On the density of exponential functionals of Lévy processes

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Abstract

In this paper, we study the existence of the density associated to the exponential functional of the Lévy process ξ,

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} e^{\xi_s}\, \mathrm{d}s,$$

where $\mathbf{e}_q$ is an independent exponential r.v. with parameter $q \geq 0$. In the case when ξ is the negative of a subordinator, we prove that the density of $I_{\mathbf{e}_q}$, here denoted by k, satisfies an integral equation that generalizes the one found by Carmona et al. [7]. Finally when $q = 0$, we describe explicitly the asymptotic behaviour at 0 of the density k when ξ is the negative of a subordinator and at ∞ when ξ is a spectrally positive Lévy process that drifts to $+\infty$.

Keywords: Lévy processes, exponential functional, subordinators, self-similar Markov processes.

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1 Introduction

A real-valued Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost sure right continuous paths with left-limits. We write $\xi = (\xi_t, t \geq 0)$ for its trajectory and $\mathbb{P}$ for its law. The law $\mathbb{P}$ of a Lévy processes is characterized by its one-time transition probabilities. In particular there always exists

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a triple (a, σ^2, Π) where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, such that, for $t \geq 0$ and $z \in \mathbb{R}$

$$\mathbb{E}[e^{iz\xi_t}] = \exp\{-\Psi(z)t\}, \tag{1.1}$$

where

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \Big(1 - e^{izx} + izx\mathbf{1}_{\{|x|<1\}}\Big)\Pi(\mathrm{d}x).$$

In the case when ξ is a subordinator, the Lévy measure Π has support on $[0, \infty)$ and fulfils the extra condition $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$. Hence, the characteristic exponent Ψ can be expressed as

$$\Psi(z) = -icz + \int_{(0,\infty)} \Big(1 - e^{izx}\Big)\Pi(\mathrm{d}x),$$

where $c \geq 0$ and is known as the drift coefficient. It is well-known that the function Ψ can be extended analytically on the complex upper half-plane, so the Laplace exponent of ξ is given by

$$\phi(\lambda) := -\log \mathbb{E}[e^{-\lambda\xi_1}] = \Psi(i\lambda) = c\lambda + \int_{(0,\infty)} \Big(1 - e^{-\lambda x}\Big)\Pi(\mathrm{d}x).$$

Similarly, in the case when ξ is a spectrally negative Lévy process (i.e. has no positive jumps), the Lévy measure Π has support on $(-\infty, 0)$ and the characteristic exponent Ψ can be written as

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{(-\infty,0)} \Big(1 - e^{izx} + izx\mathbf{1}_{\{x>-1\}}\Big)\Pi(\mathrm{d}x).$$

It is also well-known that the function Ψ can be extended analytically on the complex lower half-plane, so its Laplace exponent satisfies

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda\xi_1}] = -\Psi(-i\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \Big(e^{\lambda x} - 1 + -\lambda x\mathbf{1}_{\{x>-1\}}\Big)\Pi(\mathrm{d}x).$$

In this article, one of our aims is to study the existence of the density associated to the exponential functional

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} e^{\xi_s}\,\mathrm{d}s,$$

where $\mathbf{e}_q$ is an exponential r.v. independent of the Lévy process ξ with parameter $q \geq 0$ If $q = 0$, then $\mathbf{e}_q$ is understood as ∞. In this case, we assume that the process ξ drifts towards $-\infty$ since it is a necessary and sufficient condition for the almost sure finiteness of $I := I_\infty$, see for instance Theorem 1 in Bertoin and Yor [4].

Up to our knowledge nothing is known about the existence of the density of $I_{\mathbf{e}_q}$ when $q > 0$. In the case when $q = 0$, the existence of the density of I has been proved by Carmona et al. [7] for Lévy processes whose jump structure is of finite variation and recently by Bertoin et al. [2] (see Theorem 3.9) for any real-valued Lévy process. In particular when ξ is the negative of a subordinator such that $\mathbb{E}[|\xi_1|] < \infty$, Carmona et al. [7] (see Proposition 2.1) proved that the r.v. I has a density, k, that is the unique (up to a multiplicative constant) L^1 positive solution to the equation

$$(1 - cx)k(x) = \int_x^\infty \overline{\Pi}(\log(y/x))k(y)\,\mathrm{d}y, \qquad x \in (0, 1/c), \tag{1.2}$$

where $\overline{\Pi}(x) := \Pi(x, \infty)$.

Here, we generalize the above equation. Indeed, we establish an integral equation for the density of $I_{\mathbf{e}_q}$, $q \geq 0$, when ξ is the negative of a subordinator and we note that when $q = 0$, the condition $\mathbb{E}[|\xi_1|] < \infty$ is not essential for the existence of its density and the validity of (1.2).

Another interesting problem is determining the behaviour of the density of the exponential functional I at 0 and at ∞. This problem has been recently studied by Kuznetzov [14] for Lévy processes with rational Laplace exponent (at 0 and at ∞), by Kuznetsov and Pardo [16] for hypergeometric Lévy processes (at 0 and at ∞) and by Patie [19] for spectrally negative Lévy processes (at ∞). In most of the applications, it is enough to have estimates of the tail behaviour $\mathbb{P}(I \leq t)$ when t goes to 0 and/or $\mathbb{P}(I \geq t)$ when t goes to ∞. The tail behaviour $\mathbb{P}(I \leq t)$ was studied by Pardo [20] in the case where the underlying Lévy process is spectrally positive and its Laplace exponent is regularly varying at infinity with index $\gamma \in (1, 2)$, and by Caballero and Rivero [6] in the case when ξ is the negative of a subordinator whose Laplace exponent is regularly varying at 0. Furthermore, the tail behaviour $\mathbb{P}(I \geq t)$ has been studied in a general setting, see [8, 18, 21, 22] . The second main result of this paper is related to this problem. Namely, we describe explicitly the asymptotic behaviour at 0 of the density of I when ξ is a subordinator which in particular implies the behaviour of $\mathbb{P}(I < t)$ near 0.

The paper is organized as follows: in Section 2 we state our main results, in particular we study the density of $I_{\mathbf{e}_q}$ and the asymptotic behaviour at 0 of the density of the exponential functional associated to the negative of a subordinator. Section 3 is devoted to the proof of the main results and in Section 4, we give some examples and some numerical results for the density of $I_{\mathbf{e}_q}$ when the driving process is the negative of a subordinator.

2 Main results

Our first main result states that $I_{\mathbf{e}_q}$ has a density, for $q > 0$. Before we establish our first Theorem, we need to introduce some notation and recall some facts about positive self-similar Markov processes (pssMp) which will be our main tool in this first part.

Let $(\xi_t^\dagger, t \geq 0)$ be the process obtained by killing ξ at an independent exponential time of parameter $q > 0$, here denoted by $\mathbf{e}_q$. The law and the lifetime of $\xi^\dagger$ are denoted by $\mathbb{P}^\dagger$ and β, respectively.

We first note that

$$\left(I, \mathbb{P}^\dagger\right) = \left(\int_0^\beta \exp\left\{\xi_t^\dagger\right\} \mathrm{d}t, \mathbb{P}^\dagger\right) \stackrel{d}{=} \left(\int_0^{\mathbf{e}_q} e^{\xi_t} \mathrm{d}t, \mathbb{P}\right).$$

For $x \geq 0$ let $\mathbb{Q}_x$ be the law of $X^{(x)}$, the positive self-similar Markov process with self-similarity index 1 issued from x, associated to $\xi^\dagger$ via its Lamperti's representation (see [17] for more details on this representation), that is for $x > 0$

$$X_t^{(x)} = \begin{cases} x \exp\left\{\xi^\dagger_{\tau(t/x)}\right\}, & \text{if } \tau(t/x) < \infty \\ 0, & \text{if } \tau(t/x) = \infty, \end{cases} \quad t \geq 0;$$

where

$$\tau(s) = \inf\left\{r > 0 : \int_0^r e^{\xi_t^\dagger} \mathrm{d}t > s\right\}, \qquad \inf\{\emptyset\} = \infty,$$

and 0 is understood as a cemetery state. The process $X^{(x)}$ is a strong Markov process and it fulfills the scaling property, i.e. for $k > 0$,

$$\left(kX^{(x)}_{t/k}, t \geq 0\right) \stackrel{d}{=} \left(X_t^{(kx)}, t \geq 0\right).$$

We denote by $T_0^{(x)} := \inf\{t > 0 : X_t^{(x)} = 0\}$, the first hitting time of $X^{(x)}$ at 0. Observe that for $s > 0$, we have the following equivalences,

$$\tau(s) < \infty \quad \text{iff} \quad \tau(s) \leq \beta \quad \text{iff} \quad s \leq \int_0^\beta e^{\xi_t^\dagger} \mathrm{d}t.$$

Hence, it follows from the construction of X that the following equality in law follows

$$\left(T_0, \mathbb{Q}_1\right) \stackrel{d}{=} \left(\int_0^{\mathbf{e}_q} e^{\xi_t} \mathrm{d}t, \mathbb{P}\right).$$

Now, we have all the elements to establish our first main result. It concerns the existence of the density of $I_{\mathbf{e}_q}$.

Theorem 2.1. *Let $q > 0$, then the function*

$$h(t) := q\mathbb{Q}_1\left[\frac{1}{X_t} \mathbf{1}_{\{t < T_0\}}\right], \qquad t \geq 0,$$

is a density for the law of $I_{\mathbf{e}_q}$.

Corollary 2.2. *Assume $q > 0$ and that ξ is a subordinator. Then the law of the r.v. $I_{\mathbf{e}_q}$ is a mixture of exponential, that is its law has a density h on $(0, \infty)$ which is completely monotone. Furthermore, $\lim_{t \downarrow 0} h(t) = q$.*

In the sequel, we assume that $\xi = -\zeta$ where ζ is a subordinator and we denote by $U_q(\mathrm{d}x)$ the renewal measure of the killed subordinator $(\zeta_t, t \le \mathbf{e}_q)$, i.e.

$$\mathbb{E}\left[\int_0^{\mathbf{e}_q} f(\zeta_t)\mathrm{d}t\right] = \int_{[0,\infty)} f(x)U_q(\mathrm{d}x), \tag{2.3}$$

where f is a positive measurable function. If the renewal measure is absolutely continuous with respect to the Lebesgue measure, the function $u_q(x) = U_q(\mathrm{d}x)/\mathrm{d}x$, is usually called the renewal density. If $q = 0$, we denote U_0 and u_0 by U and u.

Before stating our first main result, which is a generalization of the integral equation (1.2) of Carmona et al. for subordinators, in the next proposition we establish that the density of $I_{\mathbf{e}_q}$ solves an integral equation in terms of the renewal measure U_q.

Proposition 2.3. *Let $q \ge 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k, and it solves the equation*

$$\int_y^\infty k(x)\mathrm{d}x = \int_0^\infty k(ye^x)U_q(\mathrm{d}x), \qquad \textit{almost everywhere.} \tag{2.4}$$

The next result generalizes (1.2).

Theorem 2.4. *Let $q \ge 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k, and it solves*

$$(1 - cx)k(x) = \int_x^\infty \overline{\Pi}(\log(y/x))k(y)\mathrm{d}y + q\int_x^\infty k(y)\mathrm{d}y \qquad x \in (0, 1/c). \tag{2.5}$$

Conversely, if a density on $(0, 1/c)$ satisfies this equation then it is the density of $I_{\mathbf{e}_q}$.

The importance of the above result will be illustrated in Theorem 2.5 where we study the asymptotic behaviour at 0 of the density k, and in Section 4 where we provide some examples where k can be computed explicitly. Further applications have been provided in Haas [11] and Haas and Rivero [12] where this equation has been used to estimate the right tail behavior of the law of I and to study the maximum domain of attraction of I.

The following corollary is another important application of equation (2.5). In particular, it says that if we know the density of the exponential functional of the negative of a subordinator, say k, then for $\rho \ge 0$, $x^\rho k(x)$ adequately normalized is the density of the exponential functional associated to the negative of a new subordinator. The proof of this fact follows easily by multiplying in both sides of equation (2.5) by x^ρ. Such result also appears in Chazal et al. [9] but in terms of the distribution of $I_{\mathbf{e}_q}$ not in terms of its density.

Corollary 2.5. *Let $q \ge 0, \rho > 0$, c_ρ a positive constant satisfying*

$$c_\rho = \int_{(0,\infty)} x^\rho k(x)\mathrm{d}x,$$

and suppose that when $q > 0$ the renewal measure U_q has a density. Then the function $h(x) := c_\rho^{-1} x^\rho k(x)$ is the density of the exponential functional of the negative of a subordinator whose Laplace exponent is given by

$$\phi_\rho(\lambda) = \frac{\lambda}{\lambda+\rho}\Big(\phi(\lambda+\rho)+q\Big). \tag{2.6}$$

Moreover, the density h solves the equation

$$(1-cx)h(x) = \int_x^\infty \overline{\Pi}_\rho(\log y/x)h(y)\mathrm{d}y \qquad x \in (0, 1/c), \tag{2.7}$$

where $\overline{\Pi}_\rho(z) = \overline{\Pi}(z)e^{-\rho z} + qe^{-\rho z}$.

We remark that the transformation studied in Chazal et al. [9] is more general than the one presented in (2.6) and that they applied such transformation to Lévy processes with one-sided jumps. We also remark that the subordinator whose Laplace exponent is given by ϕ_ρ has an infinite lifetime in any case.

Our next goal is to study the behavior of the density of $I_{\mathbf{e}_q}$ near 0. When $q = 0$, we work with the following assumption:

(A) *The Lévy measure Π belongs to the class $\mathcal{L}_\alpha$ for some $\alpha \geq 0$, that is to say that the tail Lévy measure $\overline{\Pi}$ satisfies*

$$\lim_{x\to\infty} \frac{\overline{\Pi}(x+y)}{\overline{\Pi}(x)} = e^{-\alpha y}, \quad \textit{for all } y \in \mathbb{R}. \tag{2.8}$$

Observe that regularly varying and subexponential tail Lévy measures satisfy this assumption with $\alpha = 0$ and that convolution equivalent Lévy measures are examples of Lévy measures satisfying (2.8) for some index $\alpha > 0$.

Theorem 2.6. *Let $q \geq 0$ and $\xi = -\zeta$, where ζ is a subordinator such that when $q = 0$ the Lévy measure Π satisfies assumption (A). The following asymptotic behaviour holds for the density function k of the exponential functional $I_{\mathbf{e}_q}$.*

i) If $q > 0$, then

$$k(x) \to q \qquad \textit{as } x \downarrow 0.$$

ii) If $q = 0$, then $\mathbb{E}[I^{-\alpha}] < \infty$ and

$$k(x) \sim \mathbb{E}\big[I^{-\alpha}\big]\overline{\Pi}(\log 1/x) \quad \textit{as } x \downarrow 0.$$

In the sequel we will assume that $q = 0$. The above result will help us to describe the behaviour at ∞ of the density of the exponential functional of a particular spectrally negative Lévy processes associated to the subordinator ζ. In order to explain such relation, we need the following assumptions. Assume that U, the renewal measure of the subordinator ζ, is absolutely continuous with respect to the Lebesgue measure with density u which is non-increasing and convex. We also suppose that $\mathbb{E}[\zeta_1] < \infty$. According to Theorem 2 in Kyprianou and Rivero [13] there exists a spectrally negative Lévy process $Y = (Y_t, t \geq 0)$ that drifts to $+\infty$, whose Laplace exponent is described by

$$\psi(\lambda) = \lambda\phi^*(\lambda) = \frac{\lambda^2}{\phi(\lambda)}, \qquad \text{for} \quad \lambda \geq 0,$$

where ϕ^* is the Laplace exponent of another subordinator and satisfies

$$\phi^*(\lambda) := q^* + c^*\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right)\Pi^*(\mathrm{d}x),$$

where

$$q^* = \left(c + \int_{(0,\infty)} x\Pi(\mathrm{d}x)\right)^{-1}, \qquad c^* = \begin{cases} 0 & c > 0, \text{ or } \Pi(0,\infty) = \infty, \\ 1/\Pi(0,\infty), & c = 0 \text{ and } \Pi(0,\infty) < \infty, \end{cases}$$

and the Lévy measure Π^* satisfies

$$u(x) = c^*\mathbf{1}_{\{x=0\}} + q^* + \overline{\Pi}^*(x), \qquad \text{for} \quad x \geq 0.$$

Let I_ψ be the exponential functional associated to $-Y$, i.e.

$$I_\psi = \int_0^\infty e^{-Y_s}\mathrm{d}s,$$

and denote its density by k_ψ. From the proof of Proposition 4 in Rivero [21] the density k_ψ satisfies

$$k_\psi(x) = q^*\frac{1}{x}k\left(\frac{1}{x}\right), \qquad \text{for} \quad x > 0. \tag{2.9}$$

The following corollary give us the asymptotic behaviour at ∞ of the density of the exponential functional of $-Y$.

Corollary 2.7. *Suppose that ζ is a subordinator satisfying assumption (A) and such that its renewal measure has a density which is non-increasing and convex and let Y be its associated spectrally negative Lévy process defined as above. Then the following asymptotic behaviour holds for the density function k_ψ,*

$$k_\psi(x) \sim q^*\mathbb{E}\left[I^{-\alpha}\right]\frac{1}{x}\overline{\Pi}(\log x) \quad \text{as } x \to \infty.$$

3 Proofs

Proof of Theorem 2.1. We start the proof by showing that the function

$$h(t,x) := q\mathbb{Q}_x\left[\frac{1}{X_t}\mathbf{1}_{\{t<T_0\}}\right], \qquad t\geq 0,\ x>0$$

is such that

$$\int_0^\infty h(t,x)\,\mathrm{d}t = 1, \qquad \text{for} \quad x>0. \tag{3.1}$$

Then the result follows from the identity (3.1) and the fact that

$$h(t+s) = q\mathbb{Q}_1\Big[h(s,X_t)\mathbf{1}_{\{t<T_0\}}\Big], \qquad \text{for} \quad s,t\geq 0,$$

which is a straightforward consequence of the Markov property.

Let us prove (3.1). From the definition of X and the change of variables $u=\tau(t/x)$, which implies that $\mathrm{d}u = x^{-1}\exp\{-\xi^\dagger_{\tau(t/x)}\}\mathrm{d}t$, we get

$$\begin{aligned}\int_0^\infty h(t,x)\mathrm{d}t &= q\int_0^\infty \mathrm{d}t\,\mathbb{E}\left[x^{-1}\exp\Big\{-\xi^\dagger_{\tau(t/x)}\Big\}\mathbf{1}_{\{\tau(t/x)<\infty\}}\right]\\ &= q\mathbb{E}\left[\int_0^\infty x^{-1}\exp\Big\{-\xi^\dagger_{\tau(t/x)}\Big\}\mathbf{1}_{\left\{t\leq x\int_0^\beta e^{\xi^\dagger_s}\mathrm{d}s\right\}}\mathrm{d}t\right]\\ &= q\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{u\leq\beta\}}\mathrm{d}u\right] = q\mathbb{E}(\beta)=1.\end{aligned}$$

We now prove that

$$\int_t^\infty h(s)\,\mathrm{d}s = \mathbb{P}\Big(I_{\mathbf{e}_q}>t\Big), \qquad t>0.$$

Indeed, let $t>0$ making a change of variables, using the semi-group property, and Fubini's theorem we have

$$\int_t^\infty h(s)\,\mathrm{d}s = \int_0^\infty h(s+t,1)\,\mathrm{d}s = \mathbb{Q}_1\left[\left(\int_0^\infty h(s,X_t)\,\mathrm{d}s\right)\mathbf{1}_{\{t<T_0\}}\right] = \mathbb{Q}_1(t<T_0).$$

The result follows from the identity $\mathbb{Q}_1(t<T_0) = \mathbb{P}\Big(I_{\mathbf{e}_q}>t\Big)$. □

Proof of Corollary 2.2. Here, we use the same notation as above and we follow similar arguments as in the proofs of Lemma 5 and Proposition 1 in [3]. We first prove that for every $0\leq t<T_0$ and $p>0$, the variable

$$X_t^p\int_t^{T_0}\frac{1}{X_s^{p+1}}\mathrm{d}s$$

is independent from $\sigma\{X_s, 0\leq s\leq t\}$ and is distributed as

$$\int_0^{\mathbf{e}_q} e^{-p\xi_s}\mathrm{d}s.$$

As a consequence of the Markov property at time t, we only need to show that under $\mathbb{Q}_x$, the variable

$$x^p \int_0^{T_0} \frac{1}{X_s^{p+1}} \mathrm{d}s$$

is distributed as $\int_0^{\mathbf{e}_q} e^{-p\xi_s} \mathrm{d}s$. Then the change of variables $t = \tau(s/x)$, $s = x\int_0^t e^{\xi_u^\dagger} \mathrm{d}u$, yields

$$\begin{aligned} x^p \int_0^{T_0} \frac{1}{X_s^{p+1}} \mathrm{d}s &= x^{-1} \int_0^{T_0} e^{-(p+1)\xi^\dagger_{\tau(s/x)}} \mathrm{d}s \\ &= \int_0^{\beta} e^{-(p+1)\xi_t^\dagger} e^{\xi_t^\dagger} \mathrm{d}t \\ &= \int_0^{\beta} e^{-p\xi_t^\dagger} \mathrm{d}t, \end{aligned}$$

which implies the desired identity in law since $(\xi_t^\dagger, 0 \le t \le \beta)$ and $(\xi_t, 0 \le t \le \mathbf{e}_q)$ have the same law. Hence, we have

$$\mathbb{Q}_1\left[\int_t^{T_0} \frac{1}{X_s^{p+1}} \mathrm{d}s\right] = \frac{\mathbb{Q}_1\left[X_t^{-p}; t < T_0\right]}{\phi(p) + q},$$

which implies

$$\frac{\partial \mathbb{Q}_1\left[X_t^{-p}; t < T_0\right]}{\partial t} = -(\phi(p) + q)\mathbb{Q}_1\left[X_t^{-(p+1)}; t < T_0\right].$$

By iteration, we get that the function $t \mapsto \mathbb{Q}_1\left[X_t^{-p}; t < T_0\right]$ is completely monotone and takes value 1 for $t = 0$. Thus taking $p = 1$, we deduce that $h(t)$ is completely monotone on $(0, \infty)$ and that $\lim_{t\downarrow 0} h(t) = q$. Finally from Theorem 51.6 and Proposition 51.8 in [23], we have that the law of $I_{\mathbf{e}_q}$ is a mixture of exponentials. $\square$

Proof of Proposition 2.3. The proof follows from the identity

$$\mathbb{E}\left[I_{\mathbf{e}_q}^n\right] = \frac{n}{\phi(n) + q}\mathbb{E}\left[I_{\mathbf{e}_q}^{n-1}\right], \qquad n > 0$$

Indeed, on the one hand it is clear that

$$\mathbb{E}\left[I_{\mathbf{e}_q}^n\right] = \int_0^\infty x^n k(x)\mathrm{d}x = n\int_0^\infty \mathrm{d}y\, y^{n-1} \int_y^\infty k(x)\mathrm{d}x.$$

On the other hand from the identity (2.3) with $f(x) = e^{-nx}$ and a change of variables, we get

$$\begin{aligned}\frac{n}{\phi(n)+q}\mathbb{E}\Big[I_{\mathbf{e}_q}^{n-1}\Big] &= n\int_0^\infty U_q(\mathrm{d}x)\, e^{-nx}\int_0^\infty y^{n-1}k(y)\mathrm{d}y\\ &= n\int_0^\infty U_q(\mathrm{d}x)\int_0^\infty y^{n-1}e^{-nx}k(y)\mathrm{d}y\\ &= n\int_0^\infty U_q(\mathrm{d}x)\int_0^\infty z^{n-1}k(ze^{-x})\mathrm{d}z\\ &= n\int_0^\infty \mathrm{d}z\, z^{n-1}\int_0^\infty k(ze^{-x})U_q(\mathrm{d}x).\end{aligned}$$

Then putting the pieces together, we have

$$\int_0^\infty \mathrm{d}y\, y^{n-1}\int_y^\infty k(x)\mathrm{d}x = \int_0^\infty \mathrm{d}y\, y^{n-1}\int_0^\infty k(ye^{-x})U_q(\mathrm{d}x), \qquad \text{for } n>0,$$

which implies the desired result because the density

$$y\mapsto \frac{1}{\mathbb{E}(I_{\mathbf{e}_q})}\int_y^\infty k(x)\mathrm{d}x,$$

is determined by its entire moments, which in turn is an easy consequence of the fact that k is so. □

Proof of Theorem 2.4. By Theorem 2.1 (when $q>0$) and Theorem 3.9 in [2] (when $q=0$), we know that there exists a density of $I_{\mathbf{e}_q}$, for $q\geq 0$, that we denote by h. Moreover, in [7] it has been proved that the moments of $I_{\mathbf{e}_q}$ are given by

$$\mathbb{E}\Big[I_{\mathbf{e}_q}^n\Big] = \frac{n!}{\prod_{i=1}^n\Big(q+\phi(i)\Big)}, \qquad n\in\mathbb{N} \tag{3.2}$$

where the product is understood as 1 when $n=0$.

We first prove that the function $\widetilde{h}:(0,\infty)\to(0,\infty)$ defined via

$$\widetilde{h}(x) = \begin{cases} cxh(x) + \int_x^\infty \overline{\Pi}(\log(y/x))h(y)\mathrm{d}y + q\int_x^\infty h(y)\mathrm{d}y, & \text{if } x\in(0,1/c),\\ 0, & \text{elsewhere,}\end{cases}$$

is a density for the law of $I_{\mathbf{e}_q}$ and hence that $h=\widetilde{h}$ a.e. Then we prove that the equality (2.5) holds. In order to do so, it is enough to verify that

$$\int_0^\infty x^n\widetilde{h}(x)\mathrm{d}x = \frac{n!}{\prod_{i=1}^n\Big(q+\phi(i)\Big)}, \qquad n\in\mathbb{N},$$

since the law of $I_{\mathbf{e}_q}$ is determined by its entire moments.

Indeed, elementary computations, identity (2.4) and the fact that

$$\int_0^\infty e^{-\theta y} U_q(\mathrm{d}y) = \frac{1}{\phi(\theta)+q}, \qquad \theta \geq 0,$$

give that for any integer $n \geq 0$,

$$\begin{aligned}
\int_0^\infty x^n \widetilde{h}(x)\mathrm{d}x &= c\int_0^\infty \mathrm{d}x\, x^{n+1}h(x) + \int_0^\infty \mathrm{d}x\, x^n \int_x^\infty \mathrm{d}y\, \overline{\Pi}(\log(y/x))h(y) \\
&\qquad + q\int_0^\infty \mathrm{d}x x^n \int_0^\infty h(xe^y)U_q(\mathrm{d}y) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}\Big(q+\phi(i)\Big)} + \int_0^\infty \mathrm{d}y\, h(y)\int_0^y \mathrm{d}x\, x^n \overline{\Pi}(\log(y/x)) \\
&\qquad + q\int_0^\infty U_q(\mathrm{d}y)\int_0^\infty \mathrm{d}x x^n h(xe^y) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}\Big(q+\phi(i)\Big)} + \int_0^\infty \mathrm{d}y\, h(y)y^{n+1}\int_0^\infty \mathrm{d}z\, e^{-(n+1)z}\overline{\Pi}(z) \\
&\qquad + q\int_0^\infty U_q(\mathrm{d}y)e^{-(n+1)y}\int_0^\infty \mathrm{d}z z^n h(z) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}\Big(q+\phi(i)\Big)} + \frac{(n+1)!}{\prod_{i=1}^{n+1}\Big(q+\phi(i)\Big)} \frac{\int_0^\infty \big(1-e^{-(n+1)z}\big)\Pi(\mathrm{d}z)}{n+1} \\
&\qquad + q\frac{n!}{\prod_{i=1}^{n}\Big(q+\phi(i)\Big)}\int_0^\infty U_q(\mathrm{d}y)e^{-(n+1)y} \\
&= \frac{n!}{\prod_{i=1}^{n}\Big(q+\phi(i)\Big)} \frac{(n+1)c + \int_0^\infty \big(1-e^{-(n+1)z}\big)\Pi(\mathrm{d}z) + q}{q+\phi(n+1)} \\
&= \frac{n!}{\prod_{i=1}^{n}\Big(q+\phi(i)\Big)}.
\end{aligned}$$

Now, let $\mathcal{N} = \Big\{x \in \mathbb{R} : h(x) \neq \widetilde{h}(x)\Big\}$. By the above arguments, we know that the Lebesgue measure of $\mathcal{N}$ is zero. Let $k : (0,\infty) \to (0,\infty)$ be the function defined by

$$k(x) = \begin{cases} h(x), & \text{if } x \in \mathcal{N}^c, \\ \dfrac{1}{1-cx}\left(\displaystyle\int_x^\infty \overline{\Pi}(\log(y/x))h(y)\mathrm{d}y + q\int_x^\infty h(y)\mathrm{d}y\right), & \text{if } x \in \mathcal{N}. \end{cases}$$

We now prove that $k(x)$ satisfies equation (2.5) everywhere. If $x \in \mathcal{N}^c$ then we have that $k(x) = h(x) = \widetilde{h}(x)$, and hence equation (2.5) is verified. Finally, if $x \in \mathcal{N}$, we have the following equalities

$$
\begin{aligned}
cxk(x) &+ \int_x^\infty \overline{\Pi}(\log(y/x))k(y)\mathrm{d}y + q\int_x^\infty k(y)\mathrm{d}y \\
&= cxk(x) + \int_x^\infty \overline{\Pi}(\log(y/x))k(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\mathrm{d}y + q\int_x^\infty k(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\mathrm{d}y \\
&= cxk(x) + \int_x^\infty \overline{\Pi}(\log(y/x))h(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\mathrm{d}y + q\int_x^\infty h(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\mathrm{d}y \\
&= \frac{cx}{1-cx}\left(\int_x^\infty \overline{\Pi}(\log(y/x))h(y)\mathrm{d}y + q\int_x^\infty h(y)\mathrm{d}y)\right) \\
&\qquad\qquad + \int_x^\infty \overline{\Pi}(\log(y/x))h(y)\mathrm{d}y + q\int_x^\infty h(y)\mathrm{d}y \\
&= k(x).
\end{aligned}
$$

Conversely if k is a density on $(0, 1/c)$ satisfying equation (2.5), from the above computations it is clear that k and $I_{\mathbf{e}_q}$ have the same entire moments. This implies that the k is a density of the exponential functional $I_{\mathbf{e}_q}$. □

Proof of Theorem 2.6. The proof consists of three steps. First we show that when $q = 0$, $\mathbb{E}\left[I^{-\alpha}\right] < \infty$, then for $q \geq 0$ we obtain a technical estimate on the maximal growth of $k(x)$ as $x \downarrow 0$, and finally the statement of the theorem.

Step 1. Here we assume that $q = 0$ and prove that $\mathbb{E}\left[I^{-\alpha}\right] < \infty$. The case $\alpha = 0$ is obvious. For $\alpha \in (0, 1)$, we have from Theorem 2 in [4] that there exists a random variable R, independent of ξ, such that $IR \overset{d}{=} \mathbf{e}$, where $\mathbf{e}$ follows a unit mean exponential distribution. Since $\mathbb{E}[\mathbf{e}^{-\alpha}] < \infty$, the result follows.

Finally let $\alpha \geq 1$. With (2.5) and some standard computations, we find

$$
\begin{aligned}
\int_0^\infty x^{-\beta-1}k(x)\,\mathrm{d}x &= c\int_0^\infty \mathrm{d}x\, x^{-\beta}k(x) + \int_0^\infty \mathrm{d}x\, x^{-\beta-1}\int_x^\infty \mathrm{d}y\,\overline{\Pi}(\log(y/x))k(y) \\
&= c\mathbb{E}\left[I^{-\beta}\right] + \int_0^\infty \mathrm{d}y\,k(y)\int_0^y \mathrm{d}x\,x^{-\beta-1}\overline{\Pi}(\log(y/x)) \\
&= c\mathbb{E}\left[I^{-\beta}\right] + \int_0^\infty \mathrm{d}y\,y^{-\beta}k(y)\int_0^\infty \mathrm{d}u\,e^{\beta u}\overline{\Pi}(u) \\
&= -\frac{1}{\beta}\mathbb{E}\left[I^{-\beta}\right]\left(-c\beta + \int_0^\infty \left(1 - e^{\beta z}\right)\Pi(\mathrm{d}z)\right),
\end{aligned}
$$

that is to say

$$
\mathbb{E}\left[I^{-\beta-1}\right] = \mathbb{E}\left[I^{-\beta}\right]\frac{\phi(-\beta)}{-\beta}, \tag{3.3}
$$

where ϕ is the Laplace exponent of ξ, which can be extended to $(-\alpha, \infty)$ since for $\beta < \alpha$

$$
\int_0^\infty (e^{\beta u} - 1)\,\Pi(\mathrm{d}u) = \beta\int_1^\infty \overline{\Pi}(\log(z))z^{\beta-1}\,\mathrm{d}z < \infty. \tag{3.4}
$$

To see that (3.4) holds, note that $\overline{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ by (2.8). Hence $\overline{\Pi}(\log z) = z^{-\alpha}\ell(z)$ for a slowly varying function ℓ and we can apply Proposition 1.5.10 from Bingham et al. [5].

Now, by iteratively using (3.3) we see that for $\mathbb{E}[I^{-\alpha}] < \infty$ it is enough to have $\mathbb{E}[I^{-\alpha'}] < \infty$ for some $\alpha' \in [0,1)$. But this obviously holds if $\alpha' = 0$, while if $\alpha' \in (0,1)$ it holds by the same argument as we used above for the case $\alpha \in (0,1)$.

Step 2. We assume that $q \geq 0$. For $q = 0$, let p be any function such that $p(0) = 0$ and $\min\{\alpha - 1, 0\} < p(\alpha) < \alpha$ for all $\alpha > 0$. When $q > 0$ the function p will be taken as zero and hence the symbol $p(\alpha)$ will be taken as 0. The goal of this step is to show

$$\frac{k(x)}{x^{p(\alpha)}} \qquad \text{stays bounded as} \qquad x \downarrow 0. \tag{3.5}$$

Observe that when $q > 0$ it follows from (2.5) that $\liminf_{x\to 0} k(x) \geq q$. Set $h(x) := k(x)/x^{p(\alpha)}$. We can write (2.5) as

$$1 - cx = x\int_1^\infty \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(xz)}{h(x)}\,\mathrm{d}z + \frac{qx^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q} > x)}{h(x)}. \tag{3.6}$$

We argue by contradiction. Take some $\hat{x} \in (0, 1/c)$. If h were not bounded at $0+$, then $\mathbf{1}_{\{x\leq\hat{x}\}}h(x)$ would keep on attaining new maxima as $x \downarrow 0$. (Note that $\hat{x}$ is present just to make sure this statement also holds if k is not bounded at $1/c-$.) In particular this means that a sequence of points $(x_n)_{n\geq 0}$ exists with $x_n \downarrow 0$ as $n \to \infty$ and such that $h(x_n) \geq \sup_{x\in[x_n,\hat{x}]} h(x)$. We will show that this implies

$$x_n\int_1^\infty \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z + \frac{qx_n^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q} > x_n)}{h(x_n)} \to 0 \qquad \text{as} \qquad n \to \infty,$$

which indeed contradicts (3.6) since $1 - cx_n \to 1$ as $n \to \infty$. Observe that if $q > 0$ and h is not bounded at $0+$ then the second term in the latter equation tends to 0, because $p(\alpha) = 0$, by construction. So we just have to prove that the first term in the latter equation tends to 0. For this, we have

$$\begin{aligned} x_n\int_1^\infty \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z &= x_n\int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z \\ &\quad + x_n\int_{\hat{x}/x_n}^\infty \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z. \end{aligned} \tag{3.7}$$

We first deal with the first integral on the right hand side of (3.7). By construction of the sequence $(x_n)_{n\geq 0}$, we have $h(x_n z) \leq h(x_n)$ for any $z \in [1, \hat{x}/x_n]$, hence

$$x_n\int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z))z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z \leq x_n\int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z))z^{p(\alpha)}\,\mathrm{d}z. \tag{3.8}$$

If $q > 0$ or $\alpha = 0$ (recall $p(0) = 0$), we can take any $1 < z_0$ and write

$$x_n \int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z))\, \mathrm{d}z = x_n \int_1^{z_0} \overline{\Pi}(\log(z))\, \mathrm{d}z + x_n \int_{z_0}^{\hat{x}/x_n} \overline{\Pi}(\log(z))\, \mathrm{d}z$$
$$\leq x_n \int_1^{z_0} \overline{\Pi}(\log(z))\, \mathrm{d}z + x_n \left(\frac{\hat{x}}{x_n} - z_0\right) \overline{\Pi}(\log(z_0)),$$

where the inequality uses that $\overline{\Pi}$ is decreasing. Letting $n \to \infty$, recalling that $x_n \downarrow 0$, we see that the first integral on the right hand side vanishes while the second term tends to $\hat{x}\overline{\Pi}(\log z_0)$. As we can make this term arbitrarily small by choosing z_0 large enough, since $\overline{\Pi}(\log z) \to 0$ as $z \to \infty$, it follows indeed that (3.8) vanishes.

Next, let $\alpha > 0$. Since $\alpha - 1 < p(\alpha) < \alpha$, we can choose some $\beta \in (0, \alpha)$ such that $p(\alpha) - \beta + 1 \in (0, 1)$. Using this we find

$$x_n \int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z)) z^{p(\alpha)}\, \mathrm{d}z = x_n \int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z)) z^{\beta-1} z^{p(\alpha)-\beta+1}\, \mathrm{d}z$$
$$\leq x_n \left(\frac{\hat{x}}{x_n}\right)^{p(\alpha)-\beta+1} \int_1^{\hat{x}/x_n} \overline{\Pi}(\log(z)) z^{\beta-1}\, \mathrm{d}z,$$

and the right hand side indeed vanishes as $n \to \infty$, again since $x_n \downarrow 0$ and by (3.4).

It remains to show that the second integral on the right hand side of (3.7) vanishes as $n \to \infty$. We have

$$x_n \int_{\hat{x}/x_n}^{\infty} \overline{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)}\, \mathrm{d}z \leq x_n \overline{\Pi}(\log(\hat{x}/x_n)) \frac{1}{h(x_n)} \int_{\hat{x}/x_n}^{\infty} z^{p(\alpha)} h(x_n z)\, \mathrm{d}z$$
$$= \frac{\overline{\Pi}(\log(\hat{x}/x_n))}{x_n^{p(\alpha)}} \frac{1}{h(x_n)} \int_{\hat{x}}^{\infty} k(u)\, \mathrm{d}u,$$

where the inequality uses that $\overline{\Pi}$ is decreasing and the equality the definition of h together with the substitution $u = x_n z$. Since k is a density and by assumption $h(x_n) \to \infty$ as n goes to ∞, for the right hand side to vanish it remains to show that $\overline{\Pi}(\log \hat{x}/x_n)/x_n^{p(\alpha)}$ stays bounded as n increases. If $q > 0$ or $\alpha = 0$ (recall $p(0) = 0$) it is immediate since $\overline{\Pi}$ is decreasing. If $\alpha > 0$, for any $1 < z_0 < z$ integration by parts yields

$$\overline{\Pi}(\log(z)) z^{p(\alpha)} = p(\alpha) \int_{z_0}^{z} \overline{\Pi}(\log(u)) u^{p(\alpha)-1}\, \mathrm{d}u + \int_{z_0}^{z} u^{p(\alpha)}\, \mathrm{d}\overline{\Pi}(\log(u)) + \overline{\Pi}(\log(z_0)) z_0^{p(\alpha)}.$$

Now, if we let z goes to ∞, then since $p(\alpha) < \alpha$ we see from (3.4) that the first integral in the right hand side stays bounded while the second integral is negative on account of the fact that $\overline{\Pi}$ is decreasing. Consequently the left hand side has to stay bounded and we are done.

Step 3, case $q = 0$. Denote $C_\alpha = \mathbb{E}[I^{-\alpha}]$, which is finite by Step 1. From (2.5) we obtain for all $x > 0$,

$$(1 - cx) \frac{k(x)}{\overline{\Pi}(\log(1/x))} = \int_x^{\infty} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\, \mathrm{d}y. \tag{3.9}$$

Using this equation together with $k \geq 0$, Fatou's lemma and the fact that Π has an exponential tail (cf. (2.8)) yields

$$\begin{aligned}\liminf_{x\downarrow 0} \frac{k(x)}{\overline{\Pi}(\log(1/x))} &= \liminf_{x\downarrow 0} \frac{cxk(x)}{\overline{\Pi}(\log(1/x))} + \liminf_{x\downarrow 0} \int_x^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \\ &\geq \int_0^\infty y^{-\alpha} k(y)\,\mathrm{d}y = C_\alpha.\end{aligned}$$

On the other hand, for any $\varepsilon > 0$ we have as $x \downarrow 0$

$$\int_\varepsilon^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \to \int_\varepsilon^\infty y^{-\alpha} k(y)\,\mathrm{d}y \leq C_\alpha.$$

If $\alpha > 0$, this follows from the fact that the convergence (2.8) is uniform over $y \in [\varepsilon, \infty)$, see e.g. Theorem 1.5.2 in [5]. If $\alpha = 0$ this uniformity holds only over intervals of the form $[\varepsilon, x_0]$, in which case we can write the left hand side as the sum of integrals over $[\varepsilon, x_0]$ and $[x_0, \infty)$, the former in the limit again is bounded above by C_α, while for the latter we can use that $\overline{\Pi}$ is decreasing to see

$$\int_{x_0}^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \leq \frac{\overline{\Pi}(\log(x_0/x))}{\overline{\Pi}(\log(1/x))} \int_{x_0}^\infty k(y)\,\mathrm{d}y,$$

then letting first $x \to \infty$, thereby using (2.8), and then $x_0 \to \infty$ it follows that this term vanishes.

So it remains to show that

$$\limsup_{x\downarrow 0} \int_x^\varepsilon \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \to 0 \quad \text{as } \varepsilon \to 0.$$

For this, we get for ε small enough and $x < \varepsilon$:

$$\begin{aligned}\frac{1}{\overline{\Pi}(\log(1/x))} \int_x^\varepsilon \overline{\Pi}(\log(y/x)) k(y)\,\mathrm{d}y &= \frac{x}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) k(xz)\,\mathrm{d}z \\ &\leq \frac{Cx}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) (xz)^{p(\alpha)}\,\mathrm{d}z \\ &= \frac{Cx^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) z^{p(\alpha)}\,\mathrm{d}z \\ &\sim \frac{C' x^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \left(\frac{\varepsilon}{x}\right)^{p(\alpha)+1} \overline{\Pi}(\log(\varepsilon/x)) \quad \text{as } x \downarrow 0,\end{aligned}$$

where C and C' are constants, the inequality holds by Step 2 (cf. (3.5)) and the asymptotics follows from Karamata's theorem (see e.g. Theorem 1.5.11 in [5]), which indeed applies here since $\overline{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ (cf. (2.8)) and by construction (see Step 2) $p(\alpha) \geq \alpha - 1$. Now, using (2.8) we see that the ultimate right hand side goes to $C' \varepsilon^{p(\alpha)+1-\alpha}$ as $x \downarrow 0$ and this indeed vanishes as $\varepsilon \to 0$ since by construction $p(\alpha) + 1 - \alpha > 0$ for all $\alpha \geq 0$.

Step 3, case $q>0$. We will prove that

$$\int_x^\infty \overline{\Pi}(\log(y/x))k(y)dy \underset{x\to 0}{\longrightarrow} 0.$$

By Step 2, we can assume that k is bounded by $K\geq q$, in a neighborhood of $0+$. Let $\delta>1$ fixed, for x small enough we have that

$$\begin{aligned}\int_x^{x\delta} \overline{\Pi}(\log(y/x))k(y)\mathrm{d}y &\leq K\int_x^{x\delta} \overline{\Pi}(\log(y/x))\mathrm{d}y\\ &= K\int_0^{\log\delta} \overline{\Pi}(u)xe^u\mathrm{d}u\\ &\leq Kx\delta\int_0^{\log\delta} \overline{\Pi}(u)\mathrm{d}u \underset{x\to 0}{\longrightarrow} 0.\end{aligned}$$

Also, we have that

$$\int_{x\delta}^\infty \overline{\Pi}(\log(y/x))k(y)\mathrm{d}y \leq \overline{\Pi}(\log\delta)\int_{x\delta}^\infty k(y)\mathrm{d}y \underset{x\to 0}{\longrightarrow} \overline{\Pi}(\log\delta),$$

We conclude by making $\delta\to\infty$. Indeed, using equation (2.5) and the above arguments we conclude that

$$(1-cx)k(x)-q\mathbb{P}(I_{\mathbf{e}_q}>x) \underset{x\to 0}{\longrightarrow} 0,$$

from where the result follows.

□

4 Examples and some numerics

In this section, we illustrate Theorem 2.3 , Corollary 2.4 and equation (2.9) with some examples and we also provide some applications of Theorem 2.5.

Example 1. Let $q>0$ and consider the case when the subordinator is just a linear drift with $c>0$. By a simple Laplace inversion, we deduce $u_q(x)=c^{-1}e^{-\frac{q}{c}x}$. Thus, from identities (2.5) and (2.4) we get

$$(1-cx)k(x)=\frac{q}{c}\int_{[0,\infty)} k(xe^y)e^{-\frac{q}{c}y}\mathrm{d}y, \qquad x\in(0,1/c).$$

After straightforward computations, we deduce that the density of $I_{\mathbf{e}_q}$ is of the form

$$k(x)=q(1-cx)^{\frac{q}{c}-1}, \qquad x\in(0,1/c).$$

Let $\rho>0$ and note that

$$\phi_\rho(\theta)=c\theta+q\frac{\theta}{\theta+\rho} \qquad \text{and} \qquad c_\rho=\frac{q}{c^{\rho+1}}\frac{\rho(\rho+1)\Gamma(q/c)}{\Gamma(\rho+q/c+1)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator whose Laplace exponent is given by ϕ_ρ, satisfies

$$h(x) = c^{\rho+1}\frac{\Gamma(\rho+q/c+1)}{\Gamma(\rho+1)\Gamma(q/c)}x^{\rho}(1-cx)^{q/c-1}, \qquad \text{for} \quad x \in (0, 1/c),$$

in other words the exponential functional has the same law as $c^{-1}B(\rho+1, q/c)$, where $B(\rho+1, q/c)$ is a beta r. v. with parameters $(\rho+1, q/c)$.

Now, let us consider the associated spectrally negative Levy process Y whose Laplace exponent is written as follows

$$\psi(\lambda) = \frac{\lambda^2}{\phi_\rho(\lambda)} = \frac{\lambda(\lambda+\rho)}{c(\lambda+\rho)+q},$$

From (2.9), we deduce that the density of the exponential functional I_ψ associated to Y satisfies

$$k_\psi(x) = \frac{\rho c^{\rho+1}}{c\rho+q}\frac{\Gamma(\rho+q/c+1)}{\Gamma(\rho+1)\Gamma(q/c)}x^{-(\rho+q/c)}(x-c)^{q/c-1} \qquad \text{for} \quad x > c.$$

Hence I_ψ has the same law as $c(B(\rho, q/c))^{-1}$.

Example 2. Let $q = c = 0$, $\beta > 0$ and

$$\overline{\Pi}(z) = \frac{\beta}{\Gamma(a+1)}e^{-\frac{(s-1)}{a}z}\left(e^{\frac{z}{a}}-1\right)^{a-1},$$

where $a \in (0, 1]$ and $s \geq a$. Thus, the Laplace exponent ϕ has the form

$$\phi(\theta) = \beta\frac{\theta\Gamma(a(\theta-1)+s)}{\Gamma(a\theta+s)}.$$

In this case, the equation (2.5) can be written as follows

$$\begin{aligned} k(x) &= \frac{\beta}{\Gamma(a+1)}\int_x^\infty (y/x)^{-\frac{s-1}{a}}\left((y/x)^{\frac{1}{a}}-1\right)^{a-1}k(y)\mathrm{d}y \\ &= \frac{\beta x}{\Gamma(a)}\int_0^\infty (z+1)^{a-s}z^{a-1}k\Big(x(z+1)^a\Big)\mathrm{d}z, \end{aligned}$$

where we are using the change of variable $z = (y/x)^{\frac{1}{a}} - 1$. After some computations we deduce that

$$k(z) = \frac{\beta^{s/a}}{a\Gamma(s)}z^{\frac{s-a}{a}}e^{-(\beta z)^{\frac{1}{a}}}, \qquad \text{for} \quad z \geq 0. \tag{4.10}$$

In other words I has the same law as $\beta^{-1}\gamma_s^a$, where γ_s is a gamma r.v. with parameter s.

If $a = 1$, the process ξ is a compound Poisson process of parameter $\beta > 0$ with exponential jumps of mean $(s-1)^{-1} > 0$. From (4.10), it is clear that the law of its associated exponential functional has the same law as $\gamma_{(s,\beta)}$, a gamma r.v. with parameters (s,β).

We now consider the associated spectrally negative Levy process Y whose Laplace exponent satisfies

$$\psi(\lambda) = \frac{\lambda^2}{\phi(\lambda)} = \frac{\lambda\Gamma(a\lambda+s)}{\beta\Gamma(a(\lambda-1)+s)}.$$

The density of the exponential functional I_ψ associated to Y is given by

$$k_\psi(x) = \frac{\beta^{\frac{s-a}{a}}}{a\Gamma(s-a)} x^{-s/a} e^{-(\beta/x)^{1/a}}, \qquad x>0.$$

We remark that when $a=1$, the process ξ is a Brownian motion with drift and that the exponential functional I_ψ has the same law as $\gamma^{-1}_{(s-1,\beta)}$. This identity in law has been established by Dufresne [10].

Next, let $\rho>0$ and note that

$$\phi_\rho(\theta) = \beta\frac{\theta\Gamma(a(\theta+\rho-1)+s)}{\Gamma(a(\theta+\rho)+s)} \qquad \text{and} \qquad c_\rho = \frac{\Gamma(a\rho+s)}{\beta^\rho\Gamma(s)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator whose Laplace exponent is given by ϕ_ρ, satisfies

$$h(x) = \frac{\beta^{(s+a\rho)/a}}{a\Gamma(a\rho+s)} x^{(a\rho+s-a)/a} e^{-(\beta x)^{1/a}} \qquad \text{for} \quad x>0,$$

i.e. it has the same law as $\beta^{-1}\gamma^a_{a\rho+s}$. In particular, the density of the exponential functional of its associated spectrally negative Levy process satisfies

$$k_\psi(x) = \frac{\beta^{(s+a\rho-a)/a}}{a\Gamma(a(\rho-1)+s)} x^{-(a\rho+s)/a} e^{-(\beta/x)^{1/a}}, \quad x>0.$$

Example 3. Finally, let $a\in(0,1)$, $\beta\geq a$, $c=0$, $q=\Gamma(\beta)/\Gamma(\beta-a)$,

$$\overline{\Pi}(z) = \frac{1}{\Gamma(1-a)}\int_z^\infty \frac{e^{(1+a-\beta)x/a}}{(e^{x/a}-1)^{1+a}}\mathrm{d}x \quad \text{and} \quad u_q(z) = \frac{1}{\Gamma(a+1)} e^{-(\beta-1)z/a}\left(e^{\frac{z}{a}}-1\right)^{a-1}.$$

The process ξ with such characteristics is a killed Lamperti subordinator with parameters $(1/\Gamma(1-a), 1+a-\beta, 1/a, a)$, see Section 3.2 in Kuznetsov et al. [15] for a proper definition. From Theorem 1.3 the density of $I_{\mathbf{e}_q}$ satisfies the equation

$$k(x) = \int_0^\infty \left(\frac{xe^y}{\Gamma(1-a)}\int_z^\infty \frac{e^{(1+a-\beta)x/a}}{(e^{x/a}-1)^{1+a}}\mathrm{d}x + \frac{\Gamma(\beta)e^{-(\beta-1)z/a}}{\Gamma(\beta-a)\Gamma(a+1)}\left(e^{\frac{y}{a}}-1\right)^{a-1}\right) k(xe^y)\mathrm{d}y.$$

Since the above equation seems difficult to solve, we use the method of moments in order to determine the law of $I_{\mathbf{e}_q}$. We first note that

$$\mathbb{E}\left[I^n_{\mathbf{e}_q}\right] = \frac{n!\Gamma(\beta)}{\Gamma(an+\beta)},$$

and that in the case $\beta = 1$, the exponential functional $I_{\mathbf{e}_q}$ has the same distribution as X_a^{-a}, where X_a is a stable random variable, i.e.

$$\mathbb{E}\Big[e^{-\lambda X_a}\Big] = \exp\{-\lambda^a\}, \qquad \lambda \geq 0,$$

see Section 3 in [3]. Recall that the negative moments of X_a are given by

$$\mathbb{E}\Big[X_a^{-n}\Big] = \frac{\Gamma(1+n/a)}{\Gamma(1+n)}, \qquad n \geq 0.$$

Now we introduce $L_{(a,\beta)}$ and A, two independent r.v. whose laws are described as follows,

$$\mathbb{P}(L_{(a,\beta)} \in \mathrm{d}y) = \mathbb{E}\left[\frac{a\Gamma(\beta)}{\Gamma(\beta/a)X_a^{\beta}}; \frac{1}{X_a^a} \in \mathrm{d}y\right],$$

and

$$\mathbb{P}(A \in \mathrm{d}y) = \Big(\beta/a - 1\Big)(1-x)^{\beta/a-2}\mathbf{1}_{[0,1]}(x)\mathrm{d}x.$$

It is important to note from example 1, that A has the same law as the exponential functional associated to the subordinator σ which is defined as follows

$$\sigma_t = t + \beta/a - 1, \qquad t \geq 0.$$

On the one hand, it is clear that

$$\mathbb{E}\Big[L_{(a,\beta)}^n\Big] = \frac{a\Gamma(\beta)}{\Gamma(\beta/a)}\mathbb{E}\Big[X_a^{-(an+\beta)}\Big] = \frac{\Gamma(\beta)}{\Gamma(\beta/a)}\frac{\Gamma(n+\beta/a)}{\Gamma(an+\beta)},$$

and on the other hand, we have

$$\mathbb{E}\Big[A^n\Big] = \frac{\Gamma(n+1)\Gamma(\beta/a)}{\Gamma(n+\beta/a)},$$

which implies the $I_{\mathbf{e}_q}$ has the same law as $L_{a,\beta}A$.

Finally we numerically illustrate the density k and its asymptotic behaviour at 0 for some particular subordinators ζ. Let us first shortly discuss the method we used. Clearly the equation (1.2) motivates the following straightforward discretisation procedure: approximate k by a step function $\tilde{k}$, i.e.

$$\tilde{k}(x) = \sum_{i=0}^{N-1} \mathbf{1}_{\{x\in[x_i,x_{i+1})\}} y_i,$$

where $0 = x_0 < x_1 < \ldots < x_N = 1/c$ forms a grid on the x-axis. The heights y_i can then be found by iterating over $i = N-1, \ldots, 0$, thereby at each step using (1.2) with $x = x_i$ and k replaced by $\tilde{k}$. Two remarks are in place here.

Firstly, as (1.2) is linear in k the condition that k is a density is required to uniquely determine the solution. This translates to the fact that the numerical procedure discussed above requires a starting point, i.e. the value $y_{N-1} > 0$ should be known. (Of course, starting with $y_N = 0$ yields $\tilde{k} \equiv 0$.) We proceed by first leaving y_{N-1} undetermined, run the iteration so that every y_i in fact becomes a linear function of y_{N-1}, and then find y_{N-1} by requiring that $\tilde{k}$ integrates to 1.

The second remark is that even though any choice of grid would in principle work, we found one in particular to be useful. Indeed, if we set $x_n = (1/c)\Delta^{N-n}$ for some Δ less than (but typically very close to) 1, equation (2.5) yields the following relation:

$$\begin{aligned}(1 - cx_n)y_n &= \int_{x_n}^{\infty} \overline{\Pi}(\log(y/x_n))\tilde{k}(y)\,\mathrm{d}y = x_n \int_1^{\infty} \overline{\Pi}(\log(z))\tilde{k}(x_n z)\,\mathrm{d}z \\ &= x_n \sum_{i=n}^{N-1} y_i \int_1^{\infty} \overline{\Pi}(\log(z))\mathbf{1}_{\{x_n z \in [x_i, x_{i+1})\}}\, vz = x_n \sum_{i=n}^{N-1} y_i \int_{\Delta^{n-i}}^{\Delta^{n-i-1}} \overline{\Pi}(\log(z))\,\mathrm{d}z.\end{aligned}$$

The approximation this setup yields is very efficient in comparison with e.g. the approximation using a standard equidistant grid, due to the fact that in this case we need to evaluate only N different integrals numerically*.

First we consider two examples for which the density k of I is explicitly known. The first one is taken from Example 2 with $a = 1$, $\beta = 2$ and $s = 3/2$. In this case from (4.10), we have

$$k(x) = \frac{2^{5/2}}{\sqrt{\pi}} x^{1/2} e^{-2x} \quad \text{for } x > 0.$$

See Figures 1-4 for plots of the density k, the difference $\tilde{k} - k$ (where $\tilde{k}$ is obtained by the above method with $\Delta = 0.998$, yielding a grid of ≈ 4500 points and a few minutes computation time on an average laptop), the ratio $k(x)/\overline{\Pi}(\log(1/x))$ and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$ respectively.

The second explicit example is taken again from Example 2 with $\beta = 1$ and $s = 1$ and $a = 1/2$. In this case from (4.10), we have

$$k(x) = 2xe^{-x^2} \quad \text{for } x > 0.$$

It is important to note that $\overline{\Pi}$ satisfies (**A**) with $\alpha = 1$. In this case, Figures 5 -8 show plots plots of the density k, the difference $\tilde{k} - k$, the ratio $k(x)/\overline{\Pi}(\log(1/x))$ and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$ respectively.

Next we look at two examples where no formula for k is available. The first one is when ξ is a stable subordinator with drift, i.e. $c = 1$ and $\Pi(\mathrm{d}x) = x^{-1-a}\,\mathrm{d}x$, where we take $a = 1/4$. See Figures 9 & 10 for a plot of $\tilde{k}$ and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$ respectively. Note that this is an example of a Lévy measure satisfying (2.8) with parameter 0.

Finally, the second example is a subordinator ξ with zero drift and Lévy measure of the form $\Pi(\mathrm{d}x) = x^{-1/4}\exp(-x^n)\,\mathrm{d}x$. Figure 11 shows $\tilde{k}$ for $n = 1$ (blue), $n = 2$ (purple)

*All computations were done in the open source computer algebra system SAGE: www.sagemath.org and $n = 3$ (green) respectively. Figure 12 shows the ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$ for the case $n = 1$, since then (**A**) is satisfied with $\alpha = 1$.

References

[1] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press, Cambridge.

[2] Bertoin, J. and Lindner, A. and Maller, R. (2008) On continuity properties of the law of integrals of Lévy processes. *Séminaire de probabilités XLI*, 137–159, Lecture Notes in Math., 1934, Springer, Berlin.

[3] Bertoin, J. and Yor, M. (2001) On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Comm. Probab.,* **6**, 95–106.

[4] Bertoin, J. and Yor, M. (2005) Exponential functionals of Lévy processes. *Probab. Surv.,* **2**, 191–212.

[5] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) Regular variation. Cambridge University Press, Cambridge.

[6] Caballero, M.E. and Rivero, V. (2009) On the asymptotic behaviour of increasing self-similar Markov processes. *Electron. J. Probab.,* **14**, 865–894.

[7] Carmona, P., Petit, F. and Yor, M. (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. *Exponential functionals and principal values related to Brownian motion*, 73–121, Bibl. Rev. Mat. Iberoamericana.

[8] Chaumont, L. and Pardo, J.C. (2006) The lower envelope of positive self-similar Markov processes. *Electron. J. Probab.,* **11**, 1321–1341.

[9] Chazal, M., Kyprianou, A. and Patie, P. (2010) A transformation for Lévy processes with one-sided jumps and applications. `http://arxiv.org/abs/1010.3819`

[10] Dufresne, D. (1990) The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, **1-2**, 39–79.

[11] Haas, B. (2010) Asymptotic behavior of solutions to the fragmentation equation with shattering: an approach via self-similar Markov processes. *Ann. Appl. Proba.*, **20** (2), 382–429.

[12] Haas, B. and Rivero, V. (2011) Quasi-stationary distributions and Yaglom limits of self-similar Markov processes. *Preprint.*

[13] KYPRIANOU, A. E. AND RIVERO, V. (2008) Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.*, **13**, 1672–1701.

[14] KUZNETSOV, A. (2010) On the distribution of exponential functionals for Lévy processes with jumps of rational transform. http://arxiv.org/abs/1011.3856

[15] KUZNETSOV, A., KYPRIANOU, A., PARDO, J. C. AND VAN SCHAIK, K. (2011) A Wiener-Hopf Monte Carlo simulation technique for Lévy processes. *To appear in Ann. of Appl. Probab.*

[16] KUZNETSOV, A. AND PARDO, J.C. (2010) Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. http://arxiv.org/abs/1012.0817

[17] LAMPERTI, J (1972) Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete,* **22**, 205–225.

[18] MAULIK, K. AND ZWART, B. (2006) Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Proc. Appl.*, **116**, 156–177.

[19] PATIE, P. (2011) Law of the absorption time of some positive self-similar Markov processes. *To appear in Ann. of Probab.*

[20] PARDO, J.C. (2006) On the future infimum of positive self-similar Markov processes. *Stochastics,* **78**, 123–155.

[21] RIVERO, V. (2003) A law of iterated logarithm for increasing self-similar Markov processes. *Stochastic Stochastic Rep.,* **75**, 443–472.

[22] RIVERO, V. (2005) Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli,* **11**, 471–509.

[23] SATO, K. (1999) *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge.

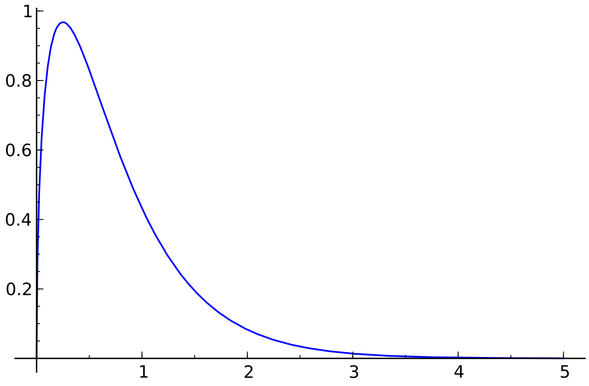

Figure 1: The density function k

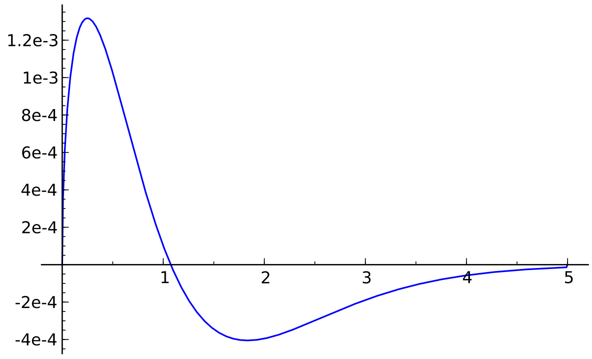

Figure 2: The difference $\tilde{k} - k$

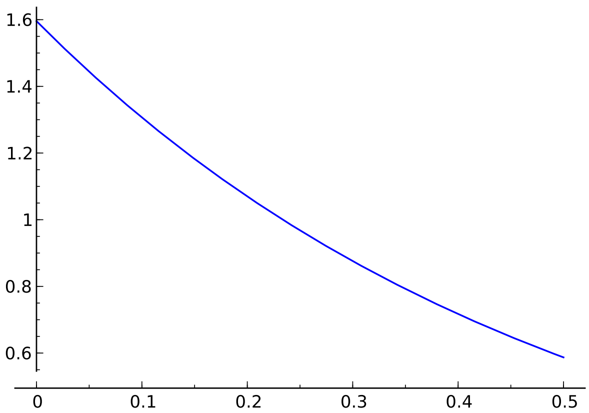

Figure 3: The ratio $k(x)/\overline{\Pi}(\log 1/x)$

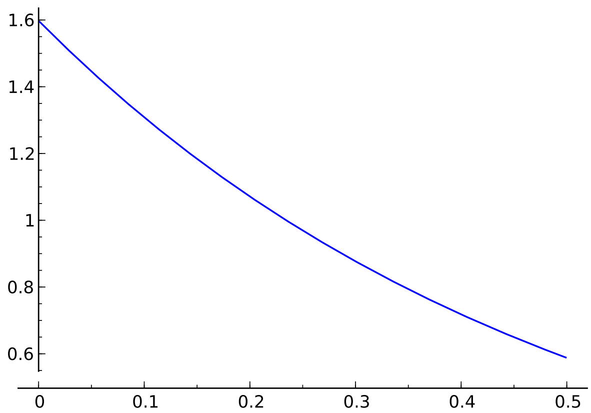

Figure 4: The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$

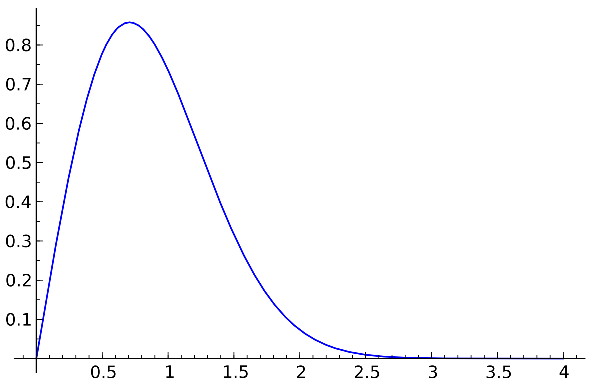

Figure 5: The density function k

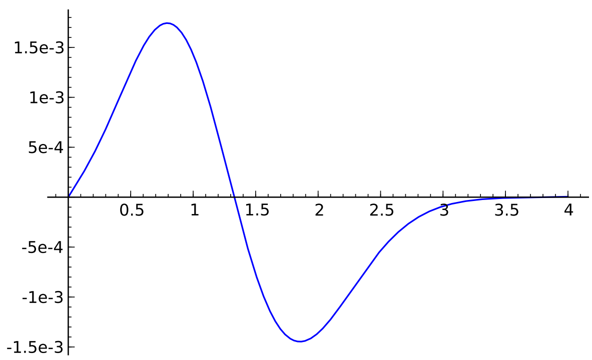

Figure 6: The difference $\tilde{k} - k$

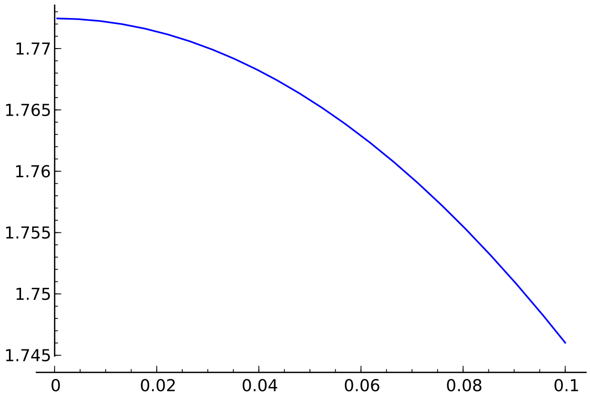

Figure 7: The ratio $k(x)/\overline{\Pi}(\log 1/x)$

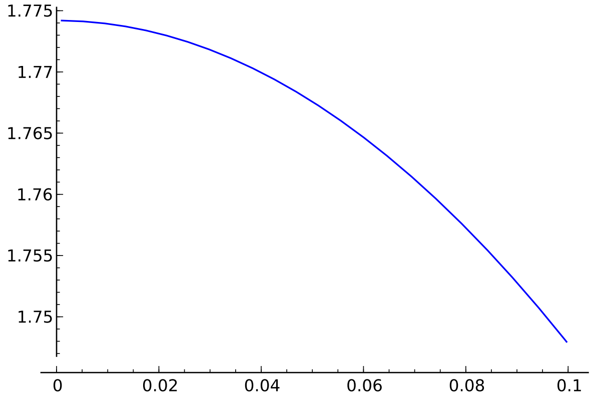

Figure 8: The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$

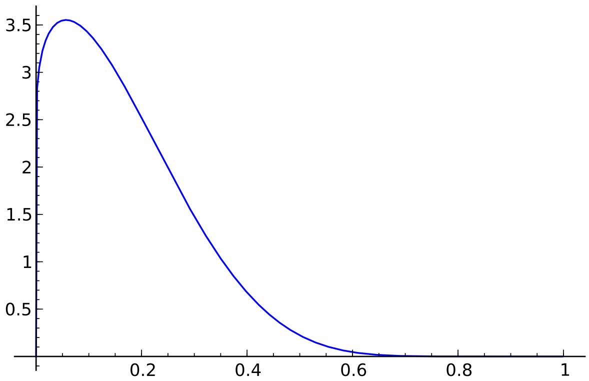

Figure 9: The density function $\tilde{k}$

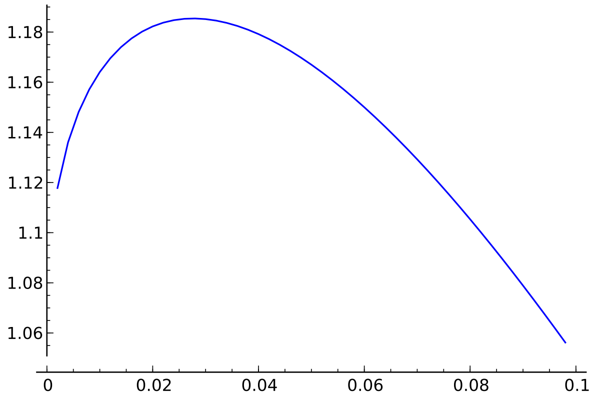

Figure 10: The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$

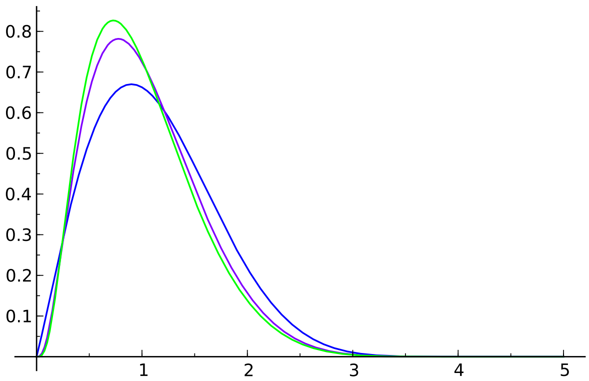

Figure 11: Density functions $\tilde{k}$

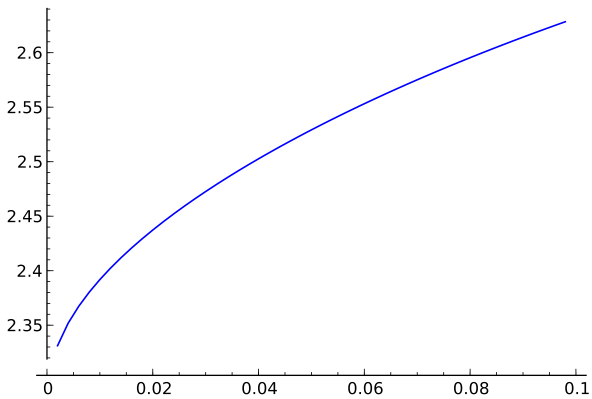

Figure 12: The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$