;
2) the answer "$kdim(G) > 3$".
begin
 $F := \emptyset$; $H := G$; $isContinue := true$;
 while ($isContinue =$ **true**)
  if there exists a vertex $v \in V(H)$ such that $l_F(v) = 2$
   $C := N(v) \cup \{v\}$;
   if C is a clique
    $F := F \cup \{C\}$; **continue** to the next iteration of the cycle;
   else the answer is "$kdim(G) > 3$"; **stop;**
  if there exists a vertex $v \in V(H)$ with $deg(v) \geq 19$
   if v is contained in a clique C with $|C| \geq 8$
    extend C to a maximal clique; $F := F \cup \{C\}$;
    continue to the next iteration of the cycle;
   else the answer is "$kdim(G) > 3$"; **stop;**
  For every non-isolated vertex $v \in V(H)$ generate all
  (v, e)-local fragments, $e = min\{ecc_H(v), 5\}$;

if there exists a vertex $v \in V(H)$ such that there is
no (v, e)-local fragments
the answer is "$kdim(G) > 3$"; **stop;**
if there exists a special clique C
$F := F \cup \{C\}$; **continue** to the next iteration of the cycle;
$isContinue := false$
endwhile;
add a pendant edge vp_v to every vertex $v \in V(H)$ with $l_F(v) = 1$;
end.

Theorem 5 *[3] Let* $lc(H) \leq s + 2$, $\Delta(H) \leq \Delta$. *Then* $treewidth(H) \leq \Delta(\Delta - 1)^{s-1}$.

Theorem 6 *The problem* $KDIM(3)$ *is polynomially solvable for chordal graph.*

Proof: The correctness of algorithm 1 follows from the considerations above. Let us show, that the Algorithm 1 is polynomial. Indeed, the procedure of finding large clique which contains the fixed vertex $v \in V(H)$ has the complexity $O(m)$. We start to generate all possible (v, e)-local fragments for a vertex $v \in V(H)$ only then $deg(v) \leq 18$. It implies $|B_e[v]| \leq const$ and thus the complexity of this procedure is constant. The outer loop of the algorithm 1 is performed at most m times.

After performing the Algorithm 1 we obtained the graph H with bounded maximum degree and the length of a longest induced cycle. By Theorem 5 H has bounded treewidth. For such a graph the problem of determining its krausz dimension is polynomially solvable [8].

□

3 m-krausz dimension of graphs

We will start with proving the NP-hardness of the problem $KDIM_m$. In order to make the proof more clear, we firstly will prove, that $KDIM_m$ is NP-hard for general graphs, and then we will use the developed construction to prove, that $KDIM_m$ is NP-hard for $(1, 2)$-colorable graphs.

Theorem 7 *The problem* $KDIM_m$ *is NP-hard for every fixed* $m \geq 1$.

Proof: Let us reduce to the problem $KDIM_m$ the following special case of the 3-dimensional matching problem (which we will call the problem A):

Given: Non-intersecting sets X, Y, Z , such that $|X| = |Y| = |Z| = q$; $M \subseteq X \times Y \times Z$, such that the following condition holds:

(*) if $(a, b, w), (a, x, c), (y, b, c) \in M$, then $(a, b, c) \in M$.

The question: Does M contain a subset $M' \subseteq M$ (*3-dimensional matching*) such, that $|M'| = q$ and every two elements of M' have not common coordinates?

It is known, that the problem A is NP-complete [7]. Let X, Y, Z, M, $|X| = |Y| = |Z| = q$, be the input of the problem A. Let us reduce the problem A to the problem of determining, if $kdim_m(G) \leq 2q$. Construct the graph G as follows:

$$V(G) = X \cup Y \cup Z \cup \{v, v_1, \ldots, v_q\}; \tag{3}$$

$$E(G) = \bigcup_{(a,b,c) \in M} \{ab, bc, ac\} \cup \{v_i v : i = 1, \ldots, q\} \cup \{vd : d \in X \cup Y \cup Z\} \tag{4}$$

(see Figure 2). Let us show that M contains the 3-dimensional matching M' if and only if there exists a krausz $(2q, m)$-partition of G.

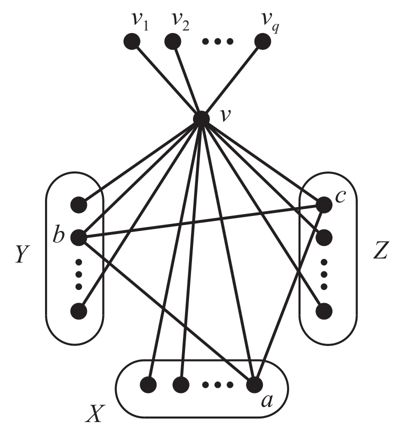

Figure 2:

Suppose, that $M' = \{(a_i, b_i, c_i) : i = 1, \ldots, q\}$ is the 3-dimensional matching. Let $Q_1 = \{\{v, a_i, b_i, c_i\} : i = 1, \ldots, q\}$, $Q_2 = \{\{v, v_i\} : i = 1, \ldots, q\}$, $Q_3 = \{\{z, t\} : zt \in E(G - (Q_1 \cup Q_2))$. Then $Q = Q_1 \cup Q_2 \cup Q_3$ is krausz $(2q, m)$-partition of G, since $deg(u) \leq 2q$ for every vertex $u \in V(G) \setminus \{v\}$ and the vertex v is covered by exactly $2q$ clusters of Q.

Let now Q be krausz $(2q, m)$-partition of G. Denote by $Q(v)$ the set of clusters of Q, which contain the vertex v. Since the vertices v_i, $i = 1, \ldots, q$, have degree 1, there exist q clusters from $Q(v)$ of the form vv_i, $i = 1, \ldots, q$. Let $C_1, \ldots, C_p$ be the remaining clusters from $Q(v)$, $p \leq q$. Then $(C_1 \cup \ldots \cup C_p) \setminus \{v\} = X \cup Y \cup Z$. Since X, Y, Z are stable sets of G, we have $|C_i| \leq 4$, $i = 1, \ldots, p$. As $|X \cup Y \cup Z| = 3q$, we have $p = q$, $|C_i| = 4$, $C_i \cap C_j = \{v\}$, $i, j = 1, \ldots, p$, $i \neq j$.

Let $C_i = \{a_i, b_i, c_i, v : a_i \in X,\ b_i \in Y,\ c_i \in Z\}$, $i = 1, \ldots, q$. The property (*) implies, that $M' = \{(a_i, b_i, c_i) : i = 1, \ldots, q\} \subseteq M$ and, by the consideration above, M' is the 3-dimensional matching. □

Corollary 8 *The problem $KDIM_m$ is NP-hard in the class of $(1, 2)$-colorable graphs for every fixed $m \geq 1$.*

Proof: Let us show, that the problem A could be reduced to the problem $KDIM_m$ in the class of $(1, 2)$-colorable graphs.

Let G be the graph constructed in the proof of Theorem 7. Let us construct the graph G' as follows: $V(G') = V(G) \cup V_1' \cup V_2'$, where

$$V_1' = \{w, w_1, \ldots, w_{2q}\}; \tag{5}$$

$$V_2' = \{f_u : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\})\}; \tag{6}$$

$E(G') = E(G) \cup E_1' \cup E_2' \cup E_3' \cup E_4'$, where

$$E_1' = \{ww_i : i = 1, \ldots, 2q\}; \tag{7}$$

$$E'_2 = \{wx : x \in X\}; \quad (8)$$

$$E'_3 = \{x_1x_2 : x_1, x_2 \in X, x_1 \neq x_2\}; \quad (9)$$

$$E'_4 = \{uf_u : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\})\} \quad (10)$$

(see Figure 3). The set $X \cup \{v\}$ is a clique, and the sets $Y \cup \{f_z : z \in Z\} \cup \{v_1, \ldots, v_q, f_v, w\}$ and $Z \cup \{f_y : y \in Y\} \cup \{w_1, \ldots, w_{2q}\}$ are stable sets of G'. So, G' is $(1, 2)$-colorable graph.

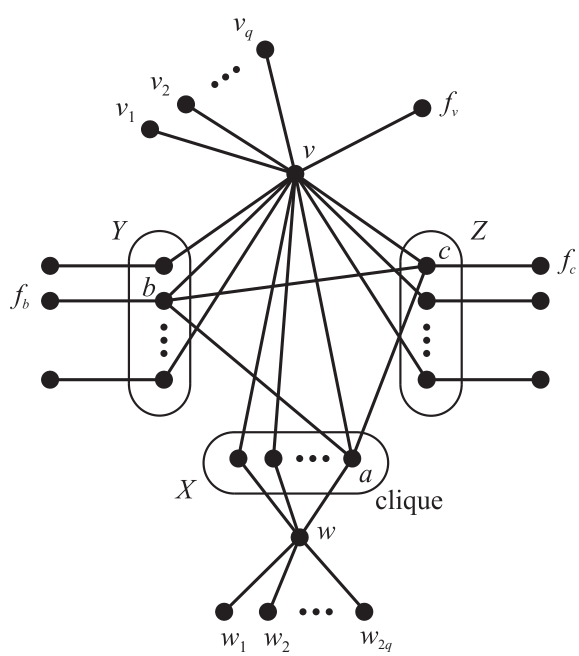

Figure 3:

It is evident, that Q is the krausz $(2q, m)$-partition of G if and only if

$$Q \cup \{X \cup \{w\}\} \cup \{\{ww_i\} : i = 1, \ldots, 2q\} \cup \{\{u, f_u\} : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\})\} \quad (11)$$

is the krausz $(2q + 1, m)$-partition of G'. □

Now we turn to the complexity of the recognition problem $KDIM_m(k)$ in the class of $(\infty, 1)$-polar graphs.

A maximal clique with at least $f(k, m) = m(k^2 - k + 1) + 1$ vertices is called a (k, m)-*large* clique.

In [13] the following two statements were proved. Since they were published only in Russian in a journal, which is difficult of access for a general reader, we repeat their proofs here.

Theorem 9 *Any (k, m)-large clique C of a graph G belongs to every krausz (k, m)-partition of G.*

Proof: Let A be a krausz (k, m)-partition of graph G, $A_1, A_2, \ldots, A_t$ be those clusters of A which have common vertices with C. Assume that $C \notin A$. Then the family $B = (B_1, B_2, \ldots, B_t)$, where

$B_i = A_i \cap C$, is a krausz (k,m)-partition of the graph $G(C)$, and (by maximality of C) $B_i \neq C$ for every $i = 1, 2, \ldots, t$.

Let us show, that $|B_i| \leq mk$ for any $i = 1, 2, \ldots, t$. Consider a cluster of B, say B_1, and a vertex $u \in C \setminus B_1$. No edge of the form ux, where $x \in B_1$, is contained in B_1. Moreover, each cluster of B different from B_1 contains at most m of such edges (by the definition of krausz (k,m)-partition). Taking into account that the vertex u belongs to at most k clusters of B, we obtain the inequality $|B_1| \leq mk$.

Now we will prove that if $B_i \setminus B_j \neq \emptyset$ for some clusters $B_j \in B$, then $|B_j \setminus B_i| \leq m(k-1)$. Consider a vertex $u \in B_i \setminus B_j$. Any edge of the form ux, where $x \in B_j \setminus B_i$ (if such one exists) is contained neither in B_i, nor in B_j. Besides, no cluster of B contains more than m of such edges by definition of krausz (k,m)-partition. Taking into account that u belongs to at most $k-1$ clusters of B different from B_i, we obtain the inequality $|B_j \setminus B_i| \leq m(k-1)$.

Consider an arbitrary vertex v of the clique C. Let, without loss of generality, it belongs to the clusters $B_1, B_2, \ldots, B_s$ of B, $s \leq t$. We show that $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$. The following equality is obvious

$$|B_1 \cup B_2 \cup \ldots \cup B_s| = |B_1| + |B_2 \setminus B_1| + |B_3 \setminus (B_1 \cup B_2)| + \ldots + |B_s \setminus (B_1 \cup B_2 \ldots \cup B_{s-1})|. \quad (12)$$

If $B_1 \setminus B_2 \neq \emptyset$, $(B_1 \cup B_2) \setminus B_3 \neq \emptyset, \ldots, (B_1 \cup B_2 \cup \ldots \cup B_{s-1}) \setminus B_s \neq \emptyset$, then by proved above each term in the right part of the equality (12), starting from the second, does not exceed $m(k-1)$. Hence we have $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$. Let, on the contrary, $i \in \{2, \ldots, s\}$ is the maximal number such, that $(B_1 \cup \ldots \cup B_{i-1}) \setminus B_i = \emptyset$. Then $B_1 \subseteq B_i$, $B_2 \subseteq B_i, \ldots, B_{i-1} \subseteq B_i$, and the sum of the first i terms in the right part of (12) is equal to $|B_1 \cup B_2 \cup \ldots \cup B_i| = |B_i| \leq mk$. Each of the other terms does not exceed $m(k-1)$ by the maximality of i. Hence

$$|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-i)m(k-1) < mk + (s-1)m(k-1).$$

So, in any case we obtain that the inequality $|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1)$ holds. Taking into account that $C = B_1 \cup B_2 \cup \ldots \cup B_s$ and $s \leq k$, we have

$$|C| \leq mk + (k-1)m(k-1) = m(k^2 - k + 1) < f(k,m).$$

The obtained contradiction proves the lemma. □

Theorem 10 *There exists a finite set $\mathcal{F}_0$ of forbidden induced subgraphs such that a split graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to an element of $\mathcal{F}_0$.*

Proof: Denote by R_p the graph obtained from the complete graph $H \cong K_{f(k,m)}$ by adding a new vertex and connecting it with exactly p vertices of H. Put $\mathcal{F}_0 = \{R_p : km+1 \leq p \leq f(k,m)-1\} \cup \{K_{1,k+1}\}$. Using Theorem 9 one can immediately verify that no graph from $\mathcal{F}_0$ belongs to L_k^m.

Let, without loss of generality, G be connected graph, and $V(G) = C \cup S$ be a bipartition of $V(G)$ into clique C and stable set S such, that C is a maximal clique. Let also no induced subgraph of G be isomorphic to an element of $\mathcal{F}_0$. Put $S = \{v_1, \ldots, v_s\}$. Consider two cases:

1) $|C| > (km-1)k + 1$.

In this case we have

$$|C| \geq (km-1)k + 2 = mk^2 - (k-1) + 1 \geq mk^2 - m(k-1) + 1 = f(k,m).$$

Then, since no induced subgraph of G is isomorphic to a graph R_p, $km+1 \leq p \leq f(k,m)-1$, we have $\deg(v_i) \leq km$ for any $i = 1,2,\ldots,s$. Since G contains no induced $K_{1,k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex u from C. Moreover, we prove that for any vertex u from C the inequality $|N(u) \cap S| \leq k-1$ holds. Assume this is not true. Let, without loss of generality, some vertex u from C be adjacent to the vertices $v_1,\ldots,v_k$ from S, $k \leq s$. Since $deg(v_i) \leq km$, $i = 1,2,\ldots,k$, and $u \in \bigcap\limits_{i=1}^{k} N(v_i)$, then $|\bigcup\limits_{i=1}^{k} N(v_i)| \leq \sum\limits_{i=1}^{k}(deg(v_i)-1)+1 \leq (km-1)k+1 < \varphi(G)$. Hence, there exists a vertex u' from C, which is not adjacent to any vertex from $v_1,\ldots,v_k$. But then $G(u,u',v_1,\ldots,v_k) \cong K_{1,k+1}$, a contradiction.

Now we can construct a krausz (k,m)-partition of G. Since $deg(v_i) \leq km$ for any $i = 1,2,\ldots,s$, then there exists a partition $N(v_i) = C_{i_1} \cup \ldots \cup C_{i_{s_i}}$, where $C_{i_j} \cap C_{i_l} = \emptyset$, $j,l \in \{1,\ldots,s_i\}$, $j \neq l$, $|C_{i_j}| \leq m$, $s_i \leq k$. Obviously, the list of cliques $\{C_{i_j} \cup \{v_i\} : i = \overline{1,s}, j = \overline{1,s_i}\}$ together with the clique C is a krausz (k,m)-partition of graph G.

2) $|C| \leq (km-1)k+1$.

Since G contains no induced $K_{1,k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex u from C. Therefore, as G is connected,

$$|G| = |C|+|S| \leq |C|+\sum_{u\in C}|N(u)\cap S| \leq ((km-1)k+1)+((km-1)k+1)k = ((km-1)k+1)(k+1),$$

i. e. the order of graph G is bounded above by a value, depending on k and m. Add to the list $\mathcal{F}_0$ all such split graphs H, that $H \notin L_k^m$ and $|H| \leq ((km-1)k+1)(k+1)$.

Obviously, the constructed in the cases 1) and 2) finite list $\mathcal{F}_0$ is a required list of forbidden induced subgraphs. □

Since $K_{1,k+1} \notin L_k^m$, the heredity of L_k^m immediately implies

Lemma 11 *A bipartite graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to $K_{1,k+1}$.*

Theorem 12 *There exists a finite set $\mathcal{F}$ of forbidden induced subgraphs such that an $(\infty,1)$-polar graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$.*

Proof: Without loss of generality we can suppose that $(\infty,1)$-polar graph G is connected. Let G have bipartition (A,B); A_i, $i = 1,2,\ldots,t$, be the vertex sets of connected components of $\overline{G}(A)$; $\mathcal{F}_0$ be the set of split graphs from Theorem 10. Denote by $\mathcal{F}_1$ the set of $(\infty,1)$-polar graphs which do not belong to the class L_k^m and have order at most $(k+1)k(f(k,m)-1)$.

Put $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{K_{1,k+1}, K_{f(k,m)+1}-e\}$, where $K_{f(k,m)+1}-e$ is the graph obtained from the complete graph $K_{f(k,m)+1}$ after deleting an edge. The set $\mathcal{F}$ is finite, since $\mathcal{F}_0$ and $\mathcal{F}_1$ are finite. According to Theorem 9, there is no krausz (k,m)-partition for $K_{f(k,m)+1}-e$. Therefore $K_{f(k,m)+1}-e \notin L_k^m$. Thus, $\mathcal{F} \cap L_k^m = \emptyset$. The necessity of the statement follows from the heredity of the class L_k^m.

Now let G contain no induced subgraph isomorphic to an element from $\mathcal{F}$. If $G(A)$ is complete, then G is split graph and by Theorem 10 $G \in L_k^m$. If $G(A)$ is empty, then G is bipartite graph and by Lemma 11 $G \in L_k^m$.

Now suppose that $G(A)$ is neither complete nor bipartite graph. Then $2 \leq t \leq |A|-1$. Since $K_{1,k+1} \in \mathcal{F}$, then $|A_i| \leq k$ for any $i = 1,2,\ldots,t$. Now we will prove that since $K_{f(k,m)+1}-e \in \mathcal{F}$, then

$t \leq f(k,m)-1$. Let, to the contrary, $t \geq f(k,m)$. As $G(A)$ is not complete graph, there exists an index $i_0 \in \{1,2,\ldots,t\}$ such that $|A_{i_0}| \geq 2$. Consider the set $S = \{a_1, a_2, \ldots, a_{i_0-1}, a'_{i_0}, a''_{i_0}, a_{i_0+1}, \ldots, a_t\}$, where $a_i \in A_i$ for any $i \in \{1,2,\ldots,t\} \setminus \{i_0\}$ and $a'_{i_0}, a''_{i_0} \in A_{i_0}$. Then $G(S)$ contains $K_{f(k,m)+1} - e$ as induced subgraph, a contradiction. Therefore

$$|A| \leq \sum_{i=1}^{t} |A_i| \leq k(f(k,m)-1).$$

Since $|N(a) \cap B| \leq k$ for any vertex $a \in A$ and G is connected, we have

$$|G| \leq |A|+|B| \leq |A|+\sum_{a\in A}|N(a)\cap B| \leq k(f(k,m)-1)+k^2(f(k,m)-1) = (k+1)k(f(k,m)-1).$$

It follows from the inclusion $\mathcal{F}_1 \subseteq \mathcal{F}$ that $G \in L_k^m$. □

Corollary 13 *The problem $KDIM_m(k)$ is polynomially solvable in the class of $(\infty,1)$-polar graphs for every fixed $k, m \geq 1$.*

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