PURE DISCRETE SPECTRUM IN SUBSTITUTION TILING SPACES

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Abstract. We introduce a procedure for establishing pure discrete spectrum for substitution tiling systems of Pisot family type and illustrate with several examples.

1. Introduction

The study of Aperiodic Order received impetus in 1985 with the discovery of physical quasicrystals. These materials have enough long range order to produce a pure point X-ray diffraction spectrum yet they lack the strict periodicity of traditional crystals. Quasicrystals are modeled by self-affine tilings of Euclidean space and these tilings are effectively studied by means of an associated topological dynamical system, the tiling space. It has been established, through a beautiful circle of ideas and in considerable generality, that a pattern of points (atoms) produces pure point X-ray diffraction if and only if the associated tiling space has pure discrete dynamical spectrum ([D],[LMS]).

We introduce in this article a procedure for determining whether (or not) a tiling dynamical system has pure discrete spectrum. The procedure applies to tiling systems that arise from substitutions. In dimension one, a necessary condition for a substitution tiling system to have pure discrete spectrum is that the expansion factor of the substitution be a Pisot number. Recent results ([LS2]) show that if a substitution tiling system in any dimension has pure discrete spectrum, then the linear expansion associated with the substitution must be generally 'Pisot' in nature and our technique is, correspondingly, restricted to so-called Pisot family substitutions (see the next section for definitions).

Our approach has its origins in the balanced pair algorithm for symbolic substitutive systems initiated by Dekking ([De]). A geometric

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version of balanced pairs (overlap coincidences) was introduced by Solomyak ([S3]) and considered for one-dimensional substitution tiling systems in [IR], [AI], [BK], [SS], [ST], and [Fo], and for higher dimensions in [L], [LM], [LMS], and [FS]. Akiyama and Lee ([AL]) have implemented overlap coincidence in computer language with an algorithm that will decide whether or not a particular substitution tiling system (with the Meyer property) has pure discrete spectrum.

Proposition 17.4 of [BK] (generalized to the 'reducible' case in [BBK]) states that it suffices to check the convergence of the balanced pair algorithm on a single balanced pair in order to determine pure discrete spectrum for a one-dimensional Pisot substitution tiling system. The main result of this article extends that result to Pisot family tiling spaces in all dimensions. (The proof of Theorem 16.3 of [BK], from which the one-dimensional result follows, contains a gap that is not easily plugged by the techniques of that paper. We recover the one-dimensional case here as Corollary 3.3.)

In the next section we briefly review the relevant definitions and background for substitution tiling spaces. Two subsequent sections establish sufficient and necessary conditions for pure discrete spectrum and a final section provides examples in which pure discrete spectrum is established.

2. Definitions and background

The basic ingredients for an n-dimensional substitution are a collection $\{\rho_1, \ldots, \rho_l\}$ of prototiles, a linear expansion $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$, and a substitution rule Φ. Each *prototile* ρ_i is an ordered pair $\rho_i = (C_i, i)$ in which C_i is a compact and topologically regular ($C_i = cl(int(C_i))$) subset of $\mathbb{R}^n$. A translate $\rho_i + v := (C_i + v, i)$ of a prototile by a $v \in \mathbb{R}^n$ is called a *tile*. The *support of a tile* $\tau = \rho_i + v$ is the set $spt(\tau) := C_i + v$ and the *interior* of τ is the interior of its support: $\mathring{\tau} := int(spt(\tau))$. A collection of tiles with pairwise disjoint interiors is called a *patch*, the *support of a patch* P is the union of the supports of its constituent tiles, $spt(P) := \cup_{\tau \in P} spt(\tau)$, and the *interior of a patch* P is $\mathring{P} := \cup_{\tau \in P}(\mathring{\tau})$ (or sometimes $int(P)$). A *tiling* is a patch whose support is all of $\mathbb{R}^n$.

Given a collection $\mathcal{A} = \{\rho_1, \ldots, \rho_l\}$ of prototiles, let $\mathcal{A}^*$ be the collection of all finite patches consisting of collections of translates of the prototiles. A *substitution* with linear expansion Λ is a map $\Phi : \mathcal{A} \to \mathcal{A}^*$ with the property: $spt(\Phi(\rho_i)) = \Lambda spt(\rho_i)$ for each i. Such a Φ extends to a map from tiles to patches by $\Phi(\rho_i + v) := \Phi(\rho_i) + \Lambda v := \{\tau + \Lambda v : \tau \in \Phi(\rho_i)\}$ and to a map from patches to patches by $\Phi(P) := \cup_{\tau \in P} \Phi(\tau)$.

A substitution Φ, as above, is *primitive* if there is $k \in \mathbb{N}$ so that $\Phi^k(\rho_i)$ contains a translate of ρ_j for each $i, j \in \{1, \ldots, l\}$. That is equivalent to saying that M^k has strictly positive entries, where $M = (m_{i,j})$ is the $l \times l$ *transition matrix* for Φ: $m_{i,j}$ is the number of translates of ρ_i that occur in $\Phi(\rho_j)$. A patch P is *allowed* for Φ if there is $k \in \mathbb{N}$ and $v \in \mathbb{R}^n$ so that $P \subset \Phi^k(\rho_i) + v$ for some i. The *tiling space* associated with a primitive substitution Φ is the set $\Omega_\Phi := \{T : T$ is a tiling and all finite patches in T are allowed for $\Phi\}$ with a topology in which two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. Let us be more precise about the topology. Given a tiling T, let $B_0[T] := \{\tau \in T : 0 \in spt(\tau)\}$ and, for $r > 0$, let $B_r[T] := \{\tau \in T : spt(\tau) \cap B_r(0) \neq \emptyset\}$. A neighborhood base at $T \in \Omega_\Phi$ is then $\{U_\epsilon\}_{\epsilon>0}$ with $U_\epsilon := \{T' \in \Omega_\Phi : B_{1/\epsilon}[T'] = B_{1/\epsilon}[T] - v$ for some $v \in B_\epsilon(0)\}$. This is a metrizable topology (see, for example, [AP]) and we denote by d any metric that induces this topology. The map $T \mapsto \Phi(T)$ and the $\mathbb{R}^n$-action $(T, v) \mapsto T - v$ are continuous and these intertwine by $\Phi(T - v) = \Phi(T) - \Lambda v$.

A tiling T has (translational) *finite local complexity* (FLC) if, for each $r \geq 0$, there are only finitely many patches, up to translation, of the form $B_r[T - v]$, $v \in \mathbb{R}^n$. The substitution Φ is said to have finite local complexity if each of its admissible tilings has finite local complexity. Finally, Φ is *aperiodic* if, for $T \in \Omega_\Phi$ and $v \in \mathbb{R}^n$, $T - v = T \implies v = 0$. Under the assumption that Φ is primitive, aperiodic, and has finite local complexity, the following facts are by now well known ([AP],[S1]):

- Ω_Φ is a continuum (i.e., a compact and connected metrizable space),
- $\Phi : \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism, and
- The $\mathbb{R}^n$-action on Ω_Φ is strictly ergodic (i.e., minimal and uniquely ergodic).

All substitutions Φ in this article will be assumed to be primitive, aperiodic, and to have finite local complexity. We will denote by μ the unique translation invariant Borel probability measure on Ω_Φ. A function $f \in L^2(\Omega_\Phi, \mu)$ is an *eigenfunction* of the $\mathbb{R}^n$-action with *eigenvalue* $b \in \mathbb{R}^n$ if $f(T - v) = e^{2\pi i \langle b, v \rangle} f(T)$ for all $v \in \mathbb{R}^n$, and almost all $T \in \Omega_\Phi$. The $\mathbb{R}^n$-action on Ω_Φ is said to have *pure discrete spectrum* if the linear span of the eigenfunctions is dense in $L^2(\Omega_\Phi, \mu)$. Solomyak proves in [S2] that every eigenfunction is almost everywhere equal to a continuous eigenfunction. It is then a consequence of the Halmos - von Neumann theory that the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum if and only if there is a compact abelian group X, an $\mathbb{R}^n$-action on X by translation, and a continuous semi-conjugacy $g : \Omega_\Phi \to X$ of $\mathbb{R}^n$-actions that is a.e. one-to-one (with respect to Haar measure).

There are certain necessary conditions on the linear expansion Λ for the substitution Φ in order for the $\mathbb{R}^n$-action on Ω_Φ to have pure discrete spectrum (pds). For there to even be a substitution associated with Λ, it's necessary that the eigenvalues of Λ be algebraic integers ([K],[LS1]). For pds, the eigenvalues of Λ must constitute a *Pisot family*: If λ is an eigenvalue of Λ and λ' is an algebraic conjugate of λ with $|\lambda'| \geq 1$, then λ' is also an eigenvalue of Λ and of the same multiplicity as λ ([LS2]). Under the additional assumptions that Λ is diagonalizable over $\mathbb{C}$ and all eigenvalues of Λ are algebraic conjugates - say, of degree d - and of the same multiplicity - say m - Lee and Solomyak prove ([LS2]) that the $\mathbb{R}^n$-action on Ω_Φ has a generous compliment of eigenfunctions. We will say that a substitution with a linear expansion whose eigenvalues are as in the previous sentence is of (m, d)-*Pisot family type* (or, if m and d are unimportant, just of *Pisot family type* - we'll also say that $\Omega = \Omega_\Phi$ is a *Pisot family tiling space*). It is proved in [BKe] that if Φ is of (m, d)-Pisot family type, then there is a continuous finite-to-one semi-conjugacy $g : \Omega_\Phi \to X$ of the $\mathbb{R}^n$-action on Ω_Φ with an action of $\mathbb{R}^n$ by translation on a group X. Moreover, there is a hyperbolic endomorphism $F : \mathbb{T}^{md} \to \mathbb{T}^{md}$ of the md-dimensional torus so that X is the inverse limit $X = \hat{\mathbb{T}}^{md} := \varprojlim F$. The $\mathbb{R}^n$-action on $\hat{\mathbb{T}}^{md}$ is a Kronecker action by translation along the n-dimensional unstable leaves of the shift homeomorphism $\hat{F} : \hat{\mathbb{T}}^{md} \to \hat{\mathbb{T}}^{md}$ and, in addition, $g \circ \Phi = \hat{F} \circ g$. (If the eigenvalues of Λ are algebraic units then F is an automorphism and $\hat{\mathbb{T}}^{md}$ is simply the md-dimensional torus.) The group $\hat{\mathbb{T}}^{md}$, with the Kronecker action, is the *maximal equicontinuous factor* of the $\mathbb{R}^n$-action on Ω_Φ: If there is a semi-conjugacy $g' : \Omega_\Phi \to Y$ with an equicontinuous $\mathbb{R}^n$-action on Y, then there is a semi-conjugacy $g'' : \hat{\mathbb{T}}^{md} \to Y$ with $g' = g'' \circ g$.

Tilings $T, T' \in \Omega$ are *regionally proximal*, $T \sim_{rp} T'$, provided there are $T_k, T'_k \in \Omega$ and $v_k \in \mathbb{R}^n$ so that $d(T, T_k) \to 0$, $d(T', T'_k) \to 0$, and $d(T_k - v_k, T'_k - v_k) \to 0$ as $k \to \infty$. It is proved in [BKe] that if Ω is a Pisot family tiling space, then T and T' in Ω are regionally proximal if and only if T and T' are *strongly regionally proximal*, $T \sim_{srp} T'$: for each $k \in \mathbb{N}$ there are $T_k, T'_k \in \Omega$ and $v_k \in \mathbb{R}^n$ with $B_k[T_k] = B_k[T]$, $B_k[T'_k] = B_k[T']$, and $B_k[T_k - v_k] = B_k[T'_k - v_k]$. The notion of regional proximality extends in an obvious way to arbitrary group actions and Auslander proves in [A] that, in the case of minimal abelian actions on compact metric spaces, regional proximality is the equicontinuous structure relation.

Theorem 2.1. (Auslander) If G is an abelian group acting minimally on the compact metric space X, and $g : X \to X_{max}$ is the maximal equicontinuous factor, then $g(x) = g(x')$ if and only if $x \sim_{rp} x'$.

We thus have:

Corollary 2.2. If Ω is a Pisot family tiling space with maximal equicontinuous factor $g : \Omega \to \hat{\mathbb{T}}^{md}$, then $g(T) = g(T')$ if and only if $T \sim_{srp} T'$.

3. Sufficient conditions for pure discrete spectrum

Given patches Q, Q' (not necessarily allowed for Φ) and $x \in int(spt(Q) \cap spt(Q'))$, we'll say that Q and Q' are *coincident* at x provided $B_0[Q - x] = B_0[Q' - x]$ and *eventually coincident* at x if there is $k \in \mathbb{N}$ so that $B_0[\Phi^k(Q - x)] = B_0[\Phi^k(Q' - x)]$. We'll say that Q and Q' are *densely eventually coincident on overlap* if Q and Q' are eventually coincident at a set of x that is dense in $int(spt(Q) \cap spt(Q'))$, and, if this happens for tilings Q, Q', we'll say that Q and Q' are *densely eventually coincident*. If Q is a finite patch and there is a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ so that $\cup_{k_1,\ldots,k_n \in \mathbb{Z}}(Q + \sum k_i v_i)$ is a tiling of $\mathbb{R}^n$, we'll denote this tiling by $\bar{Q}$. A vector $v = \sum_{i=1}^{n} a_i v_i$ is *completely rationally independent* of the basis $\{v_1, \ldots, v_n\}$ if the numbers $a_1, \ldots, a_n$ are independent over $\mathbb{Q}$. This is the same thing as saying that the map $x + L \mapsto x + v + L$ is a minimal homeomorphism of the torus $\mathbb{R}^n/L$, L being the lattice spanned by $\{v_1, \ldots, v_n\}$ over $\mathbb{Z}$.

Theorem 3.1. Suppose that Φ is a substitution of Pisot family type and that Q is a finite patch such that $\bar{Q} = \cup_{k_1,\ldots,k_n \in \mathbb{Z}}(Q + \sum k_i v_i)$ is a tiling of $\mathbb{R}^n$. Suppose also that $v \in \mathbb{R}^n$ is completely rationally independent of $\{v_1, \ldots, v_n\}$ and that $\bar{Q}$ and $\bar{Q} - v$ are densely eventually coincident. Then the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum.

Proof. Let L be the lattice $L := \{\sum k_i v_i : k_i \in \mathbb{Z}\}$ and let $D \subset spt(Q)$ be a fundamental domain for $\mathbb{R}^n/L$. For $x, y \in \mathbb{R}^n$, by $x \oplus y = z$ we will mean that $z \in D$ and $x + y - z \in L$. Note that eventual coincidence at a point is an open condition, is translation equivariant, and only depends on local structure: if Q_1 is eventually coincident with Q_1' at x, $B_0[Q_2 - y] = B_0[Q_1]$, and $B_0[Q_2' - y] = B_0[Q_1']$, then Q_2 is coincident with Q_2' at $x + y$. We first prove that if $x, w \in \mathbb{R}^n$ are such that $\bar{Q}$ and $\bar{Q} - w$ are eventually coincident at x, then $\bar{Q}$ and $\bar{Q} - w$ are densely eventually coincident. To see that this is the case, we may suppose that $x \in D$. Now there is a neighborhood U of x in D so that $\bar{Q}$ and $\bar{Q} - w$ are eventually coincident at y for all $y \in U$. With v as hypothesized, there is a dense open subset U_1 of U so that $\bar{Q}$ is eventually coincident with $\bar{Q} - v$ at each $y \in U_1$. Let us introduce the notation $y \sim_c y'$ to mean $\bar{Q}-y$ is coincident with $\bar{Q}-y'$ at 0. Note that $\sim_c$ is an equivalence relation and if $y \sim_c y+z$ then $y \sim_c y \oplus z$. We have $y \sim_c y \oplus v$ and $y \sim_c y \oplus w$ for all $y \in U_1$. Now, there is a dense open subset U_2 of U_1 so that $y \oplus w \sim_c y \oplus w \oplus v$ for all $y \in U_2$ (there is a dense open subset U_2' of $U_1 \oplus w$ with $y' \sim_c y'+v$ for all $y' \in U_2'$; let $U_2 \subset U_1$ satisfy $U_2 \oplus w = U_2'$). For $y \in U_2$, $y \sim_c y \oplus w$, and $(y \oplus v) \sim_c (y \oplus v) \oplus w$. Inductively, we construct dense open subsets $U_k \subset U_{k-1} \subset \cdots \subset U_1$ of U so that if $y \in U_k$, then $(y \oplus jv) \sim_c (y \oplus jv) \oplus w$ for $j = 1, \dots, k$. Now pick $y \in \cap_k U_k$ (such exists by the Baire Category Theorem). The assumption of the independence of v from the v_i guarantees that $\{y \oplus jv : j \in \mathbb{N}\}$ is dense in D. Thus $\bar{Q}$ and $\bar{Q}-w$ are densely eventually coincident.

If $\bar{Q}$ and $\bar{Q}-v$ are eventually coincident at x, then $\Phi^k(\bar{Q}) = \bar{Q}_k$ and $\Phi^k(\bar{Q}-v) = \bar{Q}_k - \Lambda^k v$, where $Q_k := \Phi^k(Q)$, are eventually coincident at $\Lambda^k x$, for all $k \in \mathbb{N}$. Also, $\Lambda^k v$ is completely rationally independent of $\{\Lambda^k v_1, \dots, \Lambda^k v_n\}$. Thus, given R, we may assume (by replacing Q with Q_k for sufficiently large k) that Q contains every allowed patch of diameter less than R. From this, we see that if S and S' are any two allowed finite patches that share a tile, then S and S' are densely eventually coincident on overlap. Indeed, we may assume that Q contains translated copies of both S and S'. Let $S-u \subset Q$ and let w be such that $Q-w \supset S'-u$. There is then $x \in spt(Q)$ so that Q and $Q-w$ are coincident at x (since Q and $Q-w$ agree on the overlap of $S-u$ and $S'-u$). From the above, $\bar{Q}$ and $\bar{Q}-w$ are densely eventually coincident; in particular, $S-u$ and $S'-u$, hence S and S', are densely eventually coincident on overlap.

Now, if the $\mathbb{R}^n$-action on Ω does not have pure discrete spectrum, then for each $T \in \Omega$ there are finitely many, and at least two, $T' \in \Omega$ so that T and T' are regionally proximal and T and T' don't share a tile (see [BKe]). Pick $T \in \Omega$ that is Φ-periodic. There is then $T' \in \Omega$ that is also Φ-periodic, that is strongly regionally proximal with T (Corollary 2.2), but disjoint from T. There are then $T_0, T_0' \in \Omega$ and $v_0 \in \mathbb{R}^n$ so that $B_0[T_0] = B_0[T]$, $B_0[T_0'] = B_0[T']$ and $B_0[T_0 - v_0] = B_0[T_0' - v_0]$. Let S and S' be the finite patches in T_0 and T_0': $S := B_0[T_0] \cup B_0[T_1 - v_0] + v_0$ and $S' := B_0[T_0'] \cup B_0[T_0' - v_0] + v_0$. Then S and S' are densely eventually coincident on overlap. In particular, there is x at which T and T' are eventually coincident. But T and T' are periodic, so eventual coincidence means that T and T' share a tile, contrary to assumption. Thus, the $\mathbb{R}^n$-action on Ω must have pure discrete spectrum. □

Suppose now that $\phi : \mathcal{A} \to \mathcal{A}^*$, $\mathcal{A} = \{1, \dots, k\}$, is a substitution on k letters. The *incidence matrix* of ϕ is the $k \times k$ matrix $M = M_\phi$ whose

i, j-th entry is the number of i's that appear in $\phi(j)$. The substitution is *primitive* if some power of M has all positive entries; in this case there is a positive left eigenvector $\omega = (\omega_1, \ldots, \omega_k)$ corresponding to the Perron-Frobenius eigenvalue λ of ϕ. We'll say that ϕ is a *Pisot substitution* if ϕ is primitive and λ is a Pisot number. Such a substitution on letters determines a geometrical substitution Φ on prototiles $\rho_i := [0, \omega_i]$, $i = 1, \ldots, k$, that is of $(1, d)$ - Pisot family type, d being the algebraic degree of λ.

Given a word $u \in \mathcal{A}^*$, the *abelianization* of u is the vector $[u] \in \mathbb{Z}^k$ with i-th entry equal to the number of i's that appear in u. A pair of words $(u, v) \in \mathcal{A}^* \times \mathcal{A}^*$ is a *balanced pair* if $[u] = [v]$. If (u_1, v_1) and (u_2, v_2) are balanced pairs, we will write $(u_1, v_1)(u_2, v_2) = (u_1u_2, v_1v_2)$ and call (u_1, v_1) and (u_2, v_2) *factors* of (u_1u_2, v_1v_2). A balanced pair (u, v) is *irreducible* if it has no (non-trivial) factors. Clearly, every balanced pair (u, v) can be written (uniquely) as a product of irreducible balanced pairs: each of these irreducible balanced pairs is called an *irreducible factor* of (u, v). A balanced pair of the form (a, a), $a \in \mathcal{A}$, is called a *coincidence*. Given a balanced pair (u, v), we will say that *the balanced pair algorithm for* (u, v) *terminates with coincidence* provided: (1) the collection $\{(x, y) : (x, y)$ is an irreducible factor of $(\phi^m(u), \phi^m(v))$ for some $m \in \mathbb{N}\}$ is finite; and (2) if (x, y) is an irreducible factor of $(\phi^m(u), \phi^m(v))$ for some $m \in \mathbb{N}$, then there is $k \in \mathbb{N}$ so that $(\phi^k(x), \phi^k(y))$ has a coincidence as a factor.

Theorem 3.2. Suppose that $\phi : \mathcal{A} \to \mathcal{A}^*$ is a Pisot substitution with left Perron-Frobenius eigenvector ω and suppose that $u, v \in \mathcal{A}^*$ are words with the properties: $\langle [u], \omega\rangle$ and $\langle [v], \omega\rangle$ are independent over $\mathbb{Q}$; and the balanced pair algorithm for (uv, vu) terminates with coincidence. Then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.

Proof. Let Q be a patch with underlying word uv. The length of the support of Q is $l := \langle [u] + [v], \omega\rangle$ and $\bar{Q} = \cup_{n\in\mathbb{Z}}(Q + nl)$ is a periodic tiling of $\mathbb{R}$. Let $l_u := \langle [u], \omega\rangle$. The hypotheses imply that l_u is completely rationally independent of $\{l\}$ and that $\bar{Q}$ and $\bar{Q} - l_u$ are densely eventually coincident. The result follows from Theorem 3.1. □

The following corollary appears in [BK]. The proof given there has a gap that is not easily fixed using the techniques of that paper.

Corollary 3.3. Suppose that ϕ is a Pisot substitution whose incidence matrix has characteristic polynomial irreducible over Q. If there are letters $a \neq b$ in the alphabet for ϕ so that the balanced pair algorithm for (ab, ba) terminates with coincidence, then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.

Proof. Irreducibility of the incidence matrix implies that $\langle [a], \omega \rangle$ and $\langle [b], \omega \rangle$ are independent over $\mathbb{Q}$. □

Remark 3.4. It may appear that the applicability of the methods presented above depends on the geometry of the tiles: as stated, Theorem 3.1 requires the existence of a patch that tiles $\mathbb{R}^n$ periodically. But in fact if P is any patch and $\{v_1, \ldots, v_n\}$ is any basis with $\sigma := \{\sum_{i=1}^{n} t_i v_i : t_i \in [0,1]\} \subset spt(P)$ then one can consider the generalized patch Q consisting of the generalized tiles $(spt(\tau) \cap \sigma, \tau)$, for $\tau \in P$ with $int(spt(\tau)) \cap \sigma \neq \emptyset$. It makes perfectly good sense to speak of $\bar{Q}$ and $\bar{Q} - v$ being densely eventually coincident and the proof of Theorem 3.1 goes through without change.

4. Necessary conditions for pure discrete spectrum

Throughout this section Φ will be an n-dimensional substitution of (m, d)-Pisot family type with tiling space $\Omega = \Omega_\Phi$, inflation Λ, and maximal equicontinuous factor $g : \Omega \to \hat{\mathbb{T}}^{md}$. A vector $v \in \mathbb{R}^n$ is a *return vector* for Φ if for some (equivalently, every) $T \in \Omega$, $T - v \cap T \neq \emptyset$. Let $\mathcal{R} = \mathcal{R}_\Phi := \{v : v \text{ is a return vector for } \Phi\}$ denote the set of return vectors for Φ.

We address the question: If the $\mathbb{R}^n$-action on Ω has pure discrete spectrum, for what $v \in \mathbb{R}^n$ and what tilings S must it be the case that S and $S - v$ are densely eventually coincident? It is clear (if $S \in \Omega$) that the collection of such v is countable: if S and $S - v$ are densely eventually coincident, then $\Phi^k(S)$ and $\Phi^k(S - v) = \Phi^k(S) - \Lambda^k v$ share a tile for some $k \in \mathbb{N}$ so that $\Lambda^k v$ is a return vector, of which there are only countably many by FLC. We will see below that for the class of all tilings in Ω, the answer to the above question is precisely $\cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}$. Under additional conditions this collection is a group and we will give it a homological interpretation.

For ease of exposition, we assume that the prototiles ρ_i of Φ are polytopes and that the tiles in tilings in Ω meet full face to full face in all dimensions (so each ρ_i has finitely many faces in each dimension, and if $\tau, \sigma \in T \in \Omega$ meet in a point x that is in the relative interior of a face of τ or σ, then they meet in that entire face). We may assume that the prototiles 'force the border' - otherwise, replace the prototiles by the collared prototiles (see [AP]). Let X be the Anderson-Putnam complex for Φ: X is a cell complex with one n-cell for each (collared) prototile and these n-cells are glued along faces by translation. If k-face f of ρ_i is glued to k-face g of ρ_j, then there is $v \in \mathbb{R}^n$ so that $f = g - v$, and the gluing is defined by $f \ni x \sim x + v \in g$. The rule for deciding what faces are glued is as follows. The k-face f of ρ_i is *adjacent* to the k-face g of ρ_j if there are $T \in \Omega$ and $u, v \in \mathbb{R}^n$ so that $\rho_i + u, \rho_j + v \in T$ and $\rho_i + u \cap \rho_j + v = f + u = g + v$. Let *gluable* be the smallest equivalence relation containing the adjacency relation. Two k-faces of prototiles are glued in X if they are gluable. The substitution Φ on the prototiles induces a continuous map $f = f_\Phi : X \to X$ with the property that the inverse limit $\varprojlim f$ is homeomorphic with Ω by a homeomorphism that conjugates the shift $\hat{f}$ on $\varprojlim f$ with Φ on Ω ([AP]).

We say that a patch Q is *admissible* if whenever tiles $\tau, \sigma \in Q$ meet along k-faces, those k-faces are gluable. Certainly every patch that is allowed for Φ is also admissible and if Q is an admissible patch, then so is $\Phi(Q)$. For each admissible patch Q there is then a continuous map $p^Q : spt(Q) \to X$ defined by $p^Q(x) = [x - v]$ provided $x \in spt(\rho_i) + v$ and $\rho_i + v \in Q$. Furthermore, if $f : X \to X$ is the continuous map induced by Φ, then $f \circ p^Q = p^{\Phi(Q)} \circ \Lambda$ on $spt(Q)$ for all admissible patches Q.

Now given a path $\gamma : [a, b] \to X$ and $x \in \mathbb{R}^n$, there is a unique (continuous) path $\bar{\gamma} : [a, b] \to \mathbb{R}^n$ with $\bar{\gamma}(a) = x$ and $p^Q(\bar{\gamma}(t)) = \gamma(t)$ whenever $\bar{\gamma}(t) \in spt(Q)$ for any admissible patch Q. We will call such a $\bar{\gamma}$ a *lift* of γ to $\mathbb{R}^n$. If γ is a loop in X then γ is also a singular 1-cycle and we denote its homology class by $[\gamma] \in H := H_1(X; \mathbb{Z})$. If $T \in \Omega$ and $\tau, \tau + v \in T$ (thus v is a return vector), there is a corresponding element $[\gamma] \in H$, where $\gamma(t) := p^T(x + tv)$ for fixed $x \in spt(\tau)$ and $t \in [0, 1]$. Note that $[\gamma]$ doesn't depend on the choice of $x \in spt(\tau)$, but it does depend on τ (not just v). We will show that the reverse of this process is better behaved.

Given a path $\gamma : [a, b] \to X$, let $\bar{\gamma}$ be a lift of γ to $\mathbb{R}^n$ and let $l(\gamma) := \bar{\gamma}(b) - \bar{\gamma}(a)$. This is well-defined, i.e., independent of lift, since all lifts of γ are translates of one another (as is easily seen by uniqueness). It is also clear that if $\gamma = \gamma_1 * \gamma_2$ is a concatenation of two paths, then $l(\gamma) = l(\gamma_1) + l(\gamma_2)$ and if γ^{-1} is the reverse of γ, then $l(\gamma^{-1}) = -l(\gamma)$. By lifting homotopies, one sees that l induces a homomorphism from the fundamental group of X to the additive group $\mathbb{R}^n$. We will see that l also passes to homology.

Lemma 4.1. If γ is a loop in X and $0 = [\gamma] \in H$ then $l(\gamma) = 0$.

Proof. By the above remark on concatenations, we may assume that $\gamma(0)$ is a vertex in X. Suppose first that $n = 1$. We may then homotope γ to a loop that doesn't turn in the interior of any edge and this can be done without affecting l. We may assume that γ itself has this property. Now $\gamma = \gamma_1 * \cdots * \gamma_k$ is a concatenation of paths, each running along a single edge exactly once. If γ_i and γ_j run along the same edge, but in different directions, then $l(\gamma_i) = -l(\gamma_j)$. Since $[\gamma] = 0$, each γ_i can be paired with a γ_j so that elements of a pair traverse the same edge, but in opposite directions. Thus $l(\gamma) = \sum l(\gamma_i) = 0$.

Suppose now that $n > 1$. After sufficient simplicial subdivision of X, we may homotope γ to a loop that lies in the 1-skeleton of this subdivision, only self-intersects at vertices of this subdivision, and that has a continuous lift that also runs from $\bar{\gamma}(0)$ to $\bar{\gamma}(1)$. Let us replace γ with such a curve. Let $\Gamma = \sum e_i$ be the simplicial 1-chain representing γ: the e_i are the oriented 1-simplices in the image of γ and we have modified γ so that $e_i \neq \pm e_j$ for $i \neq j$. Since γ is homologous to zero in X, there is a simplicial 2-chain $\Sigma = \sum n_j \sigma_j$ in the subdivided simplicial structure of X with $\partial\Sigma = \Gamma$. After further subdivision, we may assume that each σ_j has at most one edge on Γ.

Now there is a lift of Γ to a 1-chain $\bar{\Gamma}$ in $\mathbb{R}^n$ with $\partial\bar{\Gamma} = \bar{\gamma}(1) - \bar{\gamma}(0)$. We may lift Σ to a 2-chain $\bar{\Sigma} = \sum n_j \bar{\sigma}_j$ in $\mathbb{R}^n$ in which each $\bar{\sigma}_j$ is a lift of σ_j and $\bar{\sigma}_j$ has an edge on $\bar{\Gamma}$ if and only if σ_j has an edge on Γ (that is, σ_j meets Γ along e_i if and only if $\bar{\sigma}_j$ meets $\bar{\Gamma}$ along $\bar{e}_i$).

Now $\partial(\bar{\Sigma}) = \bar{\Gamma} + \bar{\Delta}$ with none of the elementary 1-simplices of $\bar{\Delta}$ in $\bar{\Gamma}$. Since $\partial^2(\bar{\Sigma}) = 0$,

$$\partial(\bar{\Delta}) = \bar{\gamma}(0) - \bar{\gamma}(1).$$

Each edge of $\bar{\Delta}$ lies over some edge (of some simplex in Σ) that does not occur in $\partial\Sigma = \Gamma$: write $\bar{\Delta} = \sum_i \sum_{k=1}^{j_i} (-1)^{m(i,k)} e'_{i,k}$ where, for each i, the $e'_{i,k}$ all lie over the same edge e'_i of one of the σ_j in Σ, and $e'_i \neq e'_s$ for $i \neq s$. Pushed forward into X, the equation $\partial(\bar{\Sigma}) = \bar{\Gamma} + \bar{\Delta}$ reads $\partial\Sigma = \Gamma + \sum_i (\sum_{k=1}^{j_i} (-1)^{m(i,k)}) e'_i$. Since $\partial\Sigma = \Gamma$, we see that

$$\sum_{k=1}^{j_i} (-1)^{m(i,k)} = 0$$

for all i. Now, from the displayed equation above, we have the 'formal' statement

$$\bar{\gamma}(0) - \bar{\gamma}(1) = \sum_i \sum_{k=1}^{j_i} (-1)^{m(i,k)} \partial e'_{i,k}.$$

But this is also valid as a statement of equality between elements in the additive group $\mathbb{R}^n$ and, as group elements , $\partial e'_{i,k} = \partial e'_i$ for each i, k. Thus the right hand side of the last displayed equation is, as a group element, $\sum_i (\sum_{k=1}^{j_i} (-1)^{m(i,k)}) \partial e'_i = 0$. Hence $l = 0$. $\square$

Thus, since each element of H is the homology class of a loop, $l : H \to \mathbb{R}^n$ given by $l([\gamma]) = l(\gamma)$ is a well-defined homomorphism. Note that if γ is a loop defined by return vector v, then $l([\gamma]) = v$.

Lemma 4.2. $l \circ f_* = \Lambda \circ l$.

Proof. It follows from $f \circ p^Q = p^{\Phi(Q)} \circ \Lambda$ that if $\bar{\gamma}$ is a lift of γ to $\mathbb{R}^n$, then $\Lambda \circ \bar{\gamma}$ is a lift of $f \circ \gamma$ to $\mathbb{R}^n$. Thus $l([f \circ \gamma]) = \Lambda \circ \bar{\gamma}(1) - \Lambda \circ \bar{\gamma}(0) = \Lambda(\bar{\gamma}(1) - \bar{\gamma}(0)) = \Lambda(l([\gamma])$. □

Let us call a vector $v \in \mathbb{R}^n$ a *generalized return vector* for Φ if there are $v_i \in \mathbb{R}^n$ and $T_i \in \Omega$, $i = 1, \ldots, k$, with $v = v_1 + \cdots + v_k$, so that $B_0[T_i + v_i] = B_0[T_{i+1}]$ for $i = 1, \ldots, k-1$, and $B_0[T_k + v_k] = B_0[T_1]$. Let $GR = GR(\Phi)$ be collection of generalized return vectors. It is a consequence of the next lemma that $GR(\Phi)$ is a subgroup of $\mathbb{R}^n$.

Lemma 4.3. Let X be the collared Anderson-putnam complex for Φ and let $H = H_1(X; \mathbb{Z})$. Then $l(H) = GR(\Phi)$.

Proof. If γ is a loop in X. Then γ is homotopic to a loop that factors as a product of paths γ_i with each γ_i a two-piece piecewise linear path from a prototile $\rho_{\alpha(i)}$ to an adjacent prototile $\rho_{\alpha(i+1)}$. Each γ_i is itself homotopic rel endpoints to a product of two-piece piecewise linear paths each making an allowed transition between prototiles (this from the definition of the Anderson-Putnam complex: the glueing of prototiles along faces is by the smallest equivalence relation that includes the relation defined by the allowed transitions). Thus the displacement of the lift of γ is a generalized return vector.

Conversely, each generalized return vector v determines a (at least one) loop γ with $l([\gamma]) = v$. □

The *stable manifold* of $T' \in \Omega$ is the set $W^s(T') := \{T \in \Omega : d(\Phi^k(T), \Phi^k(T')) \to 0 \text{ as } k \to \infty\}$.

Lemma 4.4. Suppose that $\Omega = \Omega_\Phi$ is an n-dimensional Pisot family tiling space whose $\mathbb{R}^n$-action has pure discrete spectrum and suppose that $T \in W^s(T')$. Then T and T' are densely eventually coincident.

Proof. Let $g : \Omega \to \hat{\mathbb{T}}^{md}$ be the map onto the maximal equicontinuous factor, let $\hat{F} : \hat{\mathbb{T}}^{md} \to \hat{\mathbb{T}}^{md}$ be the hyperbolic automorphism with $g \circ \Phi = \hat{F} \circ g$, and let $z \mapsto z - v$ denote the Kronecker action on $\hat{\mathbb{T}}^{md}$, so that $g(T - v) = g(T) - v$ for all $T \in \Omega$ and $v \in \mathbb{R}^n$. Suppose that $T \in W^s(T')$ but that $\emptyset \neq U \subset \mathbb{R}^n$ is such that $T - x \notin W^s(T' - x)$ for all $x \in U$. For fixed $x_0 \in U$, let $T_k := \Phi^k(T - x_0)$, $T'_k := \Phi^k(T' - x_0)$, $z_k := g(T_k)$, and $z'_k := g(T'_k)$. Then $d(z_k, z'_k) \to 0$ as $k \to \infty$. Choose k_i with $T_{k_i} \to S$, $T'_{k_i} \to S'$; then also $z_{k_i}, z'_{k_i} \to z := g(S) = g(S')$. Since the $\mathbb{R}^n$-action on Ω has pure discrete spectrum, S and S' are proximal. Thus, given $r > 0$, there is $y = y(r) \in \mathbb{R}^n$ so that $B_r[S - y] = B_r[S' - y]$ (see [BKe]). From the local product structure on $\hat{\mathbb{T}}^{md}$, there is $\epsilon > 0$ so that if

$0 < |w| < \epsilon$ and $w \in \mathbb{R}^n$, then $\bar{z}-w \notin W^s_\epsilon(\bar{z}) := \{\bar{z}' : d(\hat{F}^k(\bar{z})', \hat{F}^k(\bar{z}) \leq \epsilon$ for all $k \geq 0\}$, for all $\bar{z} \in \hat{\mathbb{T}}^{md}$. Let r be large enough so that if $\bar{T}, \bar{T}' \in \Omega$ and $B_r[\bar{T}] = B_r[\bar{T}']$, then $g(\bar{T}) \in W^s_\epsilon(g(\bar{T}'))$. Let $y = y(r)$, as above, and let i be large enough so that $B_r[T_{k_i} - y - w] = B_r[T'_{k_i} - y]$ with $|w| < \epsilon$. We may also take i large enough so that $x_0 + \Lambda^{-k_i} y \in U$. In $\hat{\mathbb{T}}^{md}$ it is the case that if $\bar{z} \in W^s_\epsilon(\bar{z}')$, then $\bar{z} - v \in W^s_\epsilon(\bar{z}' - v)$ for all $v \in \mathbb{R}^n$. Thus , we may further increase i so that $z_{k_i} - y \in W^s_\epsilon(z'_{k_i} - y)$. But $z_{k_i} - y - w \in W^s_\epsilon(z'_{k_i} - y)$ and $|w| < \epsilon$. Thus $w = 0$ so that $T_{k_i} - y \in W^s(T'_{k_i} - y)$, and then $T - x \in W^s(T' - x)$ for $x := x_0 + \Lambda^{-k_i} y \in U$, contradicting the choice of U. Thus $T - x \in W^s(T' - x)$ for a dense set of $x \in \mathbb{R}^n$ and it only remains to observe that T is eventually coincident with T' at x if and only if $T \in W^s(T')$ (see, for example [BO]). □

Proposition 4.5. Suppose that the $\mathbb{R}^n$-action on Ω has pure discrete spectrum and $T \in \Omega$. Then

$$\{v \in \mathbb{R}^n : T - v \text{ is densely eventually coincident with } T\} = \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}.$$

Proof. We have already observed that

$$\{v \in \mathbb{R}^n : T - v \text{ is densely eventually coincident with } T\} \subset \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}.$$

Suppose that $v \in \mathcal{R}$ and $k \in \mathbb{Z}$. There is then $\tau \in \Phi^{-k}(T) - v \cap \Phi^{-k}(T)$. Pick x in the interior of the support of τ. Then $\Phi^{-k}(T) - v - x \in W^s(\Phi^{-k}(T) - x)$. It follows that $T - \Lambda^k v - \Lambda^k x \in W^s(T - \Lambda^k x)$ so that, by Lemma 4.4, $T - \Lambda^k v - \Lambda^k x$ is densely eventually coincident with $T - \Lambda^k x$. Thus $T - \Lambda^k v$ is densely eventually coincident with T. □

Proposition 4.6. Suppose that Ω_Φ is an n-dimensional Pisot family tiling space whose $\mathbb{R}^n$-action has pure discrete spectrum. Then for any admissible tiling S of $\mathbb{R}^n$ for Φ, and any $v \in GR(\Phi)$, S and $S - v$ are densely eventually coincident.

Proof. Let $\sim_{dc}$ be the densely eventually coincident relation ($T \sim_{dc} T' \Leftrightarrow T - x$ is eventually coincident with $T' - x$ for a dense set of $x \in \mathbb{R}^n$). Since eventual coincidence at a point is an open condition, $\sim_{dc}$ is an equivalence relation. Suppose that $T_i \in \Omega$ and $v_i \in \mathbb{R}^n$, $i = 1, \ldots, k$, are such that $B_0[T_i + v_i] = B_0[T_{i+1}]$, for $i = 1, \ldots, k-1$, and $B_0[T_k + v_k] = T_1$. From the above lemma, $T_{i+1} \sim_{dc} T_i + v_i$, for $i = 1, \ldots, k$, with $T_{k+1} := T_k + v_k$, and $T_1 \sim_{dc} T_{k+1}$. Clearly, $T \sim_{dc} T' \Rightarrow T - w \sim_{dc} T' - w$, so we have $T_1 + v \sim_{dc} T_{k+1}$, with $v := v_1 + \cdots + v_k$. Thus $T_1 + v \sim_{dc} T_1$. Now let T be any element of Ω and let T_1 and v be as above. Let $w \in \mathbb{R}^n$ be so that $B_0[T - w] = B_0[T_1]$. Then $T - w \sim_{dc} T_1 \sim_{dc} T_1 + v$. But also, $T - w - v \sim_{dc} T_1 - v$, so $T - v \sim_{dc} T$. That is, $T - v \sim_{dc} T$ for all $T \in \Omega$ and all $v \in GR(\Phi)$.

Now suppose that S is admissible for Φ and let $v \in GR(\Phi)$. Let $\sigma, \tau \in S$ be such that $U = U(\sigma,\tau) := int(spt(\sigma) \cap (int(spt(\tau)) - v) \neq \emptyset$. Since S is admissible, there are $T_1, \ldots, T_k \in \Omega_\Phi$ with $\sigma \in T_1$, $\tau \in T_2$, and $T_{i+1} \cap T_i \neq \emptyset$ for $i = 1, \ldots, k-1$. Restricting $\sim_{dc}$ to U (that is, by $T \sim_U T'$ we mean $T - x$ is eventually coincident with $T' - x$ for a dense set of $x \in U$) we have: $S \sim_U T_1 \sim_U T_1 + v \sim_U T_2 + v \sim_U T_3 + v \sim_U \cdots \sim T_k + v \sim_U S + v$. As $\cup_{\sigma,\tau} U(\sigma,\tau)$ is dense in $\mathbb{R}^n$, we have $S \sim_{dc} S + v$. $\square$

Example 4.7. (**Table substitution**) The Table substitution Ψ has inflation $\Lambda = 2I$ and prototiles $\rho_1 = [0,2] \times [0,1]$, $\rho_2 = [0,1] \times [0,2]$ that substitute as pictured. Let $Q = \{\rho_1\}$ and $v = (1,0)$. Then $\bar{Q}$ is

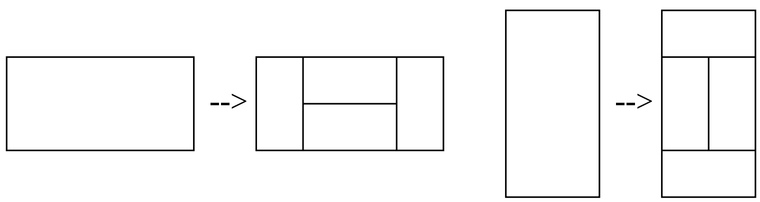

FIGURE 1. Subdivision for the table

admissible and v is a return vector, but $\bar{Q}$ and $\bar{Q} - v$ are easily seen to be nowhere eventually coincident. Thus the $\mathbb{R}^2$-action on Ω_Ψ does not have pure discrete spectrum.

Proposition 4.8. Suppose that Ω_Φ is an n-dimensional substitution tiling of (m,d)-Pisot family type whose $\mathbb{R}^n$ action has pure discrete spectrum. If the product of the nonzero eigenvalues of $f_* : H_1(X;\mathbb{Q}) \to H_1(X;\mathbb{Q})$ is ± 1, then $GR(\Phi) = \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}$.

Proof. Suppose that $u \in GR(\Phi)$. If $T \in \Omega$ then T and $T - u$ are densely eventually coincident by Proposition 4.6. By Proposition 4.5, $u \in \cup_{k \in \mathbb{Z}} \Lambda^k v$.

Conversely, if the product of the nonzero eigenvalues of $f_* : H_1(X;\mathbb{Q}) \to H_1(X;\mathbb{Q})$ is ± 1, there is $j \in \mathbb{N}$ and a subgroup H_{ER} of the free part H_{free} of $H_1(X;\mathbb{Z})$ so that $f_*^j(H_{free}) = H_{ER}$ and $f_*|_{H_{ER}} : H_{ER} \to H_{ER}$ is invertible. Let $v \in \mathcal{R}$ and let $[\gamma] \in H_1(X,\mathbb{Z})$ be so that $l([\gamma]) = v$. There are then $[\alpha] \in H_{ER}$ and $[\beta] \in H_1(X;\mathbb{Z})$ so that $[\gamma] = [\alpha] + [\beta]$ and $f_*^j([\beta])$ is a torsion element. It follows from Lemma 4.2 that $l([\beta]) = 0$ and hence $l([\alpha]) = v$. We have $\Lambda^k v = \Lambda^k l([\alpha]) = l((f_*|_{H_{ER}})^k([\alpha])) \in GR(\Phi)$, by Lemmas 4.2 and 4.3. $\square$

Even when the $\mathbb{R}^n$ action on an n-dimensional Pisot family tiling space has pure discrete spectrum, it is not necessarily the case that $GR(\Phi) = \cup_{k\in\mathbb{Z}}\Lambda^k\mathcal{R}$. For example, the Period Doubling substitution with tiles of unit length and inflation factor 2 (presented symbolically by $a \mapsto ab$, $b \mapsto aa$) has $GR = \mathbb{Z}$ and $\cup_{k\in\mathbb{Z}}\Lambda^k\mathcal{R} = \mathbb{Z}[1/2]$. To obtain equality, it is necessary that the eigenvalues of Λ be algebraic units.

Substitutions satisfying the first set of hypotheses in the following corollary are called *unimodular, homological Pisot* and it is conjectured (a version of the 'Pisot Conjecture') that their associated $\mathbb{R}^n$-actions always have pure discrete spectrum (see [BBJS]).

Corollary 4.9. Suppose that Φ is an n-dimensional substitution of (m, d)-Pisot family type with $dim(\check{H}^1(\Omega_\Phi;\mathbb{Q})) = md$ and also that the eigenvalues of the expansion Λ are algebraic units. If the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum then $GR(\Phi) = \cup_{k\in\mathbb{Z}}\Lambda^k\mathcal{R}$.

Proof. Since $\varprojlim f : X \to X$ is homeomorphic with Ω_Φ, $\check{H}^1(\Omega_\Phi;\mathbb{Q})$ is isomorphic with $\varinjlim f^* : H^1(X;\mathbb{Q}) \to H^1(X;\mathbb{Q})$. This direct limit is isomorphic with the eventual range of f^* and hence has dimension md. It is proved in [BKS] that $g^* : \check{H}^1(\mathbb{T}^{md};\mathbb{Q}) \to \check{H}^1(\Omega_\Phi;\mathbb{Q})$ is an injection and that the eigenvalues of $F^* : \check{H}^1(\mathbb{T}^{md};\mathbb{Q}) \to \check{H}^1(\mathbb{T}^{md};\mathbb{Q})$ are precisely the eigenvalues of Λ, together with all their algebraic conjugates, all of multiplicity m ($\hat{\mathbb{T}}^{md} = \mathbb{T}^{md}$ and $\hat{F} = F$ since the eigenvalues of Λ are units). Thus g^* is surjective as well and $\hat{f}^* : \check{H}^1(\varprojlim f;\mathbb{Q}) \to \check{H}^1(\varprojlim f;\mathbb{Q})$, being conjugate with Φ^* and hence with F^*, has those same eigenvalues. Since $\hat{f}^*$ is conjugate with f^* restricted to its eventual range, and the latter is (by the Universal Coefficient Theorem) just the transpose (dual) of $f_* : H^1(X;\mathbb{Q}) \to H^1(X;\mathbb{Q})$, restricted to its eventual range, we conclude that the eigenvalues of f_* are precisely those of Λ, together with their algebraic conjugates, all of multiplicity m. As these eigenvalues are assumed to be units, the product of the nonzero eigenvalues of f_* is ± 1 and Proposition 4.8 applies. □

5. Applications

Given a substitution Φ and tilings S and S' by prototiles for Φ, an *overlap* for S and S' is a pair of tiles $\{\tau, \tau'\}$ with $\tau \in S$, $\tau' \in S'$, and $\mathring{\tau} \cap \mathring{\tau}' \neq \emptyset$. The overlap $\{\tau, \tau'\}$ *leads to coincidence* if there is $x \in \mathring{\tau} \cap \mathring{\tau}'$ so that S and S' are eventually coincident at x. It is clear that if all pairs $\{\tau, \tau'\}$ that occur as overlaps for $\Phi^k(S)$ and $\Phi^k(S')$, $k \in \mathbb{N}$, lead to coincidence, then S and S' are densely eventually coincident. Pisot family substitution tilings have the Meyer property (see [LS2]) from which it follows that if Q is an admissible patch for Φ that tiles periodically and $v \in GR(\Phi) \cup \cup_{k\in\mathbb{Z}}\Lambda^k\mathcal{R}$, then the collection of all translation equivalence classes of overlaps for $\Phi^k(\bar{Q}), \Phi^k(\bar{Q}-v)$, $k \in \mathbb{N}$, is finite. In the examples of this section we implement Theorem 3.1 by finding appropriate Q and v so that each of the finitely many types of overlap that occur for $\Phi^k(\bar{Q}), \Phi^k(\bar{Q}-v)$, $k \in \mathbb{N}$, leads to coincidence.

Example 5.1. (**Octagonal substitution**)

The (undecorated) octagonal substitution is a 2-dimensional Pisot family substitution with inflation $\Lambda = (1+\sqrt{2})I$. There are 20 prototiles: four unmarked rhombi, $\rho_i = r^i(\rho_0)$, $i = 0, \dots 3$, with $r^4(\rho_0) = \rho_0$, r being rotation through $\pi/4$, and sixteen marked isosceles right triangles $\tau_i = r^i(\tau_0), \tau_i' = r^i(\tau_0')$, $i = 0, \dots, 7$, with τ_i and τ_i' having the same support but bearing different marks. Let ρ_0 have vertices $(0,0), (1,0), (\sqrt{2}/2, \sqrt{2}/2)$, and $(1+\sqrt{2}/2, \sqrt{2}/2)$, and let τ_0 and τ_0' have vertices $(0,0), (1,0)$, and $(1,1)$. The octagonal substitution, Φ_O, is described in the following figure. Let Q be the unit square patch

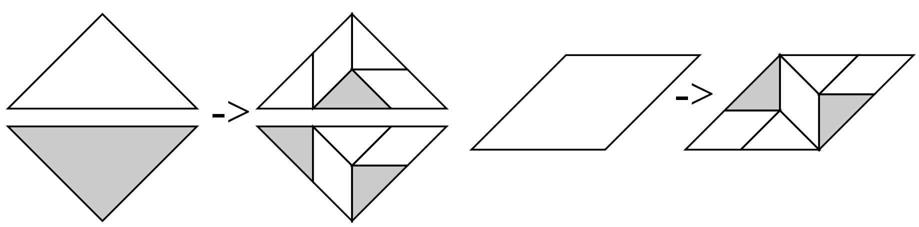

FIGURE 2. Octagon subdivision rule. The large shaded and unshaded triangles are τ_7' and τ_3, resp.

$Q = \{\tau_0, \tau_4' + (1,1)\}$ and let $v = (1-\sqrt{2}/2, \sqrt{2}/2)$. Then v is completely rationally independent of $\{(1,0),(0,1)\}$ and the following sequence of pictures proves that $\bar{Q}$ and $\bar{Q}-v$ are densely eventually coincident. Indeed, we see that each overlap that occurs for $\Phi_O^3(\bar{Q}), \Phi_O^3(\bar{Q}-v)$ already occurs (up to translation) for $\Phi_O^2(\bar{Q}), \Phi_O^2(\bar{Q}-v)$ and leads to coincidence in $\Phi_O^4(\bar{Q}), \Phi_O^4(\bar{Q}-v)$. Thus the $\mathbb{R}^2$-action on Ω_{Φ_O} has pure discrete spectrum. (This conclusion is well known: octagonal tilings can be obtained by a cut-and-project method with canonical window and such a provenance always guarantees pure discrete spectrum - see, for example, [BM].)

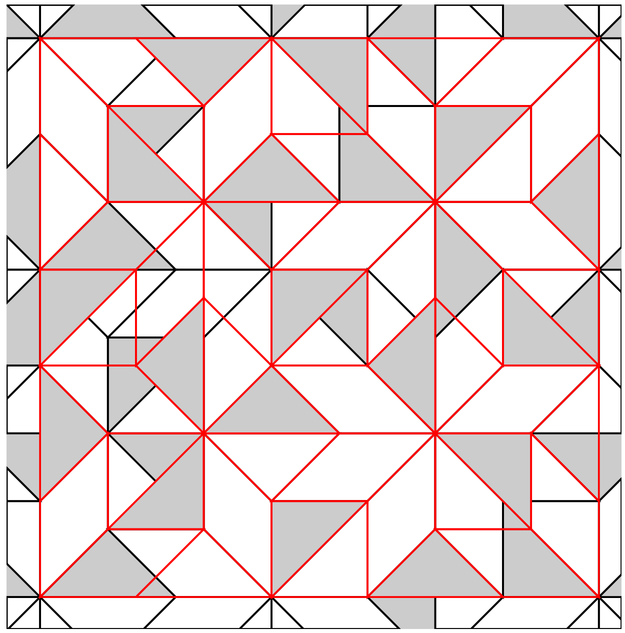

FIGURE 3. Red overlay of the patch $\Phi_O^2(Q)$ with background $\Phi_O^2(\bar{Q}-v)$.

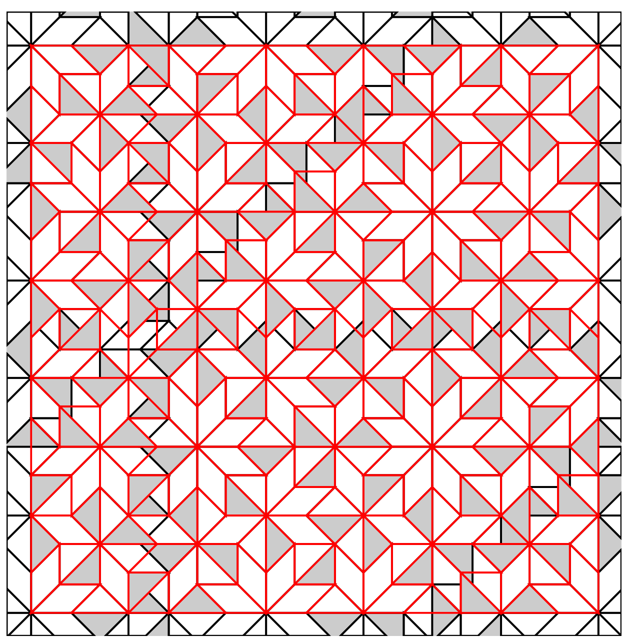

FIGURE 4. $\Phi_O^3(Q)$ and $\Phi_O^3(\bar{Q}-v)$.

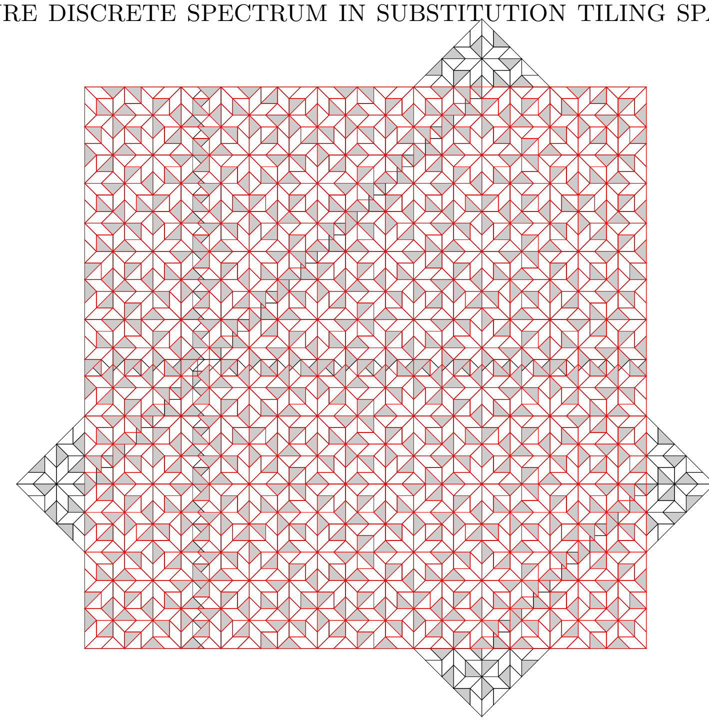

FIGURE 5. $\Phi_O^4(Q)$ and $\Phi_O^4(\bar{Q} - v)$.

Example 5.2. (**Arnoux-Rauzy substitutions**)

V. Berthé, T. Jolivet, and A. Siegel have obtained the result of this subsection by different methods ([BJS]).

We will say that words w and w' in the alphabet $\mathcal{A} = \{1, \ldots, d\}$ are *cyclically equivalent* if there are non-degenerate words u, v with $w = uv$ and $w' = vu$. Such a cyclic equivalence is *non-parallel* if the abelianizations $[u]$ and $[v]$ are non-parallel vectors. For each $i \in \mathcal{A}$, let $\sigma_i : \mathcal{A} \to \mathcal{A}^*$ be the substitution defined by $\sigma_i(j) = ji$, for $j \neq i$, and $\sigma_i(i) = i$.

Lemma 5.3. Suppose that $u^i \in \mathcal{A}^*$, $i = 1, \ldots, d$, are cyclically equivalent, $k \in \{2, \ldots, d\}$ and $j \in \{1, \ldots, k\}$. Then $\sigma_j(u^1) = v^1, \ldots, \sigma_j(u^k) = v^k$, $\sigma_j((k+1)u^{k+1}) = (k+1)v^{k+1}, \ldots, \sigma_j(du^d) = dv^d$, with v^i, $i = 1, \ldots, d$, cyclically equivalent.

Proof. Let $k \in \{2, \ldots, d\}$, $j \in \{1, \ldots, k\}$ and $l \in \{k+1, \ldots, d\}$. Then $\sigma_j(lu^l) = lj\sigma_j(u^l) = lj(w^l j) = l(jw^l)j$. Therefore, v^i, $i \in \{1, \ldots, k\}$, and $w^l j$, $l = k+1, \ldots, d$, are cyclically equivalent, and hence v^i, $i \in \{1, \ldots, k\}$, and $v^l = jw^l$, $l = k+1, \ldots, d$, are cyclically equivalent. □

Lemma 5.4. Suppose that u^i, $i = 1, \ldots, d$, are cyclically equivalent and $k \in \{3, \ldots, d\}$. Then $\sigma_k(u^1) = v^1, \ldots, \sigma_k(u^{k-1}) = v^{k-1}$, $\sigma_k(ku^k) =$

v^k $\sigma_k((k+1)u^{k+1}) = (k+1)v^{k+1}, \ldots, \sigma_k(du^d) = dv^d$, with v^i, $i = 1, \ldots, d$, cyclically equivalent.

Proof. Let $k \in \{3, \ldots, d\}$ and $l \in \{k+1, \ldots, d\}$. Then $\sigma_k(ku^k) = k\sigma_k(u^k) = k(w^k k) = (kw^k)k$ and $\sigma_k(lu^l) = lk\sigma_k(u^l) = lk(w^l k) = l(kw^l)k$. Therefore, v^i, $i \in \{1, \ldots, k-1\}$, $w^k k$ and $w^l k$, $l = k+1, \ldots, d$, are cyclically equivalent, and hence v^i, $i \in \{1, \ldots, k-1\}$, $v^k = kw^k$ and $v^l = kw^l$, $l = k+1, \ldots, d$, are cyclically equivalent. □

Given a word $w = w_1 w_2 \cdots w_k \in \mathcal{A}^*$, let $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_k}$. A substitution $\phi : \mathcal{A} \to \mathcal{A}^*$ is *Arnoux-Rauzy* if $\phi = \sigma_w$ for some w with $[w]$ strictly positive.

Proposition 5.5. If $\phi = \sigma_w$ is Arnoux-Rauzy on d letters, then there are prefixes P_i of the fixed words $\phi^\infty(i) = P_i \ldots$, $i = 1, \ldots, d$ so that there is a non-parallel cyclic equivalence between P_i and P_j for $i \neq j \in \{1, \ldots, d\}$.

Proof. By symmetry of σ_i's we may assume, after renumbering, that $w = Y d X_{d-2} d-1 \ldots 3 X_1 2 1^n$, with X_j having only 1's, …, $(j+1)$'s.

Note that

$$\begin{aligned}
\sigma_{21^n}(1) &= \mathbf{12} \\
\sigma_{21^n}(2) &= \mathbf{21}212\ldots \\
\sigma_{21^n}(3) &= 3\mathbf{21}2\ldots \\
&\vdots \\
\sigma_{21^n}(d) &= d\mathbf{21}2\ldots.
\end{aligned}$$

By Lemma 5.3 (for $k = 2$) we have that

$$\begin{aligned}
\sigma_{X_1}\sigma_{21^n}(1) &= v_1^1 \\
\sigma_{X_1}\sigma_{21^n}(2) &= v_1^2 \\
\sigma_{X_1}\sigma_{21^n}(3) &= 3v_1^3\ldots \\
&\vdots \\
\sigma_{X_1}\sigma_{21^n}(d) &= dv_1^d\ldots.
\end{aligned}$$

with v_1^i, $i = 1, \ldots, d$, cyclically equivalent.

By Lemma 5.4 (for $k = 3$) we have that

$$\begin{array}{rcl} \sigma_3\sigma_{X_1}\sigma_{21^n}(1) & = & v_2^1 \\ \sigma_3\sigma_{X_1}\sigma_{21^n}(2) & = & v_2^2 \\ \sigma_3\sigma_{X_1}\sigma_{21^n}(3) & = & v_2^3 \ldots \\ \sigma_3\sigma_{X_1}\sigma_{21^n}(4) & = & 4v_2^4 \ldots \\ & \vdots & \\ \sigma_3\sigma_{X_1}\sigma_{21^n}(d) & = & dv_2^d \ldots . \end{array}$$

with v_2^i, $i = 1, \ldots, d$, cyclically equivalent, etc.

Let $P_i := \sigma_Y(v_{2d}^i)$. Then P_i and P_j are cylically equivalent for $i \neq j$. If $P_1 = uv$ with u, v non-degenerate, then $M_\phi[1] = [\phi(1)] = [P_1] = [P_i] = [u] + [v]$. So if $[v] = t[u]$, then $\frac{1}{1+t}[1] = M_\phi^{-1}[u]$. But $M_\phi^{-1}[u]$ is in $\mathbb{Z}^d$ and $\frac{1}{1+t}[1]$ is not. Thus the cyclic equivalences are non-parallel. □

If ϕ is an Arnoux-Rauzy substitution on d letters with Pisot inflation factor (that is, the Perron-Frobenius eigenvalue of λ of M_ϕ is Pisot) we'll say that ϕ is *Pisot Arnoux-Rauzy* and *irreducible Pisot Arnoux-Rauzy* if M_ϕ is irreducible. It is shown in [AI] that all Arnoux-Rauzy substitutions on three letters are irreducible Pisot. If ϕ is Pisot Arnoux-Rauzy, the maximal equicontinuous factor for the $\mathbb{R}$-action on the associated 1-dimensional tiling space Ω_Φ is an $\mathbb{R}$-action on the m-torus $\mathbb{T}^m = \mathbb{R}^m$, $m = deg(\lambda)$ (this is because $det(M_\phi) = 1$, so λ is a unit). The maximal equicontinuous factor map $g : \Omega_\Phi \to \mathbb{T}^m$ semi-conjugates Φ with a hyperbolic toral automorphism $F : \mathbb{T}^m \to \mathbb{T}^m$.

Lemma 5.6. If ϕ is a Pisot Arnoux-Rauzy substitution, T and T' in Ω_Φ are tilings that have a common vertex, and $n_k \to \infty$ is such that $\Phi^{n_k}(T) \to S \in \Omega_\phi$ and $\Phi^{n_k}(T') \to S' \in \Omega_\Phi$, then $g(S) = g(S')$.

Proof. Suppose that T and T' have a common vertex at t_0. Then $\Phi(T)$ and $\Phi(T')$ share a tile (of type 1) with right vertex at λt_0. Then, for t slightly bigger than t_0, $T - t$ is in the Φ-stable manifold of $T' - t$ and $g(T - t) = g(T) - \tilde{t}$ is in the F-stable manifold of $g(T' - t) = g(T') - \tilde{t}$. But then $g(T)$ is in the same F-stable manifold as $g(T')$ so that $g(S) = g(\lim_{k\to\infty} \Phi^{n_k}(T)) = \lim_{k\to\infty} F^{n_k}(g(T)) = \lim_{k\to\infty} F^{n_k}(g(T')) = g(\lim_{k\to\infty} \Phi^{n_k}(T')) = g(S')$. □

Let $cr = cr(\phi)$ be the *coincidence rank* of the Pisot substitution ϕ: $cr := max\sharp\{T_1, \ldots, T_m : g(T_i) = g(T_j) \text{ and } T_i \cap T_i = \emptyset \text{ for } i \neq j\}$. The coincidence rank is finite and g is a.e. cr-to-1 (see [BK] or [BKe]). Furthermore, if $g(T_i) = g(T_j)$ and $T_i \cap T_i = \emptyset$, then T_i and T_j are nowhere eventually coincident ([BKe]). Let $T_1, \ldots, T_{cr} \in \Omega_\phi$ be tilings with $g(T_i) = g(T_j)$ and T_i nowhere eventually coincident with T_j for $i \neq j$, let V_i be the collection of vertices of T_i, let $V := \cup_{i=1}^{cr} V_i$, and write $V = \{v_n : v_n < v_{n+1}, n \in \mathbb{Z}\}$. Note that if ϕ is Arnoux-Rauzy, then $V_i \cap V_j = \emptyset$ for $i \neq j$ (if T and T' share a vertex, then $\Phi(T)$ and $\Phi(T')$ share a tile of type 1). For each $n \in \mathbb{Z}$ and $i \in \{1, \ldots, cr\}$, let $\tau_{n(i)}$ be the tile of T_i whose support meets (v_n, v_{n+1}). The pair $C_n := ([v_n, v_{n+1}], \{\tau_{n(1)}, \ldots, \tau_{n(cr)}\})$ is called a *configuration.* Configurations C_n and C_m *have the same type* if there is $t \in \mathbb{R}$ so that $[v_n, v_{n+1}] - t = [v_m, v_{m+1}]$ and $\{\tau_{n(1)} - t, \ldots, \tau_{n(cr)} - t\} = \{\tau_{m(1)}, \ldots, \tau_{m(cr)}\}$. By a result of [BKe], there are only finitely many types of configurations.

Lemma 5.7. If ϕ is Arnoux-Rauzy, there are $m < n$ so that the configurations C_n and C_m have the same type, but C_{n+1} and C_{m+1} have different types.

Proof. Otherwise, there is $L > 0$ so that C_n and C_{n+L} have the same type for all $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ let $i(n) \in \{1, \ldots . cr\}$ be (uniquely) defined by: v_n is a vertex of a tile in $T_{i(n)}$. There are then $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ so that $i(n) = i(n + kL)$. But then $i(n + jkL) = i(n)$ for all $j \in \mathbb{Z}$ and the tiling $T_{i(n)}$ is periodic. $\square$

For an Arnoux-Rauzy substitution ϕ on d letters, let $S_i \in \Omega_\Phi$ denote the tiling fixed by Φ containing a tile of type i with the origin at its left endpoint.

Lemma 5.8. If ϕ is a Pisot Arnoux-Rauzy substitution, then there are $i \neq j$ so that the fixed tilings $S_i, S_j \in \Omega_\Phi$ are densely eventually coincident.

Proof. By Lemma 5.7, there are $m < n$ so that the configurations C_n and C_m have the same type, but C_{n+1} and C_{m+1} have different types. Let i be such that vertex v_{n+1} is the initial vertex of a tile τ of type i in $T_{i(n+1)}$ and let j be such that vertex v_{m+1} is the initial vertex of a tile τ' of type j in $T_{i(m+1)}$. Note that $\{\tau_{n+1(1)} - v, \ldots, \tau_{n+1(cr)} - v\} \setminus (\{\tau - v\}) = \{\tau_{m+1(1)}, \ldots, \tau_{m+1(cr)}\} \setminus \{\tau'\}$, with $v := v_{n+1} - v_{m+1}$. Let $b := min\{v_{m+2}, v_{n+2} - v\}$. Suppose that there are t_0 and $\epsilon > 0$ so that $(t_0 - \epsilon, t_0 + \epsilon) \subset (v_{m+1}, b)$ and $\tau_{m+1(1)}, \ldots, \tau_{m+1(cr)}, \tau - v$ are pairwise nowhere eventually coincident on $(t_0 - \epsilon, t_0 + \epsilon)$. There is then $n_k \to \infty$ so that $\Phi^{n_k}(T_j - t_0) \to T'_j \in \Omega_\Phi$ for $j = 1, \ldots, cr$ and $\Phi^{n_k}(T_{i(n+1)} - v - t_0) \to T'_{cr+1} \in \Omega_\Phi$. The tilings T'_j, $j = 1, \ldots, cr + 1$, are then pairwise nowhere eventually coincident and they all have the same image under g, by Lemma 5.6. This violates the definition of cr. As the elements of $\{\tau_{m+1(1)}, \ldots, \tau_{m+1(cr)}\}$, and of $(\{\tau_{m+1(1)}, \ldots, \tau_{m+1(cr)}\} \setminus \{\tau'\}) \cup \{\tau - v\}$

are pairwise nowhere eventually coincident, it must be the case that there is $t \in (t_0 - \epsilon, t_0 + \epsilon)$ with τ' and $\tau - v$ eventually coincident at t. That is, τ' and $\tau - v$ are densely eventually coincident on (v_{m+1}, b). This means that there is $\delta > 0$ so that S_i and S_j are densely eventually coincident on $(0, \delta)$. As $\Phi^k(S_i) = S_i$ and $\Phi^k(S_j) = S_j$ for all k, we have that S_i and S_j are densely eventually coincident on $(0, \lambda^k \delta)$ for all k, so S_i and S_j are densely eventually coincident. □

Theorem 5.9. If ϕ is an irreducible Pisot Arnoux-Rauzy substitution, then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.

Proof. Let S_i and S_j, $i \neq j$, be the fixed tilings in Ω_Φ that are densely eventually coincident (Lemma 5.8), and let Q_i and Q_j be their initial patches, corresponding to the prefixes P_i and P_j that are non-parallel cyclically equivalent (Proposition 5.5). Let u and v be non-degenerate words with non-parallel abelianizations so that $P_i = uv$ and $P_j = vu$, let l_u be the length of the patch corresponding to u, and let l be the length of Q_i. Suppose that $rl_u - sl = 0$ for integers r, s. The length of a tile of type i is given by $\langle [i], \omega \rangle$ where $\omega = (\omega_1, \ldots, \omega_d)$ is a positive left Perron eigenvector of M_ϕ. We have $0 = rl_u - sl = r\langle [u], \omega\rangle - s\langle [u + v], \omega\rangle = \langle (r-s)[u] - s[v], \omega\rangle$. But the ω_i are independent over $\mathbb{Q}$ (by irreducibility), so the only integer vector orthogonal to ω is $\mathbf{0}$. Thus $(r-s)[u] = s[v]$, and since $[u]$ and $[v]$ are not parallel, $r = 0 = s$. Thus, l_u is irrationally related to l. We have that $\bar{Q}_i$ and $\bar{Q}_j = \bar{Q}_i - l$ are densely eventually coincident. By Theorem 3.1, the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum. □

Example 5.10. (**Arbitrary compositions of the Rauzy and modified Rauzy substitutions**)

Let us consider the Rauzy substitution τ_1: $1 \to 12$, $2 \to 13$ and $3 \to 1$, and modified Rauzy substitutions defined as follows, τ_2: $1 \to 12$, $2 \to 31$ and $3 \to 1$; τ_3: $1 \to 21$, $2 \to 13$ and $3 \to 1$; and τ_4: $1 \to 21$, $2 \to 31$ and $3 \to 1$. It is easy to check that applying τ_1, τ_2, τ_3 or τ_4 to any pair in the list below results in a pair that can be factored as a product of pairs in the list below (or of the duals (v, u) of pairs (u, v) in the list) and coincidences.

List of irreducible pairs: (12, 21), (13, 31), (123, 231), (321, 132), (213, 312), (1123, 3112), (3211, 2113), (1213, 3112), (3121, 2113), (1213, 3121), (1231, 3112), (1321, 2113), (2131, 3112), (1312, 2113), (11231, 31112), (12123, 23112), (32121, 21132), (12311, 31112), (11321, 21113), (121213, 311212), (312121, 212113), (121123, 311212), (321121, 212113), (121213, 312112), (312121, 211213), (121231, 231112), (132121, 211132), (1121231, 3112112), (1321211, 2112113), (12121231, 23111212), (13212121, 21211132)

Note that for any $i, j \in \{1, 2, 3, 4\}$ and any $a, b \in \{1, 2, 3\}$, $(\tau_i \circ \tau_j(a), \tau_i \circ \tau_j(b))$ is either of the form $(uk\ldots, vk\ldots)$ or $(\ldots ku, \ldots kv)$ for some $k \in \{1, 2, 3\}$ and (possibly empty) words u, v with $[u] = [v]$. Thus if τ is any finite composition of the τ_i's, the balanced pair algorithm for τ applied to $(12, 21)$ terminates with coincidence. Since such τ are Pisot with irreducible incidence matrix, the following is a consequence of Corollary 3.3.

Theorem 5.11. If τ is any finite composition of Rauzy and modified Rauzy substitutions, the $\mathbb{R}$-action on Ω_τ has pure discrete spectrum.

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