Orbital stability of standing waves of some m-coupled nonlinear Schrödinger equations

Hichem Hajaiej

Abstract

We extend the notion of orbital stability to systems of nonlinear Schrödinger equations, then we prove this property under suitable assumptions of the local nonlinearity involved.

1 Intoduction

In [4], the author has studied the following Cauchy problem :

$$\begin{cases} i\partial_t\Phi_1 + \Delta\Phi_1 + h_1(x, |\Phi_1|^2, ..., |\Phi_\ell|^2)\Phi_1 & = 0 \\ \vdots \quad \vdots \quad \vdots & \\ i\partial_t\Phi_\ell + \Delta\Phi_\ell + h_\ell(x, |\Phi_1|^2, ..., |\Phi_\ell|^2)\Phi_\ell & = 0 \\ \Phi_j(0,x) = \Phi_j^0(x) & \text{for } 1 \le j \le \ell \end{cases} \tag{1.1}$$

$\Phi_j^0 : \mathbb{R}^N \to \mathbb{C}, h_j : \mathbb{R} \times \mathbb{R}_+^\ell \to \mathbb{R}$ and $\Phi_j : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ (1.1) has numerous applications in engineering and physics. It appears in the study of spatial solitons in nonlinear wave guides, the theory of Bose-Einstein condensates, optical pulse propagation in briefringent fibers, interactions of m-wave packets, wavelength division multiplexed optical systems, see [3] and references therein. Physically Φ_j is the jth component of the beam in Kerr-like photorefractive media. In these contexts it is always possible to write (1.1) in a compact vectorial form :

$$\begin{cases} i\frac{\partial\vec{\Phi}}{\partial t} & = \hat{E}'(\vec{\Phi}) \\ \vec{\Phi}(0,x) & = \vec{\Phi}^0 = (\Phi_1^0, ..., \Phi_\ell^0) \end{cases}$$

where

$$\hat{E}(\vec{\Phi}) = \frac{1}{2}\left\{|\nabla\vec{\Phi}|_2^2 - \int H(x, \Phi_1, .., \Phi_\ell)\right\} \tag{1.3}$$

H is such that :

$$\frac{\partial H}{\partial s_j}(x, s_1, ..., s_\ell) = 2h_j(x, s_1^2, ..., s_\ell^2)s_j \tag{1.4}$$

Thus

$$\frac{\partial H}{\partial s_j}(x, s_1, ..., s_\ell) = \frac{\partial H}{\partial s_j}(x, |s_1|, ..., |s_\ell|) \tag{1.5}$$

Note that when $\ell = 1$, $H(x, s) = \int_0^{s^2} h(x, t)dt$.
A soliton or standing wave of (1.1) is a solution of (1.1) having the particular form : $\vec{\Phi}(t, x) = (\Phi_1(t, x), ..., \Phi_\ell(t, x))$ where $\Phi_j(t, x) = u_j(x)e^{-i\lambda_j t}$; λj are real numbers.
Hence $\vec{u} = (u_1, ..., u_\ell)$ is a solution of the following $m \times m$ elliptic eigenvalue problem :

$$\begin{cases} \Delta u_1 + h_1(x, u_1^2, ..., u_\ell^2)u_1 + \lambda_1 u_1 & = 0 \\ \vdots & \vdots \ \vdots \\ \Delta u_\ell + h_\ell(x, u_1^2, ..., u_\ell^2)u_\ell + \lambda_\ell u_\ell & = 0 \end{cases} \tag{1.6}$$

when $\ell = 1$, (1.6) becomes :

$$\Delta w + h(x, w^2)w + \lambda w = 0 \tag{1.7}$$

where $w \in H^1(\mathbb{R}^N, \mathbb{C})$; which can be written as a 2×2 real elliptic system for (u, v) where $w = (u, v) = u + iv$; namely

$$\begin{cases} \Delta u + h(x, u^2 + v^2)u + \lambda u & = 0 \\ \Delta v + h(x, u^2 + v^2)v + \lambda v & = 0 \end{cases} \tag{1.8}$$

where $v \equiv 0$; (1.8) leads to the scalar equation

$$\Delta u + h(x, u^2)u + \lambda u = 0. \tag{1.9}$$

(1.9) constitues in itself an important chapter of nonlinear analysis in which many brilliant mathematicians as Berger, Cazenave, Berestyski, Nehari and Lions, have intensively contributed. *A* special attention was addressed to the case $h(x, s^2) = |s|^{p-1}$. The famous concentration-compactness principle was built up by Lions to study the orbital stability of standing waves of (1.9) [8],[1].
In the scalar setting, there are two approaches to determine the orbital stability of standing waves of (1.1) . The first one reduces this question to the checking of the strict inequality $\frac{d}{d\lambda}\int u_\lambda^2 < 0$ for certain solutions u_λ of

(1.9). For non-autonomous equations, it is hard to establish conditions on the nonlinearity h ensuring the latter monotonicity property, [7] and references therein. In the vectorial setting, it does not seem possible to extend this approach for (1.6). The second alternative exploits the hamiltonian structure of (1.1) when $\ell = 1$ via the characterization of standing waves as constrained minimum. We will adapt this approach to generalize the notion of orbital stability of standing waves of (1.1). We will then establish stability of the latter particular solutions under general assumptions on H including the most relevant physical situations where $H(x, \vec{s})$ converges to a function $H^\infty(x, \vec{s})$ that depends periodically on x.
Before formulating the notion of orbital stability of standing waves of (1.1), let us first introduce some useful notation :

$$\begin{aligned}
\vec{H} &= H^1(\mathbb{R}^N, \mathbb{C}) \times ... \times H^1(\mathbb{R}^N, \mathbb{C}); H^1(\mathbb{R}^N, \mathbb{C}) = H \\
\vec{H}^1 &= H^1(\mathbb{R}^N, \mathbb{R}) \times ... \times H^1(\mathbb{R}^N, \mathbb{R}); H^1(\mathbb{R}^N, \mathbb{R}) = H^1 \\
\text{For } z &= (u, v)\ ; |z|_H^2 = |z|_2^2 + |\nabla z|_2^2 \\
|z|_2^2 &= |u|_2^2 + |v|_2^2\ ; \quad |\nabla z|_2^2 = |\nabla u|_2^2 + |\nabla v|_2^2
\end{aligned}$$

$| \, |_p$ denotes the usual norm on $L^p(\mathbb{R}^N, \mathbb{R}) = L^p$

$$\vec{z} = (z_1, ..., z_\ell) = ((u_1, v_1), ..., (u_\ell, v_\ell)) = (\vec{u}, \vec{v})$$

where

$$z_j = u_j + iv_j = (u_j, v_j).$$

The modulus of the vector $\vec{z}$, denoted by $|\vec{z}|$ is the vector

$$|\vec{z}| = (|\vec{z}_1|, ..., |\vec{z}_\ell|)\ ; \quad |z_j| = (u_j^2 + v_j^2)^{1/2}$$

Let us now define the following functionals : $\hat{E} : \vec{H} \to \mathbb{R}$ and $E : \vec{H}^1 \to \mathbb{R}$.

$$\begin{aligned}
\hat{E}(\vec{z}) = \hat{E}(\vec{u}, \vec{v}) &= \frac{1}{2}\{|\nabla \vec{z}|_2^2 - \int H(x, |\vec{z}|)\} \\
&= \frac{1}{2}\{\sum_{j=1}^{\ell}\{|\nabla u_j|_2^2 + |\nabla v_j|_2^2\} - \int H(x, |\vec{z}|)\} \\
&= \frac{1}{2}\{\sum_{j=1}^{\ell} |\nabla u_j|_2^2 + |\nabla v_j|_2^2 - \int H(x, (u_1^2 + v_1^2)^{1/2}, ..., (u_\ell^2 + v_\ell^2)^{1/2}\}
\end{aligned}$$

$$E(u) = \vec{E}(u, 0) = \frac{1}{2}\{|\nabla \vec{u}|_2^2 - \int H(x, |\vec{u}|)\}$$

For $c_1, .., c_\ell > 0$, we set $c^2 = \sum_{i=1}^{\ell} c_i^2$ and :

$$\vec{S}_c = \{\vec{z} \in \vec{H} : |z_i|_2^2 = c_i^2 \quad 1\, ; \leq i \leq \ell\}$$

$$S_c = \{\vec{u} \in \vec{H}^1 : |u_i|_2^2 = c_i^2 \quad 1 \leq i \leq \ell\}$$

$$\hat{I}_{c_1,\ldots,c_\ell} = \inf\{\hat{E}(\vec{z}) : \vec{z} \in \vec{S}_c\}$$

and

$$I_{c_1,\ldots,c_\ell} = \inf\{E(\vec{u}); \vec{u} \in S_c\}$$

$$\hat{O}_c = \{\vec{z} \in \hat{S}_c : \hat{E}(\vec{z}) = \hat{I}_{c_1,\ldots,c_2}\}$$

From now on we fix $c_1, \ldots, c_\ell > 0$ and $c^2 = \sum_{i=1}^{\ell} c_i^2$.
Following the definition in the scalar setting, we will say that $\hat{O}_c$ is stable if it is not empty and :

$$\begin{cases} \forall\, \vec{w} \in \hat{O}_c \text{ and } \forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ such that} \\ \text{ for any } \vec{\Phi}_0 \in \vec{H} \text{ such that } |\vec{\Phi}_0 - \vec{z}|_{\vec{H}} < \delta, \text{ it follows that} \\ \inf_{\vec{z}\in\hat{O}_c} |\vec{\Phi}(t,.) - \vec{w}|_{\vec{H}} < \varepsilon \quad \forall\, t \in \mathbb{R} \end{cases} \tag{1.10}$$

$\vec{\Phi}(t,.)$ designs the solution of (1.1) corresponding to the initial condition $\vec{\Phi}_0$. Hence we take advantage of the recent result established in [4], in which the author has determined assumptions on h_j ensuring the existence and uniqueness of global solutions of (1.1). Under slight modifications of Theorem 2.11 and Theorem 3.1 of [4], we have the following result.
Theorem 0.1 : Let $H : \mathbb{R} \times \mathbb{R}_+^\ell \to \mathbb{R}$ be a Carathéodory function such that:
(H_0) There exist $K > 0$ and $0 < \ell_1 < \frac{4}{N}$ such that

$$0 \leq H(x, \vec{s}) \leq K(|\vec{s}|^2 + |\vec{s}|^{\ell_1+2})$$

for any $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}_+^\ell$.
(H_1) If $N \geq 2$, there exist constants $c' > 0$ and $\alpha \in [0, \frac{4}{N-2})$; for $N \geq 3, \alpha \in [0, \infty)$ for $N = 2$ such that

$$|h_j(x, |\vec{s}|^2)s_j - h_j(x, |\vec{r}|^2)r_j| \leq c'\{1 + |\vec{s}|^\alpha + |\vec{r}|^\alpha\}|\vec{s} - \vec{r}|$$

for all $1 \leq j \leq \ell, \vec{r}, \vec{s} \in \mathbb{R}_+^\ell$.

If $N = 1$, for any $R > 0$, there exists a constant $L(R) > 0$ such that $|h_j(x, |\vec{s}|^2)s_j - h_j(x, |\vec{r}|^2)r_j| \leq L(R)|\vec{s} - \vec{r}|$ for all $\vec{s}, \vec{r} \in \mathbb{R}_+^\ell$ such that $|\vec{r}| + |\vec{s}| \leq R$.
Then for every $\vec{\Phi}_0 \in \vec{H}$, the initial value, the Cauchy problem (1.1) has a

unique solution $\vec{\Phi} \in C(\mathbb{R}, \vec{H}) \cap C^1(\mathbb{R}, (\vec{H}^{-1})$. Furthermore $\sup_{t\in\mathbb{R}} |\vec{\Phi}(t,.)|_{\vec{H}} < \infty$; and we have conservation of charges and energy ; namely

$$(C_1) \quad |\Phi_j(t,.)|_2 = |\Phi_0^j|_2 \quad \forall\, 1 \le j \le \ell \text{ and } \forall\, t \in \mathbb{R}$$

$$(C_2) \qquad \hat{E}(\vec{\Phi}(t,.)) = \hat{E}(\vec{\Phi}_0) \quad \forall\, t \in \mathbb{R}.$$

Once one knows that (1.1) admits a unique solution, it is worth to argue by contradiction to establish (1.10) :
Suppose that $\hat{O}_c$ is not stable, then either $\hat{O}_c$ is empty or :
There exist $\vec{w} \in \hat{O}_c, \varepsilon_0 > 0$ and a sequence $\{\vec{\Phi}_0^n\} \in \vec{H}$ such that :

$$|\vec{\Phi}_0^n - \vec{w}|_{\vec{H}} \to 0 \text{ as } n \to \infty \text{ but } \inf_{\vec{z}\in\hat{O}_c} |\vec{\Phi}^n(t_n,.) - \vec{z}|_{\vec{H}} \ge \varepsilon_0\} \tag{1.11}$$

for some sequence $\{t_n\} \subset \mathbb{R}$, where $\Phi^n(t_n,.)$ is the solution of (1.1) corresponding to the initial condition $\vec{\Phi}_0^n$.
Let $\vec{w}_n = \vec{\Phi}^n(t_n,.)$; since $\vec{w} \in \hat{S}_c$ and $\hat{E}(\vec{w}) = \hat{I}_{c_1,\ldots,c_\ell}$ it follows from the continuity of $|\ |_2$ and $\hat{E}$ on $\vec{H}$ (Proposition 2.1) that : $|\Phi_{0,j}^n|_2 \to c_j\ \forall\, 1 \le j \le \ell$ and $\hat{E}(\vec{w}_n) = \hat{E}(\Phi_0^n) = \hat{I}_{c_1,\ldots,c_\ell}$. Thus it follows from Theorem 0.1 that

$$|w_{n,j}|_2 = |\Phi_{0,j}^n|_2 \to c_j \quad \forall\, 1 \le j \le \ell$$

and

$$\hat{E}(\vec{w}_n) = \hat{E}(\vec{\Phi}_0^n) \to \hat{I}_{c_1,\ldots,c_\ell}.$$

If $\{\vec{w}_n\}$ admits a subsequence converging to an element $\vec{w} \in \vec{H}\ \vec{w} = (w_1, \ldots, w_\ell)$ then $|w_j|_2 \to c_j$ and $\hat{E}(\vec{w}) = \hat{I}_{c_1,\ldots,c_\ell}$ showing that $\vec{w} \in \hat{O}_c$ but $\inf_{\vec{z}\in\hat{O}_c} |\vec{\Phi}^n(t_n,.) - \vec{z}|_{\vec{H}} \le \vec{w}_n - \vec{w}|_{\vec{H}}$ contradicting (1.11) . Hence to show the orbital stability of $\hat{O}_c$, one has to prove that $\hat{O}_c$ is not empty and :

$$\begin{cases} \text{Every sequence } \{\vec{w}_n\} \subset \vec{H} \text{ such that } |w_{n,j}|_2 \to c_j \\ \text{for } 1 \le j \le \ell \text{ and } \hat{E}(\vec{w}_n) \to \hat{I}_{c_1,\ldots,c_\ell} \end{cases} \tag{1.12}$$

is relatively compact in $\vec{H}$.
In the following $\{\vec{w}_n\}$ denotes a sequence satisfying (1.12). Our objective is to prove that $\{\vec{w}_n\}$ admits a subsequence converging to an element $\vec{w} \in \vec{H}$.
Our line of attack consists in the following steps :
Step 1 : If $\{\vec{w}_n\}$ satisfies (1.12) then the sequence

$$|\vec{w}_n| = (|\vec{w}_{n,1}|, \ldots, |\vec{w}_{n,\ell}|) \quad \text{is such that } E(|\vec{w}_n|) \to I_{c_1,\ldots,c_\ell} \text{ and } |w_{n,j}|_2^2 \to c_j$$

In [5], the author has established assumptions on H ensuring that such a sequence is relatively compact in $\vec{H}^1$. It can be easily deduced from Theorem

1.1 of [5] that :
Theorem 0.2 : Suppose that H satisfies (H_0), (H_1) and
(H_2) there exists $B > 0$ such that

$$|\partial_j H(x,\vec{s})| \leq B(|\vec{s}| + |\vec{s}|^{\ell_1+1}) \quad \text{for all } x \in \mathbb{R}^N$$

and $\vec{s} \in \mathbb{R}^\ell_+; 1 \leq j \leq \ell$
(H_3) $\exists\, \Delta > 0, R > 0, s > 0, \alpha_1, ..., \alpha_\ell > 0, \quad t \in [0,2)$
such that $H(x,\vec{s}) > \Delta|x|^{-t}|s_1|^{\alpha_1}...|s_\ell|^{\alpha_\ell}$ for all $|x| \geq R$ and $|\vec{s}| < S$ where $N + 2 > \frac{N}{2}\alpha + t\,; \alpha = \sum_{j=1}^{\ell}\alpha_j$.
$(H_4) H(x, \theta_1 s_1, ..., \theta_\ell s_\ell) \geq \theta^2_{max} H(x, s_1, ..., s_\ell)$ for all $x \in \mathbb{R}^N$ $s_i \in \mathbb{R}, \theta_i \geq 1$ where $\theta_{max} = \max_{1\leq j\leq \ell}\theta_j$.
There exists a periodic function $H^\infty(x,\vec{s})$ (i.e, $\exists\, T \in \mathbb{Z}^N$ such that $H^\infty(x,+T,\vec{s}) = H^\infty(x,\vec{s}), \forall\, x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^\ell_+$) satisfying (H_3) and such that :
(H_5) There exists $0 < \Gamma < \frac{4}{N}$ such that

$$\lim_{|x|\to\infty} \frac{H(x,\vec{s}) - H^\infty(x,\vec{s})}{|\vec{s}|^2 + |\vec{s}|^{\Gamma+2}} = 0 \quad \text{uniformly for any } \vec{s}$$

(H_6) There exist $A', B' > 0$ and $0 < \beta < \ell_1 < \frac{4}{N}$ such that

$$0 \leq H^\infty(x,\vec{s}) \leq A'(|\vec{s}|^{\beta+2} + |\vec{s}|^{\ell_1+2}$$

and $\forall\, 1 \leq j \leq \ell$:

$$\partial_i H^\infty(x,\vec{s}) \leq \vec{B}'(|\vec{s}|^{\beta+1} + |\vec{s}|^{\ell_1+1}) \quad \forall\, x \in \mathbb{R}^N\,; \vec{s} \in \mathbb{R}^\ell_+\,.$$

(H_7) There exists $\sigma \in (0, \frac{4}{N})$ such that :

$$H^\infty(x, \theta_1 s_1, ..., \theta_\ell s_\ell) \geq \theta^{\sigma+2}_{max} H^\infty(x, s_1, ..., s_\ell)$$

for any $\theta_i \geq 1, x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^\ell_+$, where $\theta_{max} = \max_{1\leq j\leq \ell}\theta_i$.
Then any sequence $\{\vec{u}_n\} \subset \vec{H}^1$ such that $|u_{n,j}|^2_2 \to c_j$ and $E(\vec{u}_n) \to I_{c_1,...,c_\ell}$ admits a subsequence converging to $\vec{u} \in S_c$. Using this important information, step 2 consists of :
Step 2 : By the latter, we now know that there exists $\vec{w} \in \vec{H}^1$ such that $(u^2_{n,j} + v^2_{n,j})^{1/2}$ converges to w_j in H^1 for any $1 \leq j \leq \ell$.
On the other hand, it follows by Proposition 2.2 that $\vec{w}_n = ((u_{n,1}, v_{n,1}), (..., (u_{n,\ell}, v_{n,\ell}))$ is bounded in $\vec{H}$. Hence up to a subsequence, we may suppose that

$$u_{n,j} > u_j \quad \text{and} \quad v_{n,j} > v_j \quad \forall\, 1 \leq j \leq \ell$$

In this step, we will prove that $w_j = (u_j^2 + v_j^2)^{1/2}\ \forall\ 1 \le j \le \ell$
Step 3 : We will establish some estimates on $|\nabla \vec{w}_n|_2^2 - |\nabla|\vec{w}_n||_2^2$, which will enable us to prove that $w_{n,j} \to w_j\ \forall\ 1 \le j \le \ell$ which concludes the proof and here is our main result :
Theorem 1.1. Suppose that (H_0) to (H_7) are satisfied then for any $c_1, ..., c_\ell > 0$, the orbit $\hat{O}_c$ is stable.

2 Preliminaries

Following the proof of Lemma 3.1 of [5], we can easily derive the following proposition.
Proposition 2.1 : Under the hypothesis (H_0), the functionals $\hat{E}$ and E are continuous and have the below properties

1. There exists a constant $C > 0$ such that
$$\hat{E}(\vec{z}) \ge \frac{1}{4}|\nabla \vec{z}|_2^2 - C(c^2 + c^\gamma)$$
for all $\vec{z} \in \hat{S}_c$ and all $c_1, ..., c_\ell > 0$ where
$$\gamma = \frac{2(2\ell_1 + 4 - N\ell_1)}{4 - N\ell_1} > 2$$
2. For all $c_1, ..., c_\ell > 0$, $I_{c_1,...,c_\ell} \ge \hat{I}_{c_1,...,c_\ell} > -\infty$ and any minimizing sequences for $I_{c_1,...,c_\ell}$ and $\hat{I}_{c_1,...,c_\ell}$ are bounded in $\vec{H}^1$ (resp. $\vec{H}$).
3. $(c_1, ..., c_\ell) \to I_{c_1,...,c_\ell}$ is continuous on $(0, \infty)^\ell$.
Now for the convenience of the under, let us recall a classical.

Proposition 2.2.
Let $u, v \in H^1$, then $(u^2 + v^2)^{1/2} \in H^1$ and for $1 \le i \le N$
$$\partial_i (u^2 + v^2)^{1/2} = \begin{cases} \frac{u\partial_i u + v\partial_i v}{(u^2+v^2)^{1/2}} & \text{if } u^2 + v^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof For $\varepsilon > 0$, set
$$\begin{array}{rcl} \psi^\varepsilon : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (s_1, s_2) & \longmapsto & (s_1^2 + s_2^2 + \varepsilon^2)^{1/2} - \varepsilon \end{array}$$
Clearly $\psi^\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}), \psi^\varepsilon(0,0) = 0$ and $\sup |\nabla \psi^\varepsilon| < \infty_s$, it then follows by [9] that
$$\int \left\{ (u^2 + v^2 + \varepsilon)^{1/2} - \varepsilon \right\} \partial_i \xi = -\int \frac{u\partial_i u + v\partial_i v}{(u^2 + v^2 + \varepsilon^2)^{1/2}} \xi.$$

for any $\xi \in C_0^\infty$.
Since $0 \leq (u^2+v^2+\varepsilon^2)^{1/2} - \varepsilon \leq (u^2+v^2)^{1/2}$ and

$$\left|\frac{u\partial_i u + v\partial_i v}{(u^2+v^2+\varepsilon^2)^{1/2}}\right| \leq |\partial_i u| + |\partial_i v|,$$

we obtain

$$\int \{u^2(x)+v^2(x)\}^{1/2}\partial_i\xi(x) = \int \lim_{\varepsilon\to 0^+} \frac{u(x)\partial_i(x) + v(x)\partial_i v(x)}{u^2(x)+v^2(x)+\varepsilon^2)^{1/2}}\xi(x)$$

thanks to the dominated convergence theorem.

3 Proof of Theorem 1.1

Let $\vec{w}_n = (w_{n,1}, ..., w_{n,\ell}) = (\vec{u}_n, \vec{v}_n) = ((u_{n,1}, v_{n,1})..., (u_{n,\ell}, v_{n,\ell}))$ be a sequence in $\vec{H}$ such that $|w_{n,j}|_2 \to c_j$ and $\hat{E}(\vec{w}_n) \to \hat{I}_{c_1,...,c_\ell}$. We will prove that $\{\vec{w}_n\}$ has subsequence converging in $\vec{H}$
Setting $|\vec{w}_n| = (c_\ell|w_{n,1}|, ..., |w_{n,\ell}|)$, it follows by Proposition 2.2 that $|w_{n,j}| = (u_{n,j}^2 + v_{n,j}^2)^{1/2} \in H^1$ and for any $1 \leq j \leq \ell$ and $1 \leq i \leq N$

$$\partial_i|w_{n,j}| = \begin{cases} \frac{u_{n,j}\partial_i u_{n,j} + v_{n,j}\partial_i v_{n,j}}{(u_{n,j}^2+v_{n,j}^2)^{1/2}} & \text{if } u_{n,j}^2 + v_{n,j}^2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, by proposition 2.1, the sequence $\{\vec{w}_n\}$ is bounded in $\vec{H}$, and hence passing to a subsequence, there exists $\vec{w} = (w_1, ..., w_\ell) = ((u_1, v_1), ..., (u_\ell, v_\ell)) \in \vec{H}$ such that

$$\begin{cases} \forall\ 1 \leq j \leq \ell \quad u_{n,j} > u_j, v_{n,j} > v_j \text{ and} \\ \lim_{n\to\infty} \int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 \quad \text{exists} \end{cases} \tag{3.1}$$

Now

$$\begin{aligned} \hat{E}(\vec{w}_n) - E(|\vec{w}_n|) &= \tfrac{1}{2}\{|\nabla\vec{w}_n|_2^2 - |\nabla|\vec{w}_n||_2^2\} \\ &= \frac{1}{2}\sum_{j=1}^{\ell} |\nabla w_{n,j}|_2^2 - |\nabla(u_{n,j}^2 + v_{n,j}^2)^{1/2}|_2^2 \\ &= \frac{1}{2}\sum_{j=1}^{\ell}\sum_{i=1}^{N} \frac{(u_{n,j}\partial_i v_{n,j} - v_{n,j}\partial_i u_{n,j})^2}{u_{n,j}^2 + v_{n,j}^2} \geq 0 \qquad (3.2) \end{aligned}$$

Proving that

$$\hat{I}_{c_1,...c_\ell} = \lim_{n\to\infty} \hat{E}(\vec{w}_n) \geq \limsup E(|\vec{w}_n|) \tag{3.3}$$

But

$$|w_{n,j}|_2^2 = ||w_{n,j}||_2^2 = c_{n,j}^2 \to c_j^2 \tag{3.4}$$

It follows by the continuity property of $I_{c_1,...,c_\ell}$ proved in Proposition 2.1 that we have :

$$\lim \hat{E}(\vec{w}_n) \geq \liminf I_{c_{n,1},...,c_{n,\ell}} = I_{c_1,...,c_\ell} \geq \hat{I}_{c_1,...,c_\ell}.$$

Hence

$$\lim_{n\to+\infty} \hat{E}(\vec{w}_n) = \lim_{n\to\infty} E(|\vec{w}_n|) = I_{c_1,...,c_\ell} = \hat{I}_{c_1,...,c_\ell} \tag{3.5}$$

(3.2) and (3.5) imply that

$$\forall\, 1 \leq j \leq \ell \quad \lim_{n\to\infty} \int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 - |\nabla (u_{n,j}^2 + v_{n,j}^2)^{1/2} = 0 \tag{3.6}$$

Thus it follows form (3.1) that :

$$\lim_{n\to\infty} \int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 = \lim_{n\to+\infty} \int |\nabla (u_{n,j}^2 + v_{n,j}^2)^{1/2}|^2 \tag{3.7}$$

which is equivalent to say that :

$$\lim_{n\to\infty} |\nabla \vec{w}_n|_2^2 = \lim_{n\to\infty} |\nabla |\vec{w}_n||_2^2 \tag{3.8}$$

(3.4) and (3.5) imply using Theorem 0.2 that $|\vec{w}_n|$ is relatively compact in $\vec{H}^1$. Thus there exists $w_j \in H^1$ such that

$$\begin{cases} (u_{n,j}^2 + v_{n,j}^2)^{1/2} \text{ converges to } w_j \text{ in } H^1 \text{ and} \\ |w_j|_2 = c_j \quad \forall\, 1 \leq j \leq \ell \end{cases} \tag{3.9}$$

and $E(w_1, ..., w_\ell) = I_{c_1,...,c_\ell}$.
Our purpose now is to prove that $w_j = (u_j^2 + v_j^2)^{1/2}$ (u_j and v_j are given in (3.1)).
Using (3.1), it follows that $u_{n,j} \to u_j$ and $v_{n,j} \to v_j$ in $L^2(B(0,R))$. Furthermore a straigthfoward computation enables us to prove that.

$$[(u_{n,j}^2 + v_{n,j}^2)^{1/2} - (u_j^2 + v_j^2)^{1/2}]^2 \leq |u_{n,j} - u_j|^2 + |v_{n,j}^2 - v_j|^2$$

from which we deduce that :

$$(u_{n,j}^2 + v_{n,j}^2)^{1/2} \longrightarrow (u_j^2 + v_j^2)^{1/2} \text{ in } L^2(B(0,R))$$

for all $R > 0$. But $(u_{n,j}^2 + v_{n,j}^2)^{1/2} \to w_j$ in L^2, thus we certainly have that $(u_j^2 + v_j^2)^{1/2} = w_j \ \forall\, 1 \leq j \leq \ell$.

On the other hand $|w_{n,j}|_2 = ||w_{n,j}||_2 \to c_j = |w_j|_2$, hence we are done if we prove that

$$\lim_{n\to\infty} |\nabla w_{n,j}|_2^2 \to |\nabla w_j|_2^2 \quad \text{for any } 1 \le j \le \ell.$$

Form (3.7) we have that $\lim_{n\to\infty} |\nabla w_{n,j}|_2^2 = \lim_{n\to\infty} |\nabla |w_{n,j}||_2^2$ and

$$\lim_{n\to\infty} |\nabla |w_{n,j}||_2^2 = |\nabla |w_j||_2^2.$$

Hence

$$|\nabla w_j|_2^2 \le \lim |\nabla |w_{n,j}||_2^2 = |\nabla |w_j||_2^2. \tag{3.9}$$

Finally replacing $w_{n,j}$ by w_j in (3.2), we see that

$$|\nabla w_j|_2^2 \ge |\nabla |w_j||_2^2 \quad \forall\, 1 \le j \le \ell. \tag{3.10}$$

By (3.1), we know that $w_{n,j} \to w_j$ in H, thus $w_{n,j} > w_j$; which completes the proof of Theorem 1.1.

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