From flows of Λ Fleming-Viot processes to lookdown processes via flows of partitions

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Abstract

We show that the lookdown process can be pathwise embedded into a stochastic flow of bridges $(F_{s,t}, s \leq t)$ associated to a Λ coalescent. Such a flow of bridges couples an infinite collection of Λ Fleming-Viot processes $(\rho_{s,t}, t \in [s,\infty))_{s\in\mathbb{R}}$ where $\rho_{s,t}$ is the probability measure whose distribution function is $F_{s,t}$. Our pathwise construction yields a collection, indexed by s, of lookdown processes on a shared lookdown graph whose limiting empirical measures are $(\rho_{s,t}, t \in [s,\infty))_{s\in\mathbb{R}}$. This construction relies on the introduction of an ancestral types process and a stochastic flow of partitions from the flow of bridges, which are objects of independent interest. We prove that the flow of partitions entirely encodes a lookdown graph. Moreover, this is the unique lookdown graph that couples the infinite collection of Λ Fleming-Viot processes $(\rho_{s,t}, t \in [s,\infty))_{s\in\mathbb{R}}$. Finally, in the cases of the Beta$(2-\alpha,\alpha)$ Fleming-Viot and the standard Fleming-Viot, we reformulate the encoding of the lookdown process into an α-stable height process in terms of the flow of partitions and the ancestral types process.

1 Introduction

A generalized Fleming-Viot process $\rho := (\rho_t, t \geq 0)$ is a Markov process that describes the evolution of an infinite population. It takes its values in the set of probability measures on $[0,1]$, where each point in $[0,1]$ should be understood as a *genetic type*. For any $a \leq b \in [0,1]^2$, $\rho_t([a,b])$ is the proportion of individuals at time $t \geq 0$ with types in $[a,b]$ and thus, ρ_t describes the composition of the population at time t. Bertoin and Le Gall in [6] show that the distribution of a generalized Fleming-Viot process is

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completely characterized by a finite measure Λ on $[0,1]$, thus ρ is also called a Λ Fleming-Viot process. A precise definition will be given later. The process starts from ρ_0 taken as the uniform distribution on $[0,1]$. A point $x \in [0,1]$ will be called an *ancestral type* if there exists $t > 0$ such that $\rho_t(x) > 0$. We will say that an ancestral type x *becomes extinct* at time $t > 0$ if

$$\exists \epsilon > 0, \forall s \in [t-\epsilon, t), \rho_s(x) > 0, \rho_t(x) = 0 \tag{1}$$

An important subclass of Λ Fleming-Viot processes are those who enjoy the following property. Almost surely, for every $t > 0$, ρ_t is a weighted sum of a finite number of Dirac masses on $[0,1]$. Such Λ Fleming-Viot processes are said to come down from infinity (CDI). Roughly speaking, it means that on any interval of time $[0,\epsilon]$ an infinity of ancestral types become extinct, and the population at time ϵ is only composed of a finite number of types. This is the case when $\Lambda(dx) = \delta_0(dx)$ (standard Fleming-Viot process, which is related to the Kingman coalescent) and when Λ is the density of a Beta$(2-\alpha,\alpha)$ r.v. with $\alpha \in (1,2)$ (Beta$(2-\alpha,\alpha)$ Fleming-Viot process, which is related to the Beta$(2-\alpha,\alpha)$ coalescent), whereas when $\Lambda(dx) = dx$ (related to the Bolthausen-Sznitman coalescent) it does not hold. When the Λ Fleming-Viot comes down from infinity, we define the following event

$$E := \{\text{There exists } t > 0 \text{ s.t. two ancestral types become extinct simultaneously at time } t\} \tag{2}$$

We prove that $\mathbb{P}(E) \in \{0,1\}$ and that the Beta$(2-\alpha,\alpha)$ Fleming-Viot verifies $\mathbb{P}(E) = 0$, for all $1 < \alpha \leq 2$ (for simplicity, the case $\alpha = 2$ designates the standard Fleming-Viot process). We conjecture that this holds for any Λ Fleming-Viot that comes down from infinity.
It is well-known that the genealogy of a Λ Fleming-Viot is given by a Λ coalescent [6]. However, giving a meaning to the genealogy of such a process requires its embedding into a larger object. In this paper, we consider two distinct embeddings.

The first one has been introduced by Bertoin and Le Gall in [6, 7, 8] and is called a stochastic flow of bridges associated with a finite measure Λ on $[0,1]$. It is a consistent collection of bridges $(F_{s,t}, -\infty < s \leq t < \infty)$ (a bridge $F_{s,t}$ is the distribution function of a random probability measure $\rho_{s,t}$ on $[0,1]$ verifying an exchangeability property, see Subsection 2.2) such that the processes $(\rho_{s,t}, t \in [s,\infty))_{s\in\mathbb{R}}$ are a collection of coupled Λ Fleming-Viot processes. The upshot of that construction is that at each time $t \in \mathbb{R}$, one can define, from the flow of bridges, a Λ coalescent process which encodes the genealogy of the population alive at time t (see Subsection 2.2).

A second approach proposed by Donnelly and Kurtz in [12, 13] is the so-called (modified) lookdown process associated with a Λ Fleming-Viot. Its definition relies on the construction of a lookdown graph. We first introduce a useful notation. For each $n \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{S}_n^2$ be the subset of $\{0,1\}^n$ whose elements have at least two coordinates equal to 1. For an element $u = (u^1, u^2, \ldots) \in \mathcal{S}_\infty^2$, we denote by $[u]_n := (u^1, \ldots, u^n) \in \{0,1\}^n$ the restriction of u to its n first coordinates. Note that $[u]_n$ is not necessarily an element of $\mathcal{S}_n^2$. More generally, for any set $A \subset \mathcal{S}_\infty^2$ and every $n \in \mathbb{N}$, we define the projection of A on $\{0,1\}^n$ as the subset of $\{0,1\}^n$ composed of the restrictions to $\{0,1\}^n$ of the elements of A. Then we denote by $A_{|\mathcal{S}_n^2}$ the trace on $\mathcal{S}_n^2$ of this projection. Remark that $A_{|\mathcal{S}_n^2}$ can eventually be empty. A deterministic lookdown graph p is a point collection on $\mathbb{R} \times \mathcal{S}_\infty^2$ - that is a countable subset of $\mathbb{R} \times \mathcal{S}_\infty^2$ - such that its restriction $p_{|[s,t]\times\mathcal{S}_n^2}$ has finitely many points for every $s \leq t$ and $n \in \mathbb{N}$. The denomination *lookdown graph* arises from its graphical representation as a set of *lines* on $\mathbb{R} \times \mathbb{N}$ (see Figure 1 for an example) due to Pfaffelhuber and Wakolbinger [17]. We give a very brief description of this representation, as it will not be useful in this paper except for giving an intuitive idea of a lookdown graph. $\mathbb{R}$ is interpreted as time whereas $\mathbb{N}$ is the set of *levels*. A line is a subset of the form

$$([s_0, s_1) \times i_0) \cup ([s_1, s_2) \times i_1) \cup \dots$$

where $(s_l)_{0\leq l<n+1}$ (resp. $(i_l)_{0\leq l<n+1}$) is an increasing sequence of $\mathbb{R}$ (resp. $\mathbb{N}$) and n is the (finite or infinite) number of jumps of the line. (s_0, i_0) is called the birth point of the line. Suppose that we have defined the graphical representation until time $s \in \mathbb{R}$. For each time $t > s$ and every element $u = \{u^1, u^2, \dots\} \in \mathcal{S}_\infty^2$ such that (t, u) is a point of p, introduce the set $I := \{i \geq 1 : u^i = 1\}$. The point (t, u) will affect the evolution of the set of lines from $t-$ to t. For each level $i \in I\backslash\{\min(I)\}$, a new line is born from the point $(t, i) \in \mathbb{R} \times \mathbb{N}$. The line that contains the point $(t-, \min(I))$ is linked to the point $(t, \min(I))$. For each level $i \notin I$, the line passing by $(t-, i)$ is pushed up to the next available level (t, j), that is, the lowest level j where no line is passing at time t. The atom (t, u) is called a birth event: the level $\min(I)$ is the parent that reproduces on all the other levels of I at time t. This ends the description of the deterministic lookdown graph.

Then, from any time $s \in \mathbb{R}$ we introduce the deterministic lookdown function $(\xi_{s,t}(i), t \in [s, \infty))_{i\geq 1}$ as follows. The initial types are given by a sequence $(\xi_{s,s}(i))_{i\geq 1} \in [0,1]^{\mathbb{N}}$. Furthermore at each time $t > s$, for each level $j \geq 1$, consider the line of the lookdown graph located at (t, j). Either this line was born at time t, from a parent located at a level $i < j$, or it was already alive at a level $i \leq j$ at time $t-$. Then $\xi_{s,t}(j)$ takes the type of $\xi_{s,t-}(i)$ (see Subsection 2.3 for further details). We will use the notation $\mathscr{L}_s(p, (\xi_{s,s}(i))_{i\geq 1})$ to denote the lookdown function with initial types $(\xi_{s,s}(i))_{i\geq 1}$ and lookdown graph p starting from time s; and $\mathscr{E}_s(p, (\xi_{s,s}(i))_{i\geq 1})$ will denote the limiting empirical measures

$$\Xi_{s,t}(.) := \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\xi_{s,t}(i)}(.) \text{ for } t \in [s, \infty)$$

when it is well-defined.

To obtain a random lookdown process, it suffices to randomize the lookdown graph and the initial types. Consider a Λ lookdown graph $\mathcal{P}$, i.e., a Poisson point process on $\mathbb{R} \times \mathcal{S}_\infty^2$ with an intensity measure depending on Λ which will be precisely defined in Subsection 2.3, and a sequence $(\xi_{s,s}(i))_{i\geq 1}$ of i.i.d. uniform$[0,1]$ r.v. Donnelly and Kurtz in [13] prove that the lookdown process $(\xi_{s,t}, t \in [s, \infty)) := \mathscr{L}_s(\mathcal{P}, (\xi_{s,s}(i))_{i\geq 1})$ admits a limiting empirical measure $\Xi_{s,t}$ simultaneously for all $t \in [s, \infty)$ almost surely, which is a Λ Fleming-Viot process started from the uniform distribution on $[0,1]$. From any given time $t > s$, one can trace backward in time the lineages of the lookdown graph from each level $i \in \mathbb{N}$

and thus obtain a Λ coalescent tree that gives a meaning to the genealogy of the population alive at time t.
From a single lookdown graph, we thus define a collection of lookdown processes indexed by their starting time $s \in \mathbb{R}$ if we are given for each of them a sequence of initial types. In this case, the genealogies of these processes are coupled (they share the same lookdown graph) but their types are not, unless the sequences of initial types were suitably coupled.

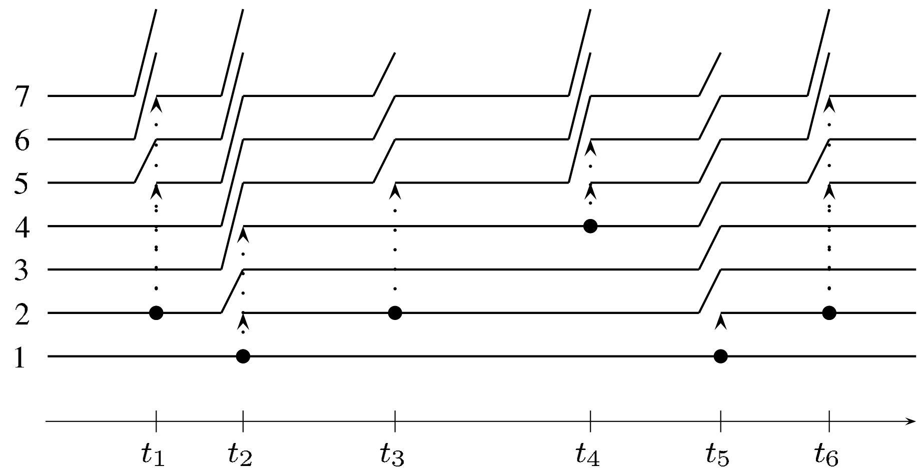


Figure 1: A lookdown graph. Each arrow corresponds to a birth event: the level carrying a dot has reproduced on the levels carrying an ending arrow. For example, at time t_1, level 2 reproduces on levels 5 and 7 while former levels 5, 6 and 7 are pushed up to the next available levels. Note that only finitely many birth events affecting the n first levels occur in any compact interval of time, for each integer n. Furthermore, by tracing back the lineages from a given time, one obtains a Λ coalescent tree.

In this paper, we show precisely how such a coupling can be achieved. Consider a stochastic flow of bridges $(F_{s,t}, -\infty < s \leq t < \infty)$ associated to a Λ measure. The questions we intend to answer are the following: at a fixed time s, how can one define a lookdown process (on the same probability space) such that its limiting empirical measure is a càdlàg modification of $(\rho_{s,t}, t \in [s, \infty))$? Is it possible to define a coupling of all these lookdown processes simultaneously, that is, for all time $s \in \mathbb{R}$? Is this construction unique ?

A first difficulty arises from the potential presence of irregularities in the flow F. Indeed, for a general measure Λ there is no Poissonian construction of a flow of bridges and many irregularities can affect the flow F, see Subsection 2.2 for further details on this point. Thus we will systematically consider a càdlàg modification $(\tilde{\rho}_{s,t}, t \in [s, \infty))$ of the Markov process $(\rho_{s,t}, t \in [s, \infty))$ (which enjoys the Feller property), for each $s \in \mathbb{R}$, and consequently the collection of modifications $(\tilde{F}_{s,t}, s \leq t)$ of the flow F. Our construction will concern two classes of measures Λ:

- **[(CDI) and $\mathbb{P}(E) = 0$].** The Λ Fleming-Viot comes down from infinity and any two of its ancestral types never become extinct simultaneously.

- **[Bolthausen-Sznitman]**. $\Lambda(dx) = dx$, the Λ Fleming-Viot does not come down from infinity and the Λ coalescent is the Bolthausen-Sznitman coalescent.

Let us introduce briefly two objects that play a fundamental role in the present work. Consider the set of ancestral types of $(\tilde{\rho}_{s,t}, t \in [s,\infty))$ for a given time $s \in \mathbb{R}$. We prove that this set is countable. In the **[(CDI) and** $\mathbb{P}(E) = 0$**]** case, one can order them by decreasing persistence. Indeed there exists an ancestral type $\mathbf{e}_s^{(1)}$ which never becomes extinct, let $d_s^1 := \infty$ be its extinction time. Then, denote by $\mathbf{e}_s^{(i)}$ the $(i-1)$-th ancestral type that becomes extinct, and let d_s^i be its extinction time, for every $i \geq 2$. Therefore $(d_s^i)_{i\geq 1}$ is a strictly decreasing sequence in $(s,\infty]$ and we have

$$\tilde{\rho}_{s,t}(.) = \sum_{j=1}^{i} \tilde{\rho}_{s,t}(\mathbf{e}_s^{(j)})\delta_{\mathbf{e}_s^{(j)}}(.) \text{ for all } t \in [d_s^{i+1}, d_s^i), i \geq 1 \tag{3}$$

The **[Bolthausen-Sznitman]** case relies on a different criterion for the ordering of the ancestral types. But in both cases, $(\mathbf{e}_s^{(i)}, s \in \mathbb{R})_{i\geq 1}$ is called the ancestral types process. This process should be seen as an extension of the primitive Eve process of Bertoin and Le Gall in [6], Section 5.3. Next, we define for all $s \leq t$ a random partition $\hat{\Pi}_{s,t}$ as follows. For all $i, j \in \mathbb{N}$

$$i \overset{\hat{\Pi}_{s,t}}{\sim} j \Leftrightarrow \tilde{F}_{s,t}^{-1}(\mathbf{e}_t^{(i)}) = \tilde{F}_{s,t}^{-1}(\mathbf{e}_t^{(j)}) \tag{4}$$

Then, we prove that $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$ is a consistent collection of exchangeable random partitions that enjoys flow properties with the coagulation operator, see Definition 3.6 and Proposition 4.8. Thus, we call it a stochastic flow of partitions. Furthermore, for each $t \in \mathbb{R}$, $(\hat{\Pi}_{t-s,t}, s \geq 0)$ is a Λ coalescent process giving the genealogy of the population alive at time t. We will prove that such a flow of partitions encodes a lookdown graph $\mathcal{P}$, see subsection 3.2.
Finally we define a collection of processes $(\xi_{s,t}(i), -\infty < s \leq t < \infty)_{i\geq 1}$ as follows. For each $s \in \mathbb{R}$ let $(\xi_{s,t}(i), t \in [s,\infty))_{i\geq 1} = \mathscr{L}_s(\mathcal{P}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ and $(\Xi_{s,t}, t \in [s,\infty)) = \mathscr{E}_s(\mathcal{P}, (\mathbf{e}_s^{(i)})_{i\geq 1})$. We thus assert our main result.

Theorem 1 *The collection of coupled lookdown processes $(\xi_{s,t}(i), s \leq t)_{i\geq 1}$ with limiting empirical measures $(\Xi_{s,t}, s \leq t)$ verify the following assertions:*

i) ***Coupling**. For each $s \in \mathbb{R}$, a.s. $(\Xi_{s,t}, t \in [s,\infty)) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$.*

ii) ***Uniqueness**. Let $\mathcal{M}$ be a Λ lookdown graph and for each $s \in \mathbb{R}$, consider a sequence $(\chi_{s,s}(i))_{i\geq 1}$ of r.v. taking distinct values in $[0,1]$. If for each $s \in \mathbb{R}$, a.s. $\mathscr{E}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1}) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$ then*

- *For each $s \in \mathbb{R}$, a.s. $(\chi_{s,s}(i))_{i\geq 1} = (\mathbf{e}_s^{(i)})_{i\geq 1}$.*
- *Almost surely, $\mathcal{M} = \mathcal{P}$.*

Remark 1.1 *One could ask for a more general uniqueness result that would concern not only Λ lookdown graph but any lookdown graph $\mathcal{M}$. This can be achieved under some technical assumptions in the* ***[(CDI) and** $\mathbb{P}(E) = 0$**]** *case. Before stating the technical assumptions required, let us give a quick idea of what configurations of lookdown processes they should exclude. Consider a lookdown graph $\mathcal{M}$ such that the level 2 (in its graphical representation) is never affected by any birth event. Therefore, the initial type carried by this level does not appear in the limiting empirical measure (when it exists). Thus, the first uniqueness result presented in the theorem cannot hold in this setting. We now give the technical assumptions needed.*

If $\mathcal{M}$ is a random lookdown graph and for each $s \in \mathbb{R}$, $(\chi_{s,s}(i))_{i\geq 1}$ is a collection of r.v. taking distinct values in $[0,1]$ such that for each $s \in \mathbb{R}$, the lookdown process $\mathscr{L}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1})$ verify for a.s. all $\omega \in \Omega$

- $(X_{s,t}, t \in [s,\infty))(\omega) := \mathscr{E}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1})(\omega)$ *exists.*
- *For each $i \geq 1$, there exists $T_i(\omega) \in (s,\infty]$ such that $X_{s,t}(\xi_{s,s}(i))(\omega) > 0$ iff $t \in (s, T_i(\omega))$ and $\chi_{s,T_i}(j) \neq \chi_{s,s}(i)$ for all $j \in \mathbb{N}$.*

Then the uniqueness result of the theorem still holds. We will not provide the proof of this result but it derives from an extension of the proof of the theorem.

This paper is organized as follows. In Section 2, we recall some basic definitions and properties concerning Λ coalescents, stochastic flows of bridges, and the lookdown process. We then study the behaviour of the Λ Fleming-Viot process, in particular the event E defined above.

In Section 3, we introduce the flows of partitions. We start with the deterministic flows of partitions and then, we randomize the flows and define the stochastic flows of partitions. Many technical results are exposed in this section. Therefore on first reading one can skip Subsection 3.2 except the Definition 3.6, which is needed in the next section. In Section 4, we develop our pathwise lookdown construction from a flow of bridges F. We define and give several properties of the ancestral types process. From this process, we are able to define a stochastic flow of partitions (see Definition 3.6). Using the result obtained in Section 3, we obtain pathwise from this flow of partitions a Poisson point process $\mathcal{P}$ on $\mathbb{R} \times \{0,1\}^{\mathbb{N}}$ which is a lookdown graph. This is the core of our pathwise construction.

Section 5 is devoted to Theorem 1. First we prove the coupling statement. Then, we focus on the uniqueness properties and prove that there exists a unique Λ lookdown graph that couples all the Λ Fleming-Viot encoded in a stochastic flow of bridges, thus obtaining the uniqueness statement of the theorem. Furthermore, we compare our lookdown construction from flows of bridges with the lookdown construction of Donnelly and Kurtz in [13] from the Moran model and give a general result on the oldest families of a Λ Fleming-Viot.

Finally, in Section 6 we reformulate results of Berestycki et al. in [1, 2] on the encoding of the lookdown process associated with the Beta$(2-\alpha,\alpha)$ Fleming-Viot into an α-stable height process, with $\alpha \in (1,2]$, in terms of the flow of partitions and the ancestral types process. The upshot of this setting is that not only the genealogy of the Beta$(2-\alpha, \alpha)$ Fleming-Viot process but also its initial types are defined in terms of the height process.

2 Preliminaries

2.1 Coalescent with multiple collisions

Let us recall the definition of the coalescents with multiple collisions, also called Λ coalescents, which are introduced in [18, 19]. As in [6], we denote by $\mathscr{P}_n$ the set of all partitions of $[n] := \{1, 2, \ldots, n\}$, with $n \in \mathbb{N} \cup \{\infty\}$. $\mathscr{P}_\infty$ is equipped with the distance $d_{\mathscr{P}}$ defined as follows. For all $\pi, \pi' \in \mathscr{P}_\infty$

$$d_{\mathscr{P}}(\pi, \pi') = 2^{-i} \Leftrightarrow i = \sup\{j \in [\infty] : \pi^{[j]} = \pi'^{[j]}\}$$

where $\pi^{[j]}$ is the restriction of π to $[j]$. The metric space $(\mathscr{P}_\infty, d_{\mathscr{P}})$ is compact.
For each $i \geq 1$, we denote by $\pi(i)$ the i-th block of a given partition $\pi \in \mathscr{P}_\infty$, where the blocks are in the increasing order of their least element. Furthermore, for each $i \geq 1$, we introduce the asymptotic frequency of the i-th block of π as

$$|\pi(i)| := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{j\in\pi(i)\}} \tag{5}$$

when the limit on the r.h.s. exists.
We define the coagulation operator $Coag : \mathscr{P}_\infty \times \mathscr{P}_\infty \to \mathscr{P}_\infty$ as follows. For any elements $\pi, \pi' \in \mathscr{P}_\infty$, $Coag(\pi, \pi')$ is the partition whose blocks are given by

$$Coag(\pi, \pi')(i) = \underset{j\in\pi'(i)}{\cup} \pi(j) \tag{6}$$

for every $i \in \mathbb{N}$. This is a Lipschitz-continuous operator and we have

$$Coag(\pi, Coag(\pi', \pi'')) = Coag(Coag(\pi, \pi'), \pi'') \tag{7}$$

for any elements $\pi, \pi', \pi'' \in \mathscr{P}_\infty$, see [4], Section 4.2 for further details.
Consider a finite measure Λ on $[0, 1]$. A Λ coalescent is a Markov process $(\Pi_t, t \geq 0)$ on $\mathscr{P}_\infty$ started from the partition $0_{[\infty]} := \{\{1\}, \{2\}, \ldots\}$ and such that, for each integer $n \geq 2$, its restriction $(\Pi_t^{[n]}, t \geq 0)$ to $\mathscr{P}_n$ is a continuous time Markov chain that evolves by coalescence events whose dynamics is the following. For any integer $2 \leq p \leq n$, consider a partition $\pi \in \mathscr{P}_n$ whose blocks are all singletons except one which has p elements. The rate at which $\Pi_t^{[n]}$ jumps to $Coag(\Pi_t^{[n]}, \pi)$ is given by

$$\lambda_{n,p} = \int_0^1 x^{p-2}(1-x)^{n-p}\Lambda(dx) \tag{8}$$

If $\Pi_t^{[n]}$ has m blocks, the total jump rate of the chain at this time is then

$$\lambda_m = \sum_{p=2}^{m} \binom{m}{p} \lambda_{m,p} \tag{9}$$

From now on, we will systematically assume that $\Lambda(1) = 0$ to avoid trivial behaviour. Indeed an atom on 1 induces coalescence events involving all the blocks simultaneously. Pitman [18] showed that a Λ coalescent could either come down from infinity (CDI), that is, the process $\#\Pi_t$ is finite at any time $t > 0$ a.s, or stay infinite, that is, $\#\Pi_t$ is infinite for all $t > 0$ a.s., where $\#\pi$ denotes the number of blocks of a partition π. A necessary and sufficient condition on the measure Λ that ensures the coming down from infinity can be found in [20]. We will denote by $\mathbb{CDI}$ the set of Λ measures for which the Λ coalescent comes down from infinity.

The Λ coalescents obtained with a measure Λ taken as the density of a Beta$(2-\alpha, \alpha)$ variable, with $0 < \alpha < 2$, are called Beta$(2-\alpha, \alpha)$ coalescents (see [2, 3, 9] for several results about such coalescents). Recall that those densities are given by

$$\Lambda(dx) = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)} x^{1-\alpha}(1-x)^{\alpha-1}dx \tag{10}$$

Those coalescents come down from infinity iff $\alpha > 1$. The Kingman coalescent is recovered (formally) when $\alpha \to 2$ (we will use the notation Beta$(0,2)(dx)$ for $\delta_0(dx)$), whereas the Bolthausen-Sznitman coalescent arises when $\alpha = 1$. Note that the latter is the only Beta$(2-\alpha, \alpha)$ coalescent that stays infinite but has no dust (or equivalently, no singleton at any given time $t > 0$) almost surely.

2.2 Stochastic flows of bridges

We recall basic definitions and properties of stochastic flows of bridges introduced by Bertoin and Le Gall in [6]. A bridge is a nondecreasing càdlàg process $B = (B(r), r \in [0,1])$ with values in $[0,1]$ such that :

- $B(0) = 0$, $B(1) = 1$
- B has exchangeable increments

Kallenberg [16] shows that for any bridge B, there exists a sequence of nonnegative r.v. $(\beta^i)_{i\in\mathbb{N}}$ with $\beta^1 \geq \beta^2 \geq \ldots$, and $\sum_{i=1}^{\infty} \beta^i \leq 1$, and a sequence of i.i.d. uniform$[0,1]$ r.v. $(U^i)_{i\in\mathbb{N}}$ independent of the sequence $(\beta^i)_{i\in\mathbb{N}}$ such that a.s. for every $r \in [0,1]$,

$$B(r) = (1 - \sum_{i=1}^{\infty} \beta^i)r + \sum_{i=1}^{\infty} \beta^i 1_{\{U^i \leq r\}} \tag{11}$$

From any bridge B and any infinite sequence $(V_p)_{p\geq 1}$ of i.i.d. uniform random variables on $[0,1]$ independent of B, one can define a random partition $\pi(B, (V_p)_{p\geq 1})$ by

$$i \overset{\pi(B,(V_p)_{p\geq 1})}{\sim} j \Leftrightarrow B^{-1}(V_i) = B^{-1}(V_j) \tag{12}$$

In [6] Bertoin and Le Gall define a consistent collection of bridges in order to obtain, using the above construction, a consistent collection of random partitions.

Definition 2.1 *A flow of bridges is a collection* $(B_{s,t}, -\infty < s \le t < \infty)$ *of bridges such that :*

- *For every* $r < s < t$, $B_{r,t} = B_{r,s} \circ B_{s,t}$ $a.s.$ *(cocycle property).*
- *The law of* $B_{s,t}$ *only depends on* $t - s$. *Furthermore, if* $s_1 < s_2 < \ldots < s_n$ *the bridges* $B_{s_1,s_2}, B_{s_2,s_3}, \ldots, B_{s_{n-1},s_n}$ *are independent.*
- $B_{0,0} = Id$ *and* $B_{0,t} \to Id$ *in probability as* $t \downarrow 0$, *in the sense of Skorohod's topology.*

Given a sequence of i.i.d. uniform $[0,1]$ variables $(V_p)_{p\ge1}$, independent of a flow of bridges $(B_{s,t}, -\infty < s \le t < \infty)$, they prove that, for each s fixed, the process $(\pi(B_{s,t},(V_p)_{p\ge1}))_{t\ge s}$ is an exchangeable coalescent, see Theorem 1 in [6]. In the particular case of a Λ coalescent, $(B_{s,t}, -\infty < s \le t < \infty)$ is called a Λ flow of bridges. When $x^{-2}\Lambda(dx)$ is a finite measure, they propose a Poissonian construction of the flow of bridges. But the Poissonian construction is not possible when $x^{-2}\Lambda(dx)$ is infinite.
Finally, let us explicit the connection between the Λ Fleming-Viot process and the Λ flow of bridges. Recall that the dual flow $(F_{s,t}, -\infty < s \le t < \infty)$ is defined by

$$F_{s,t} = B_{-t,-s} \tag{13}$$

To clarify notation, we will say that $(B_{s,t}, -\infty < s \le t < \infty)$ is a backward flow of bridges and $(F_{s,t}, -\infty < s \le t < \infty)$ is a forward flow of bridges. Remark that the forward flow verifies a dual cocycle property:

$$F_{r,t} = F_{s,t} \circ F_{r,s} \text{ a.s. for all } r < s < t \tag{14}$$

From now on, we will systematically work with forward flows of bridges, in particular a Λ flow of bridges will denote implicitly a forward Λ flow of bridges. Denote by $\mathscr{M}_1$ the space of all probability measures on $[0,1]$, equipped with its weak topology. Fix a time s, and define the Λ Fleming-Viot process as the $\mathscr{M}_1$-valued process $(\rho_{s,t}, t \in [s,\infty))$ where

$$\rho_{s,t}([0,x]) = F_{s,t}(x) \text{ , for all } x \in [0,1]$$

Bertoin and Le Gall in [6] prove that this process is a Markov process with a Feller semigroup which is characterized by a martingale problem (based on a duality argument with the Λ coalescent) that we do not recall here. Therefore the process $(\rho_{s,t}, t \in [s,\infty))$ admits a càdlàg modification denoted by $(\tilde{\rho}_{s,t}, t \in [s,\infty))$. The collection of bridges asociated to this càdlàg modification is denoted by $(\tilde{F}_{s,t}, t \in [s,\infty))$. For all $s < t$, we have

$$\tilde{F}_{s,t} = F_{s,t} \text{ , } \tilde{\rho}_{s,t} = \rho_{s,t} \text{ a.s.} \tag{15}$$

One should realize that a stochastic flow of bridges F, except when it arises from a Poissonian construction, may have many irregularities, that is, (random) exceptional times where the cocycle property does not hold. This will be a difficulty throughout this work and will require to consider modification of the flow F. Moreover, all the objects defined pathwise from the flow will suffer from thoses irregularities and will need themselves to be regularized, as we will see later.

Let us now describe briefly the behaviour of a Λ flow of bridges. The jump locations of the bridges $(B_{0,t}, t > 0)$ evolve as time passes, whereas the jump sizes only coagulate. The forward behaviour is quite different. Indeed, although for all $t \geq 0$

$$F_{0,t} \overset{(d)}{=} B_{0,t} \tag{16}$$

this equality does not hold in terms of processes. Roughly speaking, jump sizes of $(F_{0,t}, t > 0)$ evolve in time, but jump locations are fixed (however some new jumps appear as time passes, if the Λ-coalescent has dust). We will investigate some properties of this forward flow in our study of the Λ Fleming-Viot processes in Subsection 2.4.

2.3 The lookdown process

Let us recall the definition of the lookdown process, introduced by Donnelly and Kurtz in [12, 13] and generalized to the case of Ξ-coalescents in [10]. Its definition requires the introduction of a lookdown graph. Let us define some notation. For each $n \in \{2, 3, ..., \infty\}$, let $\mathcal{S}_n^2$ be the subset of $\{0,1\}^n$ whose elements have at least two coordinates $1 \leq i < j \leq n$ equal to 1. For all $n, m \in \{2, 3, ..., \infty\}$ such that $n < m$, we denote by $[u]_n := (u^1, \dots, u^n)$ the restriction of an element $u = (u^1, \dots, u^m) \in \mathcal{S}_m^2$ to its n first coordinates. Remark that $[u]_n$ is not necessarily an element of $\mathcal{S}_n^2$. Thus, for a given subset A of $\mathcal{S}_m^2$, we denote by $A_{|\mathcal{S}_n^2}$ the restriction to $\mathcal{S}_n^2$ of its projection on $\{0,1\}^n$, using the restriction map described above. Note that $A_{|\mathcal{S}_n^2}$ can be empty.

Definition 2.2 *A deterministic lookdown graph is a deterministic point collection p on $\mathbb{R} \times \mathcal{S}_\infty^2$ such that for each $n \in \mathbb{N}$, for all $s \leq t$, $p_{|[s,t] \times \mathcal{S}_n^2}$ has finitely many points.*

The point collection p should be seen as a collection of points $(t, u) \in \mathbb{R} \times \mathcal{S}_\infty^2$ called birth events, where t designates the birth time and u determines the individuals that participate to this event. More precisely, the set

$$I_{t,u} := \{i \geq 1 : u^i = 1\} \tag{17}$$

is called the set of individuals that participate to the birth event.
Fix $s \in \mathbb{R}$ and consider a vector of initial types $(\xi_{s,s}(i))_{i \geq 1} \in [0,1]^{\mathbb{N}}$. For each $n \in \mathbb{N}$, one can define a particle system $\xi_{s,t}^n = (\xi_{s,t}^n(1), \dots, \xi_{s,t}^n(n)), t \in [s, \infty)$ with values in $[0,1]^n$, by:

- $\xi_{s,s}^n(i) := \xi_{s,s}(i)$ for all $i \in [n]$.
- At any birth event $(t, u) \in p$ with $t > s$ and such that $\#\{I_{t,u} \cap [n]\} \geq 2$, for each $i \in [n]$, $r \mapsto \xi_{s,r}^n(i)$ evolves as follows

$$\begin{cases} \xi_{s,t}^n(i) = \xi_{s,t-}^n(\min(I_{t,u})) & \text{for all } i \in I_{t,u} \\ \xi_{s,t}^n(i) = \xi_{s,t-}^n(i - (\#\{I_{t,u} \cap [i]\} - 1) \vee 0) & \text{for all } i \notin I_{t,u} \end{cases} \tag{18}$$

Remark that this is the deterministic lookdown construction with push-up, that is, instead of killing particles located at birth levels, they are pushed up to the next available level (see Figure 1). It corresponds to the modified lookdown construction of Donnelly and Kurtz [13].

From this definition, one can easily deduce that the trajectory of each particle only depends on lower particles and conclude to the compatibility of the particle systems $(\xi^n_{s,t}, t \in [s,\infty))$ with $n \in \mathbb{N}$. Hence, there exists a $[0,1]^\infty$-valued particle system $(\xi_{s,t}(i), t \in [s,\infty))_{i\geq 1}$ such that for all $n \in \mathbb{N}$, $(\xi_{s,t}(i), t \in [s,\infty))_{i\in[n]} = (\xi^n_{s,t}(i), t \in [s,\infty))_{i\in[n]}$. It is called a deterministic lookdown function. We thus introduce the following notation.

Definition 2.3 *We denote by $\mathscr{L}_s(p, (\xi_{s,s}(i))_{i\geq 1})$ the deterministic lookdown function $(\xi_{s,t}(i), t \in [s,\infty))_{i\geq 1}$ defined from the point collection p and the initial types $(\xi_{s,s}(i))_{i\geq 1}$.*

When it exists, for every $t \in [s,\infty)$ let

$$\Xi_{s,t}(.) := \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\xi_{s,t}(i)}(.) \tag{19}$$

be the limiting empirical measure of this deterministic lookdown function taken at time t.

Definition 2.4 *We denote by $\mathscr{E}_s(p, (\xi_{s,s}(i))_{i\geq 1})$ the collection of limiting empirical measures $(\Xi_{s,t}, t \in [s,\infty))$ defined from the point collection p and the initial types $(\xi_{s,s}(i))_{i\geq 1}$, when it exists.*

We now explain how one can define a random lookdown process such that its limiting empirical measures are almost surely defined, and form a Λ Fleming-Viot process, where Λ is a finite measure on $[0,1)$. We take $\mathcal{P}$ as a Poisson point process on $\mathbb{R} \times \mathcal{S}^2_\infty$ with intensity measure $dt \otimes (\mu_K + \mu_\Lambda)$ where μ_K and μ_Λ are defined as follows.

Let us define the intensity measure μ_Λ on $\mathcal{S}^2_\infty$ corresponding to resampling events with positive frequency, that is, birth events of a positive proportion of individuals. Let $\nu^x(.)$ be the distribution on $\mathcal{S}^2_\infty$ of a sequence of i.i.d. Bernoulli random variables with parameter x, for each $x \in (0,1)$.

$$\mu_\Lambda(.) := \int_{(0,1)} x^{-2}\Lambda(dx)\nu^x(.) \tag{20}$$

We now define the intensity measure for Kingman's birth events, that is, birth events involving only two individuals at once. For each $1 \leq i < j$, let $s_{i,j}$ be the element of $\mathcal{S}^2_\infty$ that has only two coordinates equal to 1: i and j. We define the measure μ_K on $\mathcal{S}^2_\infty$ by

$$\mu_K(.) := \Lambda(0) \sum_{1\leq i<j} \delta_{s_{i,j}}(.) \tag{21}$$

Definition 2.5 *A lookdown graph associated with the measure Λ - or a Λ lookdown graph in short - is a Poisson point process $\mathcal{P}$ on $\mathbb{R} \times \mathcal{S}^2_\infty$ with intensity measure $dt \otimes (\mu_K + \mu_\Lambda)$.*

Remark 2.6 *Consider such a Poisson point process. For all $s < t$ and each $n \in \mathbb{N}$, $\mathcal{P}_{|[s,t]\times\mathcal{S}_n^2}$ has finitely many points almost surely.*

Consider a lookdown graph $\mathcal{P}$ associated with the measure Λ. Fix a time $s \in \mathbb{R}$ and a sequence of i.i.d. uniform$[0,1]$ r.v. $(\xi_{s,s}(i))_{i\geq 1}$. Donnelly and Kurtz prove in [13] that $\mathscr{L}_s(\mathcal{P}, (\xi_{s,s}(i))_{i\geq 1})$ admits a limiting empirical measure $\Xi_{s,t}$ simultaneously for all $t \in [s,\infty)$, almost surely. Moreover, the process $(\Xi_{s,t}, t \in [s,\infty))$ is a càdlàg Λ Fleming-Viot process.

The lookdown graph can then be used to define a collection of lookdown processes indexed by their starting time $s \in \mathbb{R}$, if we are given for each of them a sequence of initial types. Hence their genealogies are coupled, but their types are not, unless the sequences of initial types were suitably coupled.

We end this subsection with a definition that will be useful in the sequel. Consider a lookdown process $(\xi_t(i), t \in [0,\infty)) = \mathscr{L}_0(\mathcal{P}, (\xi_0(i))_{i\geq 1})$ (for simplicity, we write ξ_t instead of $\xi_{0,t}$ to alleviate notation).

Definition 2.7 *We define $Y_t(i)$ as the lowest level at time t that carries the type $\xi_0(i)$.*

$$Y_t(i) := \inf\{j \geq 1 : \xi_t(j) = \xi_0(i)\} \tag{22}$$

2.4 Lambda Fleming-Viot process

In this subsection, we study some properties of Λ Fleming-Viot processes. We denote by $(\rho_t, t \geq 0)$ such a process assumed to be càdlàg (this Markov process enjoys the Feller property), started from the uniform distribution on $[0,1]$. From the flow of bridges representation of ρ, it should be clear that for each $t > 0$ there exists $(U_t^i, \beta_t^i)_{i\geq 1}$ defined as in subsection 2.2 such that

$$\rho_t(dx) = \sum_{i\in\mathbb{N}} \beta_t^i \delta_{U_t^i}(dx) + (1 - \sum_{i\in\mathbb{N}} \beta_t^i)dx \tag{23}$$

The set $\{U_t^i : t > 0, i \in \mathbb{N}, \beta_t^i > 0\}$ is countable. Indeed, in the lookdown representation, there is a countable number of initial types. Since any point in $\{U_t^i : t > 0, i \in \mathbb{N}, \beta_t^i > 0\}$ corresponds to an initial type of the lookdown process, we deduce that the former set of points is countable.
Those points are called the *ancestral types* of ρ. When ρ does not charge an ancestral type anymore from a given time t (see Equation (1)), we say that this ancestral type becomes extinct at t. If all ancestral types but one are extinct, we say that the remaining ancestral type has fixed. Finally, we say that an ancestral type $x \in [0,1]$ *emerges from dust* at time t if

$$\forall s \in [0,t), \rho_s(x) = 0,\ \rho_t(x) > 0 \tag{24}$$

Denote by $\#\mu$ the number of atoms of a measure μ. The following proposition is a compilation of results; part of them are a consequence of known facts (see [6]).

Proposition 2.8 *If $\Lambda \in \mathbb{CDI}$, then for all $t > 0$ the following properties hold a.s.:*

- *The measure ρ_t has no continuous part.*
- *Only a finite number of ancestral types have not become extinct at time t.*
- *One ancestral types fixes in finite time.*

If $\Lambda \notin \mathbb{CDI}$, none of the ancestral types become extinct in finite time, almost surely. Moreover, if the Λ coalescent has dust, then for all $t > 0$, ρ_t has a continuous part and ancestral types emerge from dust as time passes.

Proof If $\Lambda \in \mathbb{CDI}$, then a Λ coalescent $(\Pi_t, t \geq 0)$ has no singleton almost surely [18]. From Equation (16), it is easy to deduce that for all $t > 0$, ρ_t has no continuous part and

$$\#\rho_t \overset{(d)}{=} \#\Pi_t \tag{25}$$

Since $\#\Pi_t$ reaches 1 in finite time a.s., we obtain the first assertion. Indeed, $\{\#\rho_t = 1\}_{t>0}$ is a nested collection of events, such that

$$\mathbb{P}(\{\#\rho_t = 1\}) = \mathbb{P}(\{\#\Pi_t = 1\}) \underset{t\to\infty}{\to} 1$$

Suppose $\Lambda \notin \mathbb{CDI}$. Let us use the lookdown representation of the Λ Fleming-Viot process. Each ancestral type is carried by a certain level i at time 0. Denote by $Y_t(i)$ the lowest level at time t that carries type $\xi_0(i)$ (see Definition 2.7). We claim that $(Y_t(i), t \geq 0)$ does not reach ∞ in finite time. Indeed, the contrary would imply that only types $\xi_0(1), \xi_0(2), \ldots, \xi_0(i-1)$ have not become extinct at a certain finite time and we would deduce that the number of blocks of the Λ coalescent is finite at this time, which contradicts our assumption. Hence, none of the ancestral types become extinct. Finally, suppose that the Λ coalescent has dust and fix $t, s > 0$. The bridge $F_{0,t}$ has a strictly positive drift $d_t = 1 - \sum_{i\geq 1} \beta_t^i$, $F_{0,t}$ and $F_{t,t+s}$ are independent and $F_{0,t+s} = F_{t,t+s} \circ F_{0,t}$. Since the jump locations of $F_{t,t+s}$ are i.i.d. uniform$[0,1]$, we deduce from the law of large numbers that a proportion d_t of these jumps define ancestral types for $F_{0,t+s}$ which have no positive descendence in $F_{0,t}$, that is, ancestral types that emerge from dust by time $t+s$. □

We now focus on the coming down from infinity case, and consider the following event

$$E := \{\text{There exists } t > 0 \text{ s.t. two ancestral types become extinct simultaneously at time } t\} \tag{26}$$

Lemma 2.9 *When $\Lambda \in \mathbb{CDI}$, the event E is trivial, that is, $\mathbb{P}(E) \in \{0, 1\}$.*

Remark 2.10 *If $\Lambda \notin \mathbb{CDI}$, the lemma still holds and the event E has probability 0 since none of the ancestral types get extinct.*

Proof Consider a Λ lookdown graph $\mathcal{P}$ and an independent sequence of i.i.d. uniform$[0,1]$ random variables $(\xi_0(i))_{i\geq 1}$. Set $(\xi_t(i), t\geq 0)_{i\geq 1} := \mathscr{L}_0(\mathcal{P}, (\xi_0(i))_{i\geq 1})$ and $(\rho_t, t\geq 0) := \mathscr{E}_0(\mathcal{P}, (\xi_0(i))_{i\geq 1})$. We know that ρ is a Λ Fleming-Viot process. We stress that E is independent of $(\xi_0(i))_{i\geq 1}$ and only depends on the lookdown graph. Thus, introduce the filtration $\mathcal{F}$ as follows.

$$\mathcal{F}_t := \sigma\{\mathcal{P}_{|[0,t]\times\mathcal{S}^2_\infty}\} \text{ for all } t\geq 0$$

One easily remarks that $\mathcal{F}_{0+}$ is a trivial σ-field under $\mathbb{P}$. Set $d^i := \inf\{t\geq 0 : Y_t(i) = \infty\}$, that is, the death time of the i-th type in the lookdown representation (see Definition 2.7), which is a stopping time of the filtration $\mathcal{F}$. Since $d^i \downarrow 0$ almost surely as $i\to\infty$, we deduce that $\bigcap\limits_{i\geq 1}\mathcal{F}_{d^i} = \mathcal{F}_{0+}$. For each $i\geq 1$, define the following event

$$E_i := \{\text{There exists } t\leq d^i \text{ s.t. two ancestral types become extinct simultaneously at time } t\} \quad (27)$$

and $E_\infty := \bigcap\limits_{i\geq 1} E_i$. Clearly, $E_\infty \in \mathcal{F}_{0+}$ so it has probability 0 or 1 under $\mathbb{P}$.

Case 1 : $\mathbb{P}(E_\infty) = 1$ Since $E_\infty \subset E$, we deduce that $\mathbb{P}(E) = 1$.

Case 2 : $\mathbb{P}(E_\infty) = 0$ Suppose there exists $n\geq 1$ such that $\mathbb{P}(E_n) > 0$. It implies that there exists $i\geq n$ and $p>0$ such that

$$\mathbb{P}(\{d^i = d^{i+1}\}) = p \quad (28)$$

For each $k\geq i$, let

$$\tau_k := \inf\{t\geq 0 : Y_t(i)\geq k\} \quad (29)$$

which is a stopping-time of the filtration $\mathcal{F}$. Remark that

$$\{d^i = d^{i+1}\} = \{Y(i) \text{ and } Y(i+1) \text{ reach } \infty \text{ simultaneously}\}$$

By applying the Markov property at time τ_k (and the fact that the distribution of the lookdown graph is invariant by shift in time), we deduce that

$$\begin{aligned} \mathbb{P}(\{d^i = d^{i+1}\}) &= \mathbb{P}(\{d^{Y_{\tau_k}(i)} = d^{Y_{\tau_k}(i+1)}\}) && (30)\\ &\leq \mathbb{P}(E_k) && (31)\end{aligned}$$

Hence, for each $k\geq i$, $\mathbb{P}(E_k)\geq p$. Taking the limit when $k\uparrow\infty$, we deduce that $\mathbb{P}(E_\infty)\geq p$, which contradicts our assumption. This implies that for each $i\geq 1$

$$\mathbb{P}(\{d^i = d^{i+1}\}) = 0 \quad (32)$$

which in turn implies that $\mathbb{P}(E) = 0$.

This ends the proof. $\square$

We now determine the probability of E for some important measures Λ.

Proposition 2.11 *For a $Beta(2-\alpha,\alpha)$ Fleming-Viot process, with $1<\alpha\leq 2$, we have $\mathbb{P}(E)=0$.*

Proof We provide a sketch of the proof, since it is based on well-known results. Further details can be found in [2, 9, 14] or in Section 6. Consider the encoding of the lookdown representation of a Beta$(2-\alpha,\alpha)$ Fleming-Viot process via the α-stable height process, developped by Berestycki et al. (see [2]), for $1<\alpha<2$. Let $(H_t, t\geq 0)$ be the height process associated to the α-stable Lévy process, as defined by Duquesne and Le Gall in [14], and $T_r := \inf\{t\geq 0 : L(t,0)>r\}$ where $L(t,x)$ is the local-time accumulated by H at level x until time t. An extension of the Ray-Knight theorem given in [14] ensures that $Z := (L(T_r,s), s\geq 0)$ is a continuous state branching process (CSBP in short) with an α-stable branching mechanism started from r. Consider for all t the random level

$$U(t) := \inf\{s>0 : \int_0^s \frac{\alpha(\alpha-1)\Gamma(\alpha)}{Z_x^{\alpha-1}}dx > t\} \tag{33}$$

Roughly speaking, U maps coalescent time scale to CSBP time scale (this is a consequence of Theorem 1.1 in [9]). Consider all the excursions of H above level 0. These excursions are distributed according to a Poisson point process on $[0, Z_0 = r]\times\mathscr{E}_+$ where $\mathscr{E}_+$ stands for the set of positive excursions, with intensity measure $dt\otimes\nu^{exc}$ where ν^{exc} is the excursion measure of the height process. Since the measure ν^{exc} gives a finite mass to the set of positive continuous functions whose supremum is greater than any given threshold $\epsilon>0$, one can order those excursions by decreasing height. The lookdown representation of [2] can then be restated as follows. Consider a sequence of i.i.d. uniform$[0,1]$ random variables $(\xi_0(i), i\geq 1)$, and associate each type $\xi_0(i)$ to the i-th highest excursion of H above level 0. Then at any time $t>0$, $\xi_t(j)$ is defined as the type of the j-th highest excursion above level $U(t)$, for each $j\geq 1$, where the type of the j-th highest excursion above level $U(t)$ is the type of the (unique) excursion above 0 in which it is embedded. Results of [2] ensure that $(\xi_t(i), t\geq 0)_{i\geq 1}$ is a lookdown representation of a Beta$(2-\alpha,\alpha)$ Fleming-Viot process.
It is then sufficient to prove that two distinct particles $(Y_t(i), t\geq 0)$ and $(Y_t(j), t\geq 0)$, with $1<i<j$, never reach ∞ simultaneously (see Definition 2.7) in order to prove our proposition. This is equivalent to saying that the i-th and j-th highest excursions above level 0 do not die simultaneously in the coalescent time scale. But it is clear that they do not die simultaneously in the CSBP time scale, since their death time in this time scale is simply their height. As U is a continuous mapping, we deduce that they do not die simultaneously in the coalescent time scale either.
The Kingman case, $\alpha=2$, follows from quite similar arguments applied to a reflected Brownian motion (see [1] for a description of this encoding or Section 6). □

Definition 2.12 *Consider a measure Λ on $[0,1)$. Suppose that there exists $1<\alpha\leq 2$ such that, $x^{-2}\Lambda(dx) = x^{-2}Beta(2-\alpha,\alpha)(dx)+\mu(dx)$ where $\mu(dx)$ is a finite measure on $(0,1)$. Then we say that Λ is an almost Beta$(2-\alpha,\alpha)$ measure, with $1<\alpha\leq 2$.*

Corollary 2.13 *Let Λ be an almost Beta$(2-\alpha,\alpha)$ measure, with $1<\alpha\leq 2$, then $\mathbb{P}(E)=0$ for the corresponding Λ Fleming-Viot process.*

Proof The proof of Lemma 2.9 has shown that $E \in \mathcal{F}_{0+}$. That is, E depends on the very initial behaviour of ρ. Adding a finite mass μ to the measure $x^{-2}\Lambda(dx)$ does not change the initial behaviour of the corresponding Λ Fleming-Viot process. Indeed, in the lookdown reprensentation, the first birth event induced by μ arrives at a time T, distributed as an exponential random variable with parameter $\mu((0,1))$, which is strictly positive almost surely. Hence until time T, the behaviour of the Λ Fleming-Viot process coincides with the behaviour of the Beta$(2-\alpha,\alpha)$ Fleming-Viot process. Then, the previous Proposition ensures the asserted result. □

Finally, we conjecture that this result is true for any Λ coalescent that comes down from infinity. We intend to prove this result in a future work.

Conjecture 2.14 *If* $\Lambda \in \mathbb{CDI}$, $\mathbb{P}(E) = 0$.

3 Flows of partitions

In this section, we introduce flows of partitions. We begin by giving a deterministic definition of a flow of partitions and we show that, under a technical assumption, it is equivalent with a deterministic lookdown graph. Then, we randomize these objects and introduce the stochastic flows of partitions. Many results are technical and will be useful later in this work. However, on first reading one can restrict oneself to Subsection 3.1 and Definition 3.6.

3.1 Deterministic flows of partitions

A càdlàg function is a right continuous function with left limits, while a làdcàg function is a left continuous function with right limits.

Definition 3.1 *A deterministic flow of partitions is a collection* $(\hat{\pi}_{s,t}, -\infty < s \leq t < \infty)$ *of partitions such that*

- *For every* $r < s < t$, $\hat{\pi}_{r,t} = Coag(\hat{\pi}_{s,t}, \hat{\pi}_{r,s})$.
- *For every* $s \in \mathbb{R}$, $(\hat{\pi}_{s,t}, t \in [s,\infty))$ *is a càdlàg* $\mathscr{P}_\infty$*-valued function and* $(\hat{\pi}_{s-r,s}, r \geq 0)$ *is a làdcàg* $\mathscr{P}_\infty$*-valued function.*

Furthermore, if for all $s \in \mathbb{R}$, $\hat{\pi}_{s-,s}$ *has at most one unique non-singleton block, then we say that* $\hat{\pi}$ *is a deterministic flow of partitions without simultaneous mergers.*

Such objects are naturally related with deterministic lookdown graphs, let us show how. We introduce, for each $n \in \mathbb{N} \cup \{\infty\}$, $\mathscr{P}_n^2$ as the subset of $\mathscr{P}_n$ whose elements have a unique non singleton block. Moreover, we introduce the map $g_n : \mathcal{S}_n^2 \to \mathscr{P}_n^2$ defined as follows. For any $u = \{u^1, \ldots, u^n\} \in \mathcal{S}_n^2$, set $I := \{i \in [n] : u^i = 1\}$. Let $g_n(u)$ be the unique element of $\mathscr{P}_n^2$ such that for all $i, j \in [n]$

$$i \overset{g_n(u)}{\sim} j \Leftrightarrow u^i = u^j = 1 \tag{34}$$

Thus $g_n(u)$ has a unique non singleton block. Obviously, g_n is a bijection from $\mathcal{S}_n^2$ to $\mathscr{P}_n^2$.

Proposition 3.2 *There exists a one-to-one correspondence between the set of deterministic flows of partitions without simultaneous mergers and the set of deterministic lookdown graphs.*

Proof Consider a deterministic lookdown graph p. For each $n \in \mathbb{N}$ and every $s < t$, let $(t_m, u_m)_{1\leq m\leq q}$ denote the finitely many atoms of $p_{|(s,t]\times\mathcal{S}_n^2}$ in the increasing order of their time coordinate and set

$$\hat{\pi}_{s,t}^{[n]} := Coag(g_n(u_q), Coag(g_n(u_{q-1}), \ldots, Coag(g_n(u_2), g_n(u_1)) \ldots)) \tag{35}$$

and $\hat{\pi}_{s,s}^{[n]} := 0_{[n]}$. Obviously, the collection of partitions $(\hat{\pi}_{s,t}^{[n]}, n \in \mathbb{N})$ is compatible and defines by a projective limit a unique partition $\hat{\pi}_{s,t}$ such that its restriction to $[n]$ is $\hat{\pi}_{s,t}^{[n]}$ for each $n \in \mathbb{N}$. Thus, it is straightforward to verfiy that the collection of partitions $(\hat{\pi}_{s,t}, -\infty < s \leq t < \infty)$ is a deterministic flow of partitions without simultaneous mergers.
Conversely consider a deterministic flow of partitions $\hat{\pi}$ without simultaneous mergers. We define the collection of its jumps $p := \underset{s:\pi_{s-,s}\neq 0_{[\infty]}}{\cup} \{s, g_\infty^{-1}(\hat{\pi}_{s-,s})\}$ which is a point collection on $\mathbb{R}\times\mathcal{S}_\infty^2$. Since for each $n \in \mathbb{N}$, the restriction $\hat{\pi}^{[n]}$ of the flow to $\mathscr{P}_n$ has a càdlàg property and that $\mathscr{P}_n$ is a finite set, we deduce that it makes finitely many jumps in any finite interval of time. Therefore, we conclude that p is a deterministic lookdown graph. $\square$

The interest of this correspondence is that the flow of partitions entirely encodes the genealogical relationships of the lookdown graph. Indeed, consider a deterministic lookdown graph p and let $\hat{\pi}$ be the deterministic flow of partitions associated via the preceding bijection. Set $(\xi_{s,t}(i), t \in [s,\infty)) = \mathscr{L}_s(p, (\xi_{s,s}(i))_{i\geq 1})$, for a given sequence of initial types $(\xi_{s,s}(i))_{i\geq 1}$. Then, we have the following identity.

Lemma 3.3 *For all $t \in [s,\infty)$ and all $i, j \in \mathbb{N}$*

$$\xi_{s,t}(j) = \xi_{s,s}(i) \Leftrightarrow j \in \hat{\pi}_{s,t}(i) \tag{36}$$

Proof This is a simple consequence of the properties of the coagulation operator and of the definition of the lookdown process. $\square$

Therefore, the partition-valued function obtained as the genealogy of the population alive at time t is given by $(\hat{\pi}_{t-r,t}, r \geq 0)$, and we can define a new notation $\mathscr{L}_s(\hat{\pi}, (\xi_{s,s}(i))_{i\geq 1}) := \mathscr{L}_s(p, (\xi_{s,s}(i))_{i\geq 1})$.
We end this section with a property of deterministic flows of partitions, which will be useful later. Consider a deterministic flow of partitions $\hat{\pi}$. Recall our notation $\pi(i)$ for the i-th block of a partition π and $|\pi(i)|$ for the asymptotic frequency of this block when it exists (see Subsection 2.1).

Lemma 3.4 *For each $i \geq 1$, for every $t \in \mathbb{R}$, $(\underset{j\in[i]}{\cup} \hat{\pi}_{t-r,t}(j), r \geq 0)$ is a non decreasing (for the inclusion) collection of subsets of $\mathbb{N}$. Therefore, for each $i \geq 1$ the function $(|\hat{\pi}_{t-r,t}(i)|, r \geq 0)$ admits right and left limits, when these asymptotic frequencies exist.*

Proof Fix $t \in \mathbb{R}$ and consider the process $(\hat{\pi}_{t-r,t}, r \geq 0)$. This process evolves through coagulation events. For each $i, n \in \mathbb{N}$ and every $r' \geq r \geq 0$, we have the following identity

$$n \in \underset{j\in[i]}{\cup} \hat{\pi}_{t-r,t}(j) \Rightarrow n \in \underset{j\in[i]}{\cup} \hat{\pi}_{t-r',t}(j)$$

Indeed, suppose that $n \in \underset{j\in[i]}{\cup} \hat{\pi}_{t-r,t}(j)$. For any $r' \geq r$, we have

$$\hat{\pi}_{t-r',t} = Coag(\hat{\pi}_{t-r,t}, \hat{\pi}_{t-r',t-r})$$

Therefore Equation (6) ensures the asserted identity, which in turn entails that $(\underset{j\in[i]}{\cup} \hat{\pi}_{t-r,t}(j), r \geq 0)$ is a non decreasing collection of subsets of $\mathbb{N}$. Finally, suppose that $(\hat{\pi}_{t-r,t}, r \geq 0)$ admits asymptotic frequencies. We deduce from the non decreasing property of $(\underset{j\in[i]}{\cup} \hat{\pi}_{t-r,t}(j), r \geq 0)$ that $(\sum_{j\in[i]} |\hat{\pi}_{t-r,t}(j)|, r \geq 0)$ admits right and left limits. Thus, $(|\hat{\pi}_{t-r,t}(i)|, r \geq 0)$ admits right and left limits at any point since it is equal to $(\sum_{j\in[i]} |\hat{\pi}_{t-r,t}(j)| - \sum_{j\in[i-1]} |\hat{\pi}_{t-r,t}(j)|, r \geq 0)$. □

Remark 3.5 *Even if* $(|\hat{\pi}_{-r,0}(i)|, r \geq 0)$ *admits right and left limits, the identities* $\lim_{r\downarrow t} |\hat{\pi}_{-r,0}(i)| = |\hat{\pi}_{(-t)-,0}(i)|$ *and* $\lim_{r\uparrow t} |\hat{\pi}_{-r,0}(i)| = |\hat{\pi}_{-t,0}(i)|$ *do not necessarily hold. For instance, consider the partition* $\hat{\pi}_{-t,0}$ *that has a unique non singleton block* $\{1, n+1, n+2, \ldots\}$ *whenever* $t \in [1/n, 1/(n+1))$, *and* $\hat{\pi}_{0,0} = 0_{[\infty]}$. *In that case,* $(\hat{\pi}_{-t,0}, t \geq 0)$ *is làdcàg, and admits asymptotic frequencies for all* $t \geq 0$. *However* $\lim_{r\downarrow 0} |\hat{\pi}_{-r,0}(1)| = 1$ *while* $\hat{\pi}_{0-,0} = \hat{\pi}_{0,0} = 0_{[\infty]}$ *and therefore* $|\hat{\pi}_{0-,0}(1)| = 0$. *This is due to the topology induced on* $\mathscr{P}_\infty$ *by the metric* $d_{\mathscr{P}}$, *which does not give any information on the asymptotic frequencies.*

3.2 Stochastic flows of partitions

In this subsection, we introduce the stochastic flows of partitions which are new objects of independent interest. We present the construction from a lookdown graph of a stochastic flow of partitions, then we show how a given stochastic flow of partitions encodes pathwise a lookdown graph.

Definition 3.6 *A stochastic flow of partitions is a random collection of partitions* $\hat{\Pi} = (\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$ *that enjoys the following properties:*

- *For every* $r < s < t$, $\hat{\Pi}_{r,t} = Coag(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$ $a.s.$ *(dual cocycle property).*
- $\hat{\Pi}_{s,t}$ *is an exchangeable random partition whose law only depends on* $t - s$. *Furthermore, for any* $s_1 < s_2 < \ldots < s_n$ *the partitions* $\hat{\Pi}_{s_1,s_2}, \hat{\Pi}_{s_2,s_3}, \ldots, \hat{\Pi}_{s_{n-1},s_n}$ *are independent.*
- $\hat{\Pi}_{0,0} = 0_{[\infty]}$ *and* $\hat{\Pi}_{-t,0} \to 0_{[\infty]}$ *in probability as* $t \downarrow 0$, *for the distance* $d_{\mathscr{P}}$.

Moreover, when the process $(\hat{\Pi}_{-t,0}, t \geq 0)$ *is a* Λ *coalescent, we say that* $\hat{\Pi}$ *is associated with the measure* Λ *or, in short, is a* Λ *flow of partitions.*

Remark that we have defined the *forward* flow of partitions. By analogy with bridges, one can define the *backward* flow of partitions $(\Pi_{s,t}, -\infty < s \leq t < \infty)$ by setting

$$\Pi_{s,t} := \hat{\Pi}_{-t,-s} \tag{37}$$

Remark 3.7 *It is worth noting that flows of partitions have been introduced separately by Foucart [15] as a population model for Ξ Fleming-Viot processes with immigration.*

One should pay attention to the fact that the trajectories of a given stochastic flow of partitions $\hat{\Pi}$ are not necessarily deterministic flows of partitions in the sense of our definition. Indeed, the coagulation of the partitions holds almost surely for every triplet $r < s < t$, but not necessarily simultaneously for every triplet $r < s < t$ almost surely. Thus many irregularities can affect a single trajectory. We now derive the construction from a lookdown graph.

Construction from a lookdown graph. Consider a random lookdown graph $\mathcal{P}$ assumed to be a Poisson point process on $\mathbb{R} \times \mathcal{S}^2_\infty$ with intensity $dt \otimes \mu$. Necessarily μ is a measure on $\mathcal{S}^2_\infty$ whose restriction to $\mathcal{S}^2_n$ has a finite mass, for all $n \in \mathbb{N}$. For each $\omega \in \Omega$, one can define a deterministic flow of partitions $\hat{\Pi}^{\mathcal{P}}(\omega)$ using Proposition 3.2 and the point collection $\mathcal{P}(\omega)$.

Proposition 3.8 *The process $\hat{\Pi}^{\mathcal{P}}$ is a stochastic flow of partitions. Furthermore, if $\mu = \mu_K + \mu_\Lambda$ then it is a Λ flow of partitions.*

Proof Fix $r \leq s \leq t$. For all $n \in \mathbb{N}$, we easily deduce from the definition that

$$\hat{\Pi}^{\mathcal{P},[n]}_{r,t} = Coag(\hat{\Pi}^{\mathcal{P},[n]}_{s,t}, \hat{\Pi}^{\mathcal{P},[n]}_{r,s})$$

This implies that $\hat{\Pi}^{\mathcal{P}}_{r,t} = Coag(\hat{\Pi}^{\mathcal{P}}_{s,t}, \hat{\Pi}^{\mathcal{P}}_{r,s})$.
The independence and continuity properties are straightforward from the definition and the fact that the Poisson point process is stationary in time. Finally, when $\mu = \mu_K + \mu_\Lambda$, the Poissonian construction of coalescent processes (see [4]) ensures that $(\hat{\Pi}^{\mathcal{P}}_{-t,0}, t \geq 0)$ is a Λ coalescent. □

Remark 3.9 *Remark that the trajectories of a stochastic flow of partitions constructed from a lookdown graph are deterministic flows of partitions.*

This concludes the Construction from a lookdown graph.

It is then natural to wonder if one can define from a stochastic flow of partitions $\hat{\Pi}$ a random lookdown graph. However the potential existence of irregularities in the flow induces many difficulties. It is then necessary to define a modification of this flow, such that the trajectories of this modification are deterministic flows of partitions a.s. Roughly speaking, we define a regularized modification $\tilde{\hat{\Pi}}$ of the original stochastic flow of partitions. This will allow us define a random lookdown graph $\mathcal{P}$ pathwise

from $\tilde{\hat{\Pi}}$, such that the stochastic flow of partitions $\hat{\Pi}^{\mathcal{P}}$ constructed from $\mathcal{P}$ verifies $\hat{\Pi}^{\mathcal{P}} = \tilde{\hat{\Pi}}$ a.s.
First, we give a general result about stochastic flows of partitions. Introduce the following filtrations.

$$\mathcal{F}_t^{\Pi} := \sigma\{\Pi_{r,s}, r \le s \le t\} \tag{38}$$
$$\mathcal{F}_t^{\hat{\Pi}} := \sigma\{\hat{\Pi}_{r,s}, r \le s \le t\} \tag{39}$$

We have the following property.

Lemma 3.10 *The process* $(\Pi_{0,t}, 0 \le t < \infty)$ *(resp.* $(\hat{\Pi}_{0,t}, 0 \le t < \infty)$*) is a* $\mathcal{F}^{\Pi}$ *(resp.* $\mathcal{F}^{\hat{\Pi}}$*) Markov process taking values in* $\mathscr{P}_\infty$ *with a Feller semigroup. For any* $\mathcal{F}^{\Pi}$*-stopping time (resp.* $\mathcal{F}^{\hat{\Pi}}$*-stopping time)* T*,* $(\Pi_{T,T+t}, 0 \le t < \infty)$ *(resp.* $(\hat{\Pi}_{T,T+t}, 0 \le t < \infty)$*) is a process independent of* $\mathcal{F}_T^{\Pi}$ *(resp.* $\mathcal{F}_T^{\hat{\Pi}}$*) with the same law as* $(\Pi_{0,t}, 0 \le t < \infty)$ *(resp.* $(\hat{\Pi}_{0,t}, 0 \le t < \infty)$*).*

Proof Consider $(\hat{\Pi}_{0,t}, 0 \le t < \infty)$. The very definition of stochastic flows of partitions ensures that this process is Markov with a semigroup Q_t such that for every $\pi \in \mathscr{P}_\infty$

$$Q_t(\pi, .) \overset{(d)}{=} Coag(\hat{\Pi}_{0,t}, \pi) \tag{40}$$

Clearly $Q_t \circ Q_s = Q_{t+s}$. Recall that $(\mathscr{P}_\infty, d_{\mathscr{P}})$ is compact and consider a bounded continuous function $f : \mathscr{P}_\infty \to \mathbb{R}$.

$$Q_t f(\pi) = \mathbb{E}[f(Coag(\hat{\Pi}_{0,t}, \pi))]$$

Since $Coag$ is continuous, by dominated convergence we get that $Q_t f$ is a bounded continuous function. Notice that $\hat{\Pi}_{0,t} \to 0_{[\infty]}$ in probability. Then, for any $\pi \in \mathscr{P}_\infty$

$$Q_t f(\pi) = \mathbb{E}[f(Coag(\hat{\Pi}_{0,t}, \pi))] \underset{t \downarrow 0}{\to} f(\pi)$$

This implies the Feller property of Q. The strong Markov property is due to the Feller property, and the stationarity and independence of the increments of $\hat{\Pi}$ ensures that $(\hat{\Pi}_{T,T+t}, 0 \le t < \infty)$ has the same distribution as $(\hat{\Pi}_{0,t}, 0 \le t < \infty)$. The proof for the process $(\Pi_{0,t}, 0 \le t < \infty)$ is quite similar. □

Regularization of a stochatic flow of partitions. For each $s \in \mathbb{Q}$, $(\hat{\Pi}_{s-r,s}, r \in [0, \infty))$ (resp. $(\hat{\Pi}_{s,t}, t \in [s, \infty))$) admits a làdcàg (resp. càdlàg) modification thanks to Lemma 3.10. There exists an event $\Omega_{\hat{\Pi}}$ of probability 1 such that on this event, we have

- For every rational values $r < s < t$, $\hat{\Pi}_{r,t} = Coag(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$.
- For every rational value s, the process $(\hat{\Pi}_{s,t}, t \in [s, \infty) \cap \mathbb{Q})$ is càdlàg.
- For every rational value s, the process $(\hat{\Pi}_{s-r,s}, r \in [0, \infty) \cap \mathbb{Q})$ is làdcàg.

We now define for every $(s, t) \in \mathbb{R}$ the partition $\tilde{\hat{\Pi}}_{s,t}$ on the event $\Omega_{\hat{\Pi}}$ as follows.

Lemma 3.11 *On the event $\Omega_{\hat{\Pi}}$, the following random partition is well-defined.*

$$\tilde{\hat{\Pi}}_{s,t} := \begin{cases} \lim\limits_{r \downarrow s, r \in \mathbb{Q}} \hat{\Pi}_{r,t} \text{ if } t \in \mathbb{Q} \\ \lim\limits_{v \downarrow t, v \in \mathbb{Q}} \hat{\Pi}_{s,v} \text{ if } s \in \mathbb{Q} \\ Coag(\tilde{\hat{\Pi}}_{q,t}, \tilde{\hat{\Pi}}_{s,q}) \text{ for any arbitrary rational } q \in (s,t) \text{ if } s,t \notin \mathbb{Q} \end{cases} \tag{41}$$

Furthermore, for every $r \leq s \leq t$, $\tilde{\hat{\Pi}}_{r,t} = Coag(\tilde{\hat{\Pi}}_{s,t}, \tilde{\hat{\Pi}}_{r,s})$.

Proof We work on the event $\Omega_{\hat{\Pi}}$ throughout this proof. Suppose that $s \in \mathbb{Q}$. Since $(\hat{\Pi}_{s,v}, v \in [s,\infty) \cap \mathbb{Q})$ is the restriction of a càdlàg modification of $(\hat{\Pi}_{s,v}, v \in [s,\infty))$ to its rational marginals, the limit is well-defined. The case $t \in \mathbb{Q}$ is obtained similarly. In both cases, for any $r < s < t$ such that either s is rational or r and t are rational, we have

$$\tilde{\hat{\Pi}}_{r,t} = Coag(\tilde{\hat{\Pi}}_{s,t}, \tilde{\hat{\Pi}}_{r,s})$$

This is due to the continuity of the coagulation operator (see Subsection 2.1) and the assumption made on the event $\Omega_{\hat{\Pi}}$.
Finally, suppose that $s,t \notin \mathbb{Q}$. It suffices to show that $Coag(\tilde{\hat{\Pi}}_{q,t}, \tilde{\hat{\Pi}}_{s,q})$ does not depend on the value $q \in (s,t)$. Consider two such values q, q', suppose that $q < q'$ and use Equation (7) to obtain

$$\begin{aligned} Coag(\tilde{\hat{\Pi}}_{q',t}, \tilde{\hat{\Pi}}_{s,q'}) &= Coag(\tilde{\hat{\Pi}}_{q',t}, Coag(\tilde{\hat{\Pi}}_{q,q'}, \tilde{\hat{\Pi}}_{s,q})) = Coag(Coag(\tilde{\hat{\Pi}}_{q',t}, \tilde{\hat{\Pi}}_{q,q'}), \tilde{\hat{\Pi}}_{s,q}) \\ &= Coag(\tilde{\hat{\Pi}}_{q,t}, \tilde{\hat{\Pi}}_{s,q}) \end{aligned}$$

Thus, the definition of $\tilde{\hat{\Pi}}_{s,t}$ does not depend on $q \in (s,t)$.
Finally, consider three irrational $r < s < t$, and two rational values q, q' such that $q \in (r,s)$ and $q' \in (s,t)$.

$$\begin{aligned} Coag(\tilde{\hat{\Pi}}_{s,t}, \tilde{\hat{\Pi}}_{r,s}) &= Coag(Coag(\tilde{\hat{\Pi}}_{q',t}, \tilde{\hat{\Pi}}_{s,q'}), Coag(\tilde{\hat{\Pi}}_{q,s}, \tilde{\hat{\Pi}}_{r,q})) \\ &= Coag(\tilde{\hat{\Pi}}_{q',t}, Coag(\tilde{\hat{\Pi}}_{s,q'}, Coag(\tilde{\hat{\Pi}}_{q,s}, \tilde{\hat{\Pi}}_{r,q}))) = Coag(\tilde{\hat{\Pi}}_{q',t}, \tilde{\hat{\Pi}}_{r,q'}) = \tilde{\hat{\Pi}}_{r,t} \end{aligned}$$

This concludes the proof. □

On the complementary of $\Omega_{\hat{\Pi}}$, set any arbitrary value to $\tilde{\hat{\Pi}}_{s,t}$.

Proposition 3.12 *The collection of partitions $\tilde{\hat{\Pi}}$ is a modification of $\hat{\Pi}$, that is, for every $s \leq t$, a.s. $\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}$. Furthermore, for each $\omega \in \Omega_{\hat{\Pi}}$, $\tilde{\hat{\Pi}}(\omega)$ is a deterministic flow of partitions.*

Proof By definition, for every rational numbers $s \leq t$, $\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}$ on the event $\Omega_{\hat{\Pi}}$, so it holds a.s. Fix $s \in \mathbb{Q}$. On the event $\Omega_{\hat{\Pi}}$, $(\tilde{\hat{\Pi}}_{s,t}, t \in [s,\infty))$ coincides with a càdlàg modification of $(\hat{\Pi}_{s,t}, t \in [s,\infty))$.

Therefore, for every $t > s$, a.s. $\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}$. Similarly, for $t \in \mathbb{Q}$, for every $s < t$, a.s. $\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}$. Finally, consider two irrationals values $s < t$. Remark that for every rational value $q \in (s,t)$ we have

$$\mathbb{P}(\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}) \geq \mathbb{P}(\tilde{\hat{\Pi}}_{s,q} = \hat{\Pi}_{s,q}; \tilde{\hat{\Pi}}_{q,t} = \hat{\Pi}_{q,t}; \hat{\Pi}_{s,t} = Coag(\hat{\Pi}_{q,t}, \hat{\Pi}_{s,q}))$$

Since the r.h.s. of the previous equation equals 1, we conclude that a.s. $\tilde{\hat{\Pi}}_{s,t} = \hat{\Pi}_{s,t}$. Therefore $\tilde{\hat{\Pi}}$ is a modification of $\hat{\Pi}$.

Let us now prove the second assertion of the Proposition. We work on the event $\Omega_{\hat{\Pi}}$. Fix $s \in \mathbb{R}$, we have to prove that $(\tilde{\hat{\Pi}}_{s,t}, t \in [s,\infty))$ is càdlàg and $(\tilde{\hat{\Pi}}_{s-r,s}, r \geq 0)$ is làdcàg. When $s \in \mathbb{Q}$, this properties hold by definition of $\tilde{\hat{\Pi}}$. Let us focus on the case $s \notin \mathbb{Q}$. We will only prove that $(\tilde{\hat{\Pi}}_{s,t}, t \in [s,\infty))$ is càdlàg, as the proof for the other property is quite similar. Fix $t \in \mathbb{R}$ and a rational $q \in (s,t)$. We have to prove that $\tilde{\hat{\Pi}}_{s,r}$ admits a limit as $r \uparrow t$, and tends to $\tilde{\hat{\Pi}}_{s,t}$ as $r \downarrow t$. For any $r > q$, we have

$$\tilde{\hat{\Pi}}_{s,r} = Coag(\tilde{\hat{\Pi}}_{q,r}, \tilde{\hat{\Pi}}_{s,q})$$

Since $\tilde{\hat{\Pi}}_{q,r}$ admits a limit as $r \uparrow t$, the continuity property of the coagulation operator implies that $\tilde{\hat{\Pi}}_{s,r}$ admits a limit as $r \uparrow t$. Similarly, $\tilde{\hat{\Pi}}_{q,r} \to \tilde{\hat{\Pi}}_{q,t}$ as $r \downarrow t$ and the continuity property of the coagulation operator implies that $\tilde{\hat{\Pi}}_{s,r} \to \tilde{\hat{\Pi}}_{s,t}$ as $r \downarrow t$.

Finally the cocycle property with the coagulation operator has been proved in the preceding lemma. □

Now we suppose that $\hat{\Pi}$ is a Λ flow of partitions. For each $\omega \in \Omega_{\hat{\Pi}}$, let $\mathcal{P}(\omega)$ be the deterministic lookdown graph obtained from $\hat{\Pi}(\omega)$ by applying Proposition 3.2. On the complementary, set any arbitrary values to $\mathcal{P}$.

Proposition 3.13 *If $\hat{\Pi}$ is a Λ flow of partitions, then $\mathcal{P}$ is a Λ lookdown graph. Moreover $\tilde{\hat{\Pi}} = \hat{\Pi}^{\mathcal{P}}$ on $\Omega_{\hat{\Pi}}$, where $\hat{\Pi}^{\mathcal{P}}$ is the flow of partitions defined from the point process $\mathcal{P}$.*

Proof The first assertion is an easy consequence of the Markov property applied to the flow of partitions $\tilde{\hat{\Pi}}$. The second is straightforward from the definition of $\hat{\Pi}^{\mathcal{P}}$. □

Remark 3.14 *From a stochastic flow of partitions, we have been able to define a regularized modification. Note that this operation does not seem possible for a stochastic flow of bridges. Indeed, a key argument in our proof relies on the continuity of the coagulation operator whereas this property does not hold with the composition operator for bridges.*

4 Ancestral types process and stochastic flow of partitions

In this section, we consider a Λ flow of bridges $(F_{s,t}, -\infty < s \leq t < \infty)$. Recall that for each $s \in \mathbb{R}$, $(\tilde{\rho}_{s,t}, t \in [s,\infty))$ is a càdlàg modification of the Λ Fleming-Viot process $(\rho_{s,t}, t \in [s,\infty))$. Under some conditions on Λ, we define an ancestral types process and a stochastic flow of partitions pathwise from the flow of bridges. These objects will allow us to introduce our collection of lookdown processes.

4.1 The primitive Eve

Consider the so-called primitive Eve introduced by Bertoin and Le Gall in [6], Section 5.3, as the random point

$$\mathbf{e}_0 := \inf\{y \in [0,1] : \lim_{t\to\infty} \tilde{F}_{0,t}(y) = 1\} = \sup\{y \in [0,1] : \lim_{t\to\infty} \tilde{F}_{0,t}(y) = 0\}$$

This point depends on the initial time of the collection of bridges considered, here 0. More generally, introduce the primitive Eve process $\mathbf{e}$ by

$$\mathbf{e}_s := \inf\{y \in [0,1] : \lim_{t\to\infty} \tilde{F}_{s,t}(y) = 1\} = \sup\{y \in [0,1] : \lim_{t\to\infty} \tilde{F}_{s,t}(y) = 0\}$$

For each $s \in \mathbb{R}$, the definition holds on an event of probability 1. On the complementary, set $\mathbf{e}_s := 0$.
The interpretation of this Eve process is the following: given two distinct times $-\infty < s < t < \infty$, all the population at time t descends from several individuals alive at time s (corresponding to the jump locations of $\tilde{F}_{s,t}$) and a continuum of individuals (corresponding to the drift part of $\tilde{F}_{s,t}$). As time passes, one jump location will carry a larger and larger proportion of the population asymptotically equal to 1.
Remark that if $\Lambda \in \mathbb{CDI}$, $(\tilde{\rho}_{s,t}(\mathbf{e}_s), t \in [s,\infty))$ reaches the value 1 at a finite random time $T > s$, a.s. This is a clear consequence of Proposition 2.8.
Let us characterize the process $(\mathbf{e}_s, s \in \mathbb{R})$. For all $s < t$, let $\tilde{F}^{-1}_{s,t}$ be the càdlàg inverse of $\tilde{F}_{s,t}$

$$\tilde{F}^{-1}_{s,t}(y) = \inf\{r \in [0,1] : \tilde{F}_{s,t}(r) > y\} \tag{42}$$

if $y \in [0,1[$ and $\tilde{F}^{-1}_{s,t}(1) = \tilde{F}^{-1}_{s,t}(1-)$. Consider two distinct times $-\infty < r < s < \infty$. From the dual cocycle property, we get that almost surely, for all $t \in [s,\infty) \cap \mathbb{Q}$

$$\tilde{F}_{s,t} \circ \tilde{F}_{r,s}(\mathbf{e}_r) - \tilde{F}_{s,t} \circ \tilde{F}_{r,s}(\mathbf{e}_r-) = \tilde{F}_{r,t}(\mathbf{e}_r) - \tilde{F}_{r,t}(\mathbf{e}_r-) = \tilde{\rho}_{r,t}(\mathbf{e}_r) \tag{43}$$

Letting $t \to \infty$, the right hand side tends to 1. Hence, by the very definition of $\mathbf{e}_s$ we get that $\mathbf{e}_s \in [\tilde{F}_{r,s}(\mathbf{e}_r-), \tilde{F}_{r,s}(\mathbf{e}_r)]$. This implies the following result due to Bertoin and Le Gall [6], Section 5.3.

Proposition 4.1 *For all $r < s$, a.s. $\mathbf{e}_r = \tilde{F}^{-1}_{r,s}(\mathbf{e}_s)$.*

This equality describes the backward evolution of the Eve process. Note that for a fixed value s the inverse dual flow $(\tilde{F}^{-1}_{r,s}, r \in (-\infty, s])$ is independent of $\mathbf{e}_s$, since the latter depends only on the future from time s. Hence, using Theorem 4 in [7], we obtain that $(\mathbf{e}_{-r}, r \in [-s,\infty))$ is a Markov process taking values in $[0,1]$ started from $\mathbf{e}_s$ with a Feller semigroup. Indeed, it suffices to remark that $\tilde{F}^{-1}_{r,s} = \Gamma_{-s,-r}$ in the notation of [7], Section 5, before applying Theorem 4 to $(\Gamma_{-s,v}(\mathbf{e}_s), v \in [-s,\infty))$.

4.2 The ancestral types process

We restrict our construction to a certain class of Λ measures specified by the following assumption

Assumption 4.2 Λ *verifies one of these two assertions*

Case 1 : CDI and $\mathbb{P}(E) = 0$ *The* Λ *coalescent comes down from infinity and* $\mathbb{P}(E) = 0$

Case 2 : Bolthausen-Sznitman $\Lambda(dx) = dx$

Recall that almost Beta$(2-\alpha,\alpha)$ measures, with $1 < \alpha \leq 2$, verify the first Assumption. This is a consequence of Corollary 2.13. Whereas the α-stable measure with $\alpha = 1$ corresponds to the Bolthausen-Sznitman coalescent (see [5] for a connection with Neveu continuous state branching process).
Fix a time $s \in \mathbb{R}$, and consider the Λ Fleming-Viot $(\tilde{\rho}_{s,t}, t \in [s,\infty))$. Let us now define the ancestral types process.

Case 1 : CDI and $\mathbb{P}(E) = 0$ There exists an event Ω_s of probability 1 such that the process $(\#\tilde{\rho}_{s,t}, t \in [s,\infty))$ is a càdlàg integer-valued process that decreases by jumps of size 1. Let us denote by $d_s^2 > d_s^3 > \ldots > s$ the sequence of jump times of this process such that

$$\#\tilde{\rho}_{s,t} = i \quad \text{for} \quad d_s^{i+1} \leq t < d_s^i,\ i \geq 2 \tag{44}$$

Then, introduce the sequence $(\mathbf{e}_s^{(i)})_{i\geq 1}$ of ancestral types of $(\tilde{\rho}_{s,t}, t \in [s,\infty))$ by

- $\mathbf{e}_s^{(1)} := \mathbf{e}_s$
- $\mathbf{e}_s^{(i)}$ is the ancestral type that becomes extinct at time d_s^i, for each $i \geq 2$

Case 2 : Bolthausen-Sznitman There exists an event Ω_s of probability 1 such that the following is well-defined. We set

$$\mathbf{e}_s^{(1)} := \mathbf{e}_s$$

We have the following result which will be proved in Section 7.

Proposition 4.3 *Recursively for each integer* $i > 1$*, the following limit exists*

$$\mathbf{e}_s^{(i)} := \inf\{y \in [0,1] : \lim_{t\to\infty} \frac{\tilde{F}_{s,t}(y) - \sum_{1\leq j\leq i-1} \mathbf{1}_{\{y \geq \mathbf{e}_s^{(j)}\}} \tilde{\rho}_{s,t}(\mathbf{e}_s^{(j)})}{1 - \sum_{1\leq j\leq i-1} \tilde{\rho}_{s,t}(\mathbf{e}_s^{(j)})} = 1\} \tag{45}$$

$$= \sup\{y \in [0,1] : \lim_{t\to\infty} \frac{\tilde{F}_{s,t}(y) - \sum_{1\leq j\leq i-1} \mathbf{1}_{\{y \geq \mathbf{e}_s^{(j)}\}} \tilde{\rho}_{s,t}(\mathbf{e}_s^{(j)})}{1 - \sum_{1\leq j\leq i-1} \tilde{\rho}_{s,t}(\mathbf{e}_s^{(j)})} = 0\} \tag{46}$$

$(\mathbf{e}_s^{(i)})_{i\geq 1}$ are then called the ancestral types at time s.

Remark 4.4 *On the event* $\Omega \backslash \Omega_s$ *of zero probability, we set any arbitrary values to the sequence* $(\mathbf{e}_s^{(i)})_{i\geq 1}$.

The following proposition exhibits the connection between this ancestral types process and the pathwise lookdown construction. Fix a time $s \in \mathbb{R}$.

Proposition 4.5 *Consider a Λ lookdown graph $\mathcal{M}$ and a sequence $(\chi_{s,s}(i))_{i\geq 1}$ of r.v. taking distinct values in $[0,1]$. Let $(X_{s,t}, t \in [s,\infty)) := \mathscr{E}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1})$ be the limiting empirical measures of the lookdown process defined from these objects. If $(X_{s,t}, t \in [s,\infty)) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$ a.s., then $(\chi_{s,s}(i))_{i\geq 1} = (\mathbf{e}_s^{(i)})_{i\geq 1}$ a.s.*

Proof For the sake of simplicity, we fix $s = 0$. Consider a lookdown process $(\chi_{0,t}(i), t \in [0,\infty))_{i\geq 1}$ fulfilling the assumptions of the proposition. Denote by Ω^* the event of probability 1 on which $(X_{0,t}, t \in [0,\infty)) = (\tilde{\rho}_{0,t}, t \in [0,\infty))$. We know that on Ω^* there exists a random permutation σ of $\mathbb{N}$ such that

$$(\chi_{0,0}(i))_{i\geq 1} = (\mathbf{e}_0^{(\sigma(i))})_{i\geq 1}$$

To prove the proposition, it suffices to prove that, on $\Omega^* \cap \Omega_0$, σ is the identical permutation of $\mathbb{N}$. We implicitly restrict ourselves to this event for the rest of the proof.

[(CDI) and $\mathbb{P}(E) = 0$] case. For all $i \geq 1$, $t \in [d_0^{i+1}, d_0^i)$, only the atoms $(\mathbf{e}_0^{(j)})_{j\leq i}$ have a positive mass in $\tilde{\rho}_{0,t}$, that is, atoms $(\mathbf{e}_0^{(j)})_{j\geq 1}$ are ordered by decreasing persistence. This also holds for the initial types of the lookdown process $(\chi_{0,t}(i), t \in [0,\infty))_{i\geq 1}$, that is, type $\chi_{0,0}(i)$ will live longer than type $\chi_{0,0}(j)$ in $\tilde{\rho}$, for any $1 \leq i < j$. We deduce that σ is the identical permutation.

[Bolthausen-Sznitman] case. The atoms $(\mathbf{e}_0^{(i)})_{i\geq 1}$ are ordered by decreasing masses at ∞, that is, for all $i \geq 1$

$$\frac{\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)})}{\sum_{j\geq i} \tilde{\rho}_{0,t}(\mathbf{e}_0^{(j)})} \underset{t\to\infty}{\longrightarrow} 1 \tag{47}$$

It suffices to prove the same result for the lookdown process. Let $(\Pi_t, t \geq 0)$ be a Λ coalescent. Recall that $(\Pi_t(i))_{i\geq 1}$ are the blocks of Π_t in the increasing order of their least element and that $(|\Pi_t(i)|)_{i\geq 1}$ are the asymptotic frequencies of the blocks of Π_t, see Equation (5). We know that the genealogy of the lookdown process at time t is given by a Λ coalescent run during t units of time. More precisely, the partition obtained by gathering the levels of the lookdown process at time t who have the same type, is distributed as Π_t. Remark that

$$X_{0,t}(\chi_{0,0}(i)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{\chi_{0,t}(j)=\chi_{0,0}(i)\}} \tag{48}$$

which is the asymptotic frequency of type $\chi_{0,0}(i)$ in the lookdown process at time t. Then we deduce that for every $t > 0$, we have

$$(X_{0,t}(\chi_{0,0}(i)))_{i\geq 1} \overset{(d)}{=} (|\Pi_t(i)|)_{i\geq 1} \tag{49}$$

From the symmetry of the flow and Proposition 4.3, we know that

$$\frac{|\Pi_t(i)|}{\sum_{j\geq i} |\Pi_t(j)|} \underset{t\to\infty}{\overset{(\mathbb{P})}{\longrightarrow}} 1 \tag{50}$$

Hence, we get

$$\frac{X_{0,t}(\chi_{0,0}(i))}{\sum_{j\geq i} X_{0,t}(\chi_{0,0}(j))} \underset{t\to\infty}{\overset{(\mathbb{P})}{\longrightarrow}} 1 \tag{51}$$

Thus on $\Omega^* \cap \Omega_0$, we deduce that $\mathbf{e}_0^{(i)} = \chi_{0,0}(i)$ for all $i \geq 1$. □

We now state a very useful property of the ancestral types process.

Proposition 4.6 *For all $s \in \mathbb{R}$, $(\mathbf{e}_s^{(i)})_{i\geq 1}$ is a sequence of i.i.d. uniform$[0,1]$ r.v., independent of $(F_{r,r'}, r \leq r' \leq s)$, i.e independent of the past of the flow up to time s, and also independent of the sequence of processes $(\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i)}), t \in [s,\infty))_{i\geq 1}$.*

Proof For the sake of simplicity, we fix $s = 0$. The independence of this sequence from the past is an immediate consequence of its definition and of the independence property of a stochastic flow of bridges. Let us now focus on the rest of the proposition. Denote by Φ the measurable map that associates to a Λ Fleming-Viot process its ancestral types according to the definition given at the beginning of this subsection. In particular, we have $\Phi(\tilde{\rho}_{0,t}, t \geq 0) = (\mathbf{e}_0^{(i)})_{i\geq 1}$. Now consider a sequence $(\chi_0(i))_{i\geq 1}$ of i.i.d. uniform$[0,1]$ r.v., and an independent Λ lookdown graph $\mathcal{M}$. Denote by $(X_t, t \geq 0) = \mathscr{E}_0(\mathcal{M}, (\chi_0(i))_{i\geq 1})$ the limiting empirical measures of the lookdown process defined from the latter objects. As recalled in Subsection 2.3, $(X_t, t \geq 0)$ is a Λ Fleming-Viot process. Hence we can define its ancestral types $\Phi(X_t, t \geq 0)$. From Proposition 4.5, we deduce that a.s.

$$\Phi(X_t, t \geq 0) = (\chi_0(i))_{i\geq 1}$$

Therefore, using the fact that $(\tilde{\rho}_{0,t}, t \geq 0) \overset{(d)}{=} (X_t, t \geq 0)$, we deduce that

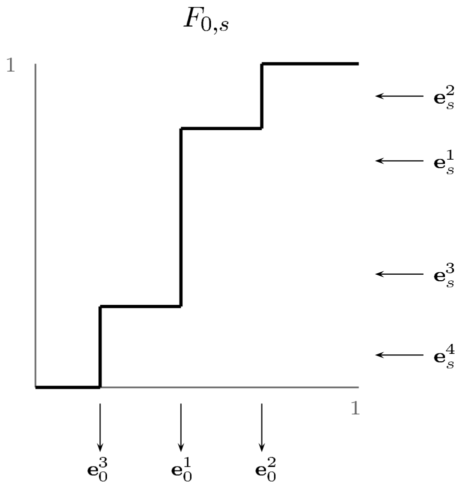
$$((\tilde{\rho}_{0,t}, t \geq 0), (\mathbf{e}_0^{(i)})_{i\geq 1}) \overset{(d)}{=} ((X_t, t \geq 0), (\chi_0(i))_{i\geq 1})$$

This implies that $(\mathbf{e}_0^{(i)})_{i\geq 1}$ is a sequence of i.i.d. uniform$[0,1]$ r.v. Moreover, note that the asymptotic frequencies $(X_t(\chi_0(i)), t \geq 0)_{i\geq 1}$ only depend on the lookdown graph $\mathcal{M}$, thus are independent of the initial types $(\chi_0(i))_{i\geq 1}$. Hence, we deduce that the sequence $(\mathbf{e}_0^{(i)})_{i\geq 1}$ is independent of $(\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)}), t \in [0,\infty))_{i\geq 1}$. □

In the next subsection, we introduce pathwise from F a stochastic flow of partitions using a key property of Bertoin and Le Gall in [6].

4.3 Key property and stochastic flow of partitions

First, let us recall a key property used by Bertoin and Le Gall to compose independent exchangeable random partitions associated to a sequence of independent bridges, as exposed in [6], Lemma 2 and Corollary 1. Consider a sequence of independent uniform$[0,1]$ variables $(V_i)_{i\geq 1}$ and an independent bridge B. Denote by $(A_j)_{j\geq 1}$ the blocks of the partition $\pi(B,(V_i)_{i\geq 1})$ ordered by their smallest element (those blocks are in finite number if B has a finite number of jumps and no drift). The key property is the following. Define a sequence of random variables $V'_j := B^{-1}(V_i)$ for an arbitrary $i \in A_j$. If there is a finite number of blocks in $\pi(B,(V_i)_{i\geq 1})$, complete the sequence with independent uniform$[0,1]$ random variables. The key property yields that the $(V'_j)_{j\geq 1}$ are i.i.d uniform$[0,1]$ variables, independent of $\pi(B,(V_i)_{i\geq 1})$. We will say that $(B,(V_i)_{i\geq 1},(V'_j)_{j\geq 1})$ follows the composition rule.



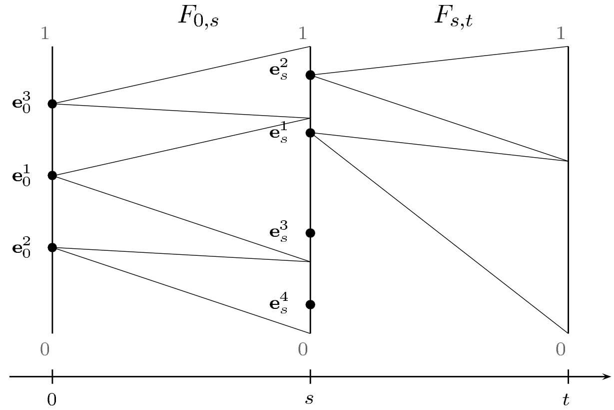


Figure 2: An illustration of Proposition 4.7 in the (CDI) case. On the left, an example of the composition rule. On the right, the genealogical structure arising from this result.

Proposition 4.7 *For all $-\infty < r < s < \infty$, almost surely $(\tilde{F}_{r,s}, (\mathbf{e}_s^{(i)})_{i\geq 1}, (\mathbf{e}_r^{(j)})_{j\geq 1})$ follows the composition rule.*

Proof To alleviate notation, we suppose $r = 0 < s$. There exists an event $\tilde{\Omega} \subset \Omega_0 \cap \Omega_s$ of probability 1 such that on this event for all $t \in \mathbb{Q}$, $\tilde{F}_{0,t} = \tilde{F}_{s,t} \circ \tilde{F}_{0,s}$. We work on this event until the end of the proof. Recall that $(\mathbf{e}_s^{(i)})_{i\geq 1}$ is a sequence of i.i.d uniform$[0,1]$ random variables independent of the past of the flow up to time s, thus independent of the bridge $\tilde{F}_{0,s}$. Those r.v. play the role of the $(V_i)_{i\geq 1}$ in the key property presented above.
According as the Λ-coalescent comes down from infinity or stays infinite, the random number k of blocks of $\pi(\tilde{F}_{0,s}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ is almost surely finite or almost surely infinite. Denote by $(A_j)_{1\leq j\leq k}$ the blocks of $\pi(\tilde{F}_{0,s}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ in the increasing order of their least element. Then, we can define a sequence of random variables $V'_j := \tilde{F}_{0,s}^{-1}(\mathbf{e}_s^{(i_j)})$ where $i_j := \min(A_j)$, for all $j \in [k]$. If k is finite, then we set $V'_j := \mathbf{e}_0^{(j)}$ for all $j > k$. The key property of Bertoin and Le Gall ensures that the $(V'_j)_{j\in[k]}$ are independent of the partition $\pi(\tilde{F}_{0,s}, (\mathbf{e}_s^{(i)})_{i\geq 1})$. To prove the Proposition, it remains to show that:

(i) $\mathbf{e}_0^{(j)} = V'_j$ for all $j \in [k]$ a.s.

(ii) $(\mathbf{e}_0^{(j)})_{j>k}$ are i.i.d. uniform$[0,1]$, independent of $(\mathbf{e}_0^{(j)})_{j\in[k]}$ and of $\pi(\tilde{F}_{0,s}, (\mathbf{e}_s^{(i)})_{i\geq 1})$.

Since k only depends on $(\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)}), t \geq 0)_{i\geq 1}$, we deduce from Proposition 4.6 that $(\mathbf{e}_0^{(j)})_{j>k}$ are i.i.d. uniform$[0,1]$, independent of $(\mathbf{e}_0^{(j)})_{j\in[k]}$. Furthermore, since $\tilde{F}_{0,s}$ only depends on $(\mathbf{e}_0^{(j)})_{j\in[k]}$ and on $(\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)}), t \geq 0)_{i\geq 1}$, we easily deduce that $(\mathbf{e}_0^{(j)})_{j>k}$ are independent of $\tilde{F}_{0,s}$. Finally since $(\mathbf{e}_0^{(j)})_{j>k}$ are independent of the future of the flow F after time s, it is clear that they are independent of $(\mathbf{e}_s^{(i)})_{i\geq 1}$. Therefore, they are independent of $\pi(\tilde{F}_{0,s}, (\mathbf{e}_s^{(i)})_{i\geq 1})$. The second assertion follows. Let us prove the first assertion.

Note that the $(\mathbf{e}_0^{(j)})_{1\le j\le k}$ are a reordering of the $(V_j')_{1\le j\le k}$. Indeed, the $(\mathbf{e}_0^{(j)})_{1\le j\le k}$ correspond to the jump locations of $\tilde{F}_{0,s}$, and the $(V_j')_{1\le j\le k}$ form the set of values taken by $(\tilde{F}_{0,s}^{-1}(\mathbf{e}_s^{(i)}))_{i\ge 1}$.

Case 1 : CDI and $\mathbb{P}(E)=0$ We stress that for each $j\in[k]$, for all $t\in[s,\infty)\cap\mathbb{Q}$ (see Figure 2)

$$\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i_j)}) \le \tilde{\rho}_{0,t}(V_j') \le \tilde{\rho}_{s,t}([0,1]\backslash\{\mathbf{e}_s^{(1)},\dots,\mathbf{e}_s^{(i_j-1)}\})$$

Denote by $\tilde{d}^j := \inf\{t\ge s : \tilde{\rho}_{0,t}(V_j')=0\}$. The previous identity ensures that for each $j\in[k]$

$$\tilde{d}^j = d_s^{i_j} \tag{52}$$

Indeed, $\tilde{\rho}_{s,t}([0,1]\backslash\{\mathbf{e}_s^{(1)},\dots,\mathbf{e}_s^{(i_j-1)}\})$ reaches 0 at time $d_s^{i_j}$, and by definition, $\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i_j)})$ reaches 0 at this same time. Since both the $(d_s^{i_j})_{j\in[k]}$ and the $(d_0^j)_{j\in[k]}$ are strictly decreasing and since the $(\mathbf{e}_0^{(j)})_{1\le j\le k}$ are a reordering of the $(V_j')_{1\le j\le k}$, we easily conclude that $V_j'=\mathbf{e}_0^{(j)}$ for all $j\in[k]$.

Case 2 : Bolthausen-Sznitman We know that :

$$\lim_{t\to\infty}\tilde{\rho}_{0,t}(\tilde{F}_{0,s}^{-1}(\mathbf{e}_s^{(1)}))=1 \tag{53}$$

By definition of the ancestral types process, it follows that $\mathbf{e}_0^{(1)}=V_1'$.
From Proposition 4.3 we know that for each $j\ge 2$

$$\lim_{t\to\infty}\frac{\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i_j)})}{1-\sum_{1\le l\le i_j-1}\tilde{\rho}_{s,t}(\mathbf{e}_s^{(l)})}=1 \tag{54}$$

thus we get

$$\lim_{t\to\infty}\frac{\tilde{\rho}_{0,t}(V_j')}{1-\sum_{1\le l\le j-1}\tilde{\rho}_{0,t}(V_l')} \ge \lim_{t\to\infty}\frac{\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i_j)})}{1-\sum_{1\le l\le i_j-1}\tilde{\rho}_{s,t}(\mathbf{e}_s^{(l)})}=1 \tag{55}$$

Since the $(\mathbf{e}_0^{(j)})_{j\ge 1}$ are a reordering of the $(V_j')_{j\ge 1}$, we deduce from the previous inequation and the Proposition 4.3 that $V_j'=\mathbf{e}_0^{(j)}$ for each $j\ge 2$.

This ends the proof of the proposition. □

Remark that, in the **(CDI) and** $\mathbb{P}(E)=0$ case, the descendents of individuals $(\mathbf{e}_0^{(j)})_{j>k}$ get extinct by time s. In the genealogical interpretation, $(\mathbf{e}_0^{(j)})_{1\le j\le k}$ are the ancestors of the k oldest families in the population alive at time s.
From now on it will be convenient to introduce, for each $s\le t\in\mathbb{R}$,

$$\hat{\Pi}_{s,t} := \pi(\tilde{F}_{s,t},(\mathbf{e}_t^{(i)})_{i\ge 1}) \tag{56}$$

Proposition 4.7 has shown that the i-th block of $\hat{\Pi}_{s,t}$ corresponds to the descendence of $\mathbf{e}_s^{(i)}$ in the population at time t, for each $i\in[k]$ (see Figure 2 for an illustration).

Proposition 4.8 *The collection of partitions $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$ is a Λ flow of partitions.*

This Proposition somehow extends the basic correspondence of Bertoin and Le Gall, Section 3.2 in [6], under our Assumption 4.2 on Λ. Moreover, this is a pathwise correspondence since the flow of partitions has been defined in terms of the flow of bridges. Remark that we have defined the *forward* flow of partitions. By analogy with bridges, one can define the *backward* flow of partitions $(\Pi_{s,t}, -\infty < s \leq t < \infty)$ by

$$\Pi_{s,t} := \hat{\Pi}_{-t,-s} \tag{57}$$

Proof The first requirement of Definition 3.6 is an easy consequence of Proposition 4.7. Indeed, the partition $\hat{\Pi}_{r,t}$ is obtained by applying the key property with the sequence $(\mathbf{e}_t^{(i)})_{i\geq 1}$ and the bridge $\tilde{F}_{r,t} = \tilde{F}_{s,t} \circ \tilde{F}_{r,s}$ a.s. From the proof of Corollary 1, in [6], we deduce that the random partition $\hat{\Pi}_{r,t}$ is equal to the coagulation of $\hat{\Pi}_{s,t}$ by the partition $\hat{\Pi}_{r,s}$ (although this Corollary asserts an equality in distribution, the proof actually defines an a.s. equality).
Let us prove the independence of the increments in the case $n = 2$, the general case is obtained by an easy induction. Fix $r < s < t$. We know that the sequence $(\mathbf{e}_s^{(i)})_{i\geq 1}$ is obtained by the key property of Bertoin and Le Gall applied to the sequence $(\mathbf{e}_t^{(i)})_{i\geq 1}$ and the independent bridge $\tilde{F}_{s,t}$. Moreover, $(\mathbf{e}_s^{(i)})_{i\geq 1}$ is independent of the partition $\hat{\Pi}_{s,t}$. Given that the partition $\hat{\Pi}_{r,s}$ depends only on the sequence $(\mathbf{e}_s^{(i)})_{i\geq 1}$ and the bridge $\tilde{F}_{r,s}$ which is also independent of $\hat{\Pi}_{s,t}$, we deduce the independence of $\hat{\Pi}_{r,s}$ and $\hat{\Pi}_{s,t}$. Furthermore, the fact that the distribution of $\hat{\Pi}_{s,t}$ only depends on $t - s$ is an immediate consequence of the stationarity of flows of bridges.
The convergence in probability of $\hat{\Pi}_{-t,0} \to 0_{[\infty]}$ for the distance $d_{\mathscr{P}}$ is a consequence of the next Lemma. Finally, since the flow of bridges F is associated with the measure Λ, we immediately deduce that $(\hat{\Pi}_{-t,0}, t \geq 0)$ is a Λ coalescent using the property recalled in Subsection 2.2. □

Lemma 4.9 *Consider a collection of bridges $(B_t)_{t>0}$ and an independent sequence of i.i.d. uniform$[0,1]$ random variables $(V_i)_{i\geq 1}$. The following conditions are equivalent*

a) The exchangeable partition $\pi(B_t, (V_i)_{i\geq 1})$ converges in probability to $0_{[\infty]}$ for the distance $d_{\mathscr{P}}$ as $t \downarrow 0$.

b) The bridge B_t converges in probability to Id in the sense of Skorohod's topology as $t \downarrow 0$.

We postpone the proof of this technical Lemma to Section 7.

5 Proof of Theorem 1

We have defined a stochastic flow of partitions $\hat{\Pi}$ pathwise from the flow of bridges F. As F may have many irregularities, so may $\hat{\Pi}$. However, using the regularization procedure described in Subsection 3.2, one obtains a modification $\tilde{\hat{\Pi}}$ of the original flow $\hat{\Pi}$ such that its trajectories are deterministic flows of partitions, almost surely. Furthermore, the collection of the jumps of $\tilde{\hat{\Pi}}$ defines a Λ lookdown graph $\mathcal{P}$

as proved in Proposition 3.13. Thus, we have defined pathwise from the flow of bridges F an ancestral types process $(\mathbf{e}_s^{(i)}, s \in \mathbb{R})_{i\geq 1}$ and a lookdown graph $\mathcal{P}$. Those two objects actually define a collection of coupled lookdown processes with limiting empirical measures the collection $(\tilde{\rho}_{s,t}, t \in [s,\infty))_{s\in\mathbb{R}}$ as we will see in this section.

Let us introduce a particle system $(\xi_{s,t}(i), s \leq t)_{i\geq 1}$ as follows. For each $s \in \mathbb{R}$, set $(\xi_{s,t}(i), t \in [s,\infty))_{i\geq 1} := \mathscr{L}_s(\mathcal{P}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ and denote by $(\Xi_{s,t}, t \in [s,\infty)) := \mathscr{E}_s(\mathcal{P}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ its limiting empirical measures. Let us recall the statement of the theorem.

Theorem 1 *The collection of coupled lookdown processes $(\xi_{s,t}(i), s \leq t)_{i\geq 1}$ with limiting empirical measures $(\Xi_{s,t}, s \leq t)$ verify the following assertions:*

i) ***Coupling**. For each $s \in \mathbb{R}$, a.s. $(\Xi_{s,t}, t \in [s,\infty)) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$.*

ii) ***Uniqueness**. Let $\mathcal{M}$ be a Λ lookdown graph and for each $s \in \mathbb{R}$, consider a sequence $(\chi_{s,s}(i))_{i\geq 1}$ of r.v. taking distinct values in $[0,1]$. If for each $s \in \mathbb{R}$, a.s. $\mathscr{E}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1}) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$ then*

- *For each $s \in \mathbb{R}$, a.s. $(\chi_{s,s}(i))_{i\geq 1} = (\mathbf{e}_s^{(i)})_{i\geq 1}$.*
- *Almost surely, $\mathcal{M} = \mathcal{P}$.*

This section is devoted to the proof of the theorem. In the first subsection, we prove the coupling statement. In the second subsection, we investigate the uniqueness properties of the lookdown construction and prove the uniqueness statement. Finally, in the third subsection we compare our lookdown construction from a flow of bridges with the original lookdown definition of Donnelly and Kurtz in [13].

5.1 Coupling

Proof (Theorem 1-Coupling) Fix $s \in \mathbb{R}$. Remark that both processes $(\Xi_{s,t}, t \in [s,\infty))$ and $(\tilde{\rho}_{s,t}, t \in [s,\infty))$ are càdlàg processes. Therefore, to prove that a.s.

$$(\Xi_{s,t}, t \in [s,\infty)) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$$

it is sufficient to prove that for each $t \in [s,\infty)$, we have a.s.

$$\Xi_{s,t} = \tilde{\rho}_{s,t}$$

Consider a time $t \in [s,\infty)$. Since $\hat{\Pi}_{s,t}^{\mathcal{P}}$ admits asymptotic frequencies, a simple application of Equation (36) ensures that

$$\Xi_{s,t}(dx) = \sum_{i=1}^{\infty} |\hat{\Pi}_{s,t}^{\mathcal{P}}(i)| \delta_{\mathbf{e}_s^{(i)}}(dx)$$

Moreover, we know that a.s. for every $i \geq 1$, $\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i)}) = |\hat{\Pi}_{s,t}(i)|$ and $\hat{\Pi}^{\mathcal{P}}_{s,t} = \hat{\Pi}_{s,t}$. Therefore, a.s.

$$\tilde{\rho}_{s,t}(dx) = \sum_{i=1}^{\infty} |\hat{\Pi}^{\mathcal{P}}_{s,t}(i)| \delta_{\mathbf{e}_s^{(i)}}(dx)$$

Thus, we get that a.s. for every $t \in [s,\infty) \cap \mathbb{Q}$, we have $\Xi_{s,t}(.) = \tilde{\rho}_{s,t}(.)$. Since both are càdlàg processes, we have proved the identity. This ensures the coupling statement of the theorem. □

5.2 Uniqueness

We now focus on the uniqueness statement of the theorem. Let $\mathcal{M}$ be a Λ lookdown graph and for each $s \in \mathbb{R}$, consider a sequence $(\chi_{s,s}(i))_{i\geq 1}$ of r.v. taking distinct values in $[0,1]$. We denote by $(\chi_{s,t}(i), t \in [s,\infty))_{i\geq 1} := \mathscr{L}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1})$ and $(X_{s,t}, t \in [s,\infty)) := \mathscr{E}_s(\mathcal{M}, (\chi_{s,s}(i))_{i\geq 1})$ its limiting empirical measures. We suppose that for each $s \in \mathbb{R}$, $(X_{s,t}, t \in [s,\infty)) = (\tilde{\rho}_{s,t}, t \in [s,\infty))$ a.s.

Proposition 4.5 implies that for each $s \in \mathbb{R}$, we have $(\chi_{s,s}(i))_{i\geq 1} = (\mathbf{e}_s^{(i)})_{i\geq 1}$ a.s. It remains to prove that $\mathcal{M} = \mathcal{P}$ a.s. To do so, we will consider the stochastic flows of partitions defined pathwise from these point processes (see Subsection 3.2), say $\hat{\Pi}^{\mathcal{M}}$ and $\hat{\Pi}^{\mathcal{P}}$. Recall that the trajectories of those flows are deterministic flows of partitions without simultaneous mergers. Using Proposition 3.2, it is equivalent to prove that a.s. $\hat{\Pi}^{\mathcal{M}} = \hat{\Pi}^{\mathcal{P}}$ in order to prove the uniqueness statement.
There exists an event Ω^* of probability 1 such that on this event, for every rational $s \leq t$ and every integer $i \geq 1$ we have

$$\tilde{\rho}_{s,t}(\mathbf{e}_s^{(i)}) = |\hat{\Pi}^{\mathcal{P}}_{s,t}(i)| = |\hat{\Pi}^{\mathcal{M}}_{s,t}(i)| \tag{58}$$

The proof of the uniqueness statement of the theorem relies on two lemmas. Some properties will hold both for $\hat{\Pi}^{\mathcal{M}}$ and $\hat{\Pi}^{\mathcal{P}}$, thus we will use the notation $\hat{\Pi}^{\times}$ to designate indifferently any of them.

Lemma 5.1 *For all $s \in \mathbb{R}$, and every $t \in (s,\infty) \cap \mathbb{Q}$, we have for every $i \geq 1$*

$$\begin{aligned} |\hat{\Pi}^{\times}_{s-,t}(i)| &= \lim_{r\downarrow 0} |\hat{\Pi}^{\times}_{s-r,t}(i)| \\ |\hat{\Pi}^{\times}_{s,t}(i)| &= \lim_{r\downarrow 0} |\hat{\Pi}^{\times}_{s+r,t}(i)| \end{aligned} \tag{59}$$

Therefore we deduce that for all $\omega \in \Omega^*$ and every $i \geq 1$

$$\begin{aligned} (|\hat{\Pi}^{\mathcal{P}}_{s-,t}(i)|(\omega), t \in (s,\infty) \cap \mathbb{Q}) &= (|\hat{\Pi}^{\mathcal{M}}_{s-,t}(i)|(\omega), t \in (s,\infty) \cap \mathbb{Q}) \\ (|\hat{\Pi}^{\mathcal{P}}_{s,t}(i)|(\omega), t \in (s,\infty) \cap \mathbb{Q}) &= (|\hat{\Pi}^{\mathcal{M}}_{s,t}(i)|(\omega), t \in (s,\infty) \cap \mathbb{Q}) \end{aligned} \tag{60}$$

Proof Fix $s \in \mathbb{R}$ and $t \in (s,\infty) \cap \mathbb{Q}$. From Lemma 3.4, we know that the r.h.s. of Equations (59) exist. Fix $\epsilon > 0$, $r_0 < t-s$ and $\omega \in \Omega$. For every $i \geq 1$, there exists $n > i$ such that $\sum_{j=1}^{n} |\hat{\Pi}^{\times}_{s+r_0,t}(j)|(\omega) > 1-\epsilon$. Remark that only a finite number of coagulation events will coalesce two or more blocks among the n first during the interval of time $[s-r_0, s+r_0]$. The jumps of $(\sum_{j=1}^{i} |\hat{\Pi}^{\times}_{s+r,t}(j)|(\omega), r \in [-r_0, r_0])$

due to such coagulation events are thus finitely many, whereas the sum of all the other jumps is lower than ϵ. Therefore we deduce that

$$\left|\sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s-,t}(j)|(\omega)-\lim_{r\downarrow 0}\sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s-r,t}(j)|(\omega)\right|<\epsilon$$
$$\left|\sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s,t}(j)|(\omega)-\lim_{r\downarrow 0}\sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s+r,t}(j)|(\omega)\right|<\epsilon$$

Since this holds for all $\epsilon > 0$, we get that the l.h.s. of the preceding equations are equal to 0. Finally, remark that for all $\omega \in \Omega$ and $i \geq 1$,

$$\begin{aligned}|\hat{\Pi}^{\times}_{s,t}(i)|(\omega) &= \sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s,t}(j)|(\omega)-\sum_{j=1}^{i-1}|\hat{\Pi}^{\times}_{s,t}(j)|(\omega)\\ &= \lim_{r\downarrow 0}\sum_{j=1}^{i}|\hat{\Pi}^{\times}_{s+r,t}(j)|(\omega)-\lim_{r\downarrow 0}\sum_{j=1}^{i-1}|\hat{\Pi}^{\times}_{s+r,t}(j)|(\omega)\\ &= \lim_{r\downarrow 0}|\hat{\Pi}^{\times}_{s+r,t}(i)|(\omega)\end{aligned}$$

We obtain the left continuity. The right limit is obtained similarly. Finally, on Ω^*, it suffices to use those limits conjointly with Equation (58) to obtain Equations (60). □

Lemma 5.2 *Let I be a subset of $\mathbb{N}$. The following assertions are equivalent*

i) $\hat{\Pi}^{\times}_{s-,s}$ *has a unique non singleton block I.*

ii)
$$\begin{cases}(|\hat{\Pi}^{\times}_{s-,t}(i)|, t\in(s,\infty)\cap\mathbb{Q}) = (\sum_{j\in I}|\hat{\Pi}^{\times}_{s,t}(j)|, t\in(s,\infty)\cap\mathbb{Q}) \text{ if } i=\min(I)\\ (|\hat{\Pi}^{\times}_{s-,t}(i)|, t\in(s,\infty)\cap\mathbb{Q}) = (|\hat{\Pi}^{\times}_{s,t}(j)|, t\in(s,\infty)\cap\mathbb{Q}) \text{ if } \begin{cases} i\neq\min(I)\\ i=j-(\#\{I\cap[j]\}-1)\vee 0\end{cases}\end{cases}$$

Proof Suppose *i)*. Since $\hat{\Pi}^{\times}_{s-,t} = Coag(\hat{\Pi}^{\times}_{s,t}, \hat{\Pi}^{\times}_{s-,s})$, the very definition of the coagulation operator implies *ii)*.
Suppose *ii)*. From the very definition of $\hat{\Pi}^{\times}$ from a Poisson point process on $dt \times \mathcal{S}^2_\infty$, we know that $\hat{\Pi}^{\times}_{s-,s}$ is a partition with at most one non singleton block.
We know that for all $\omega \in \Omega^*$, $(|\hat{\Pi}^{\times}_{s-,t}(i)|(\omega), t \in (s,\infty)\cap\mathbb{Q})_{i\geq 1}$ are all distinct. Indeed, in the **[(CDI) and** $\mathbb{P}(E) = 0$**]** case, the extinction times of the asymptotic frequencies are strictly distinct while in the **[Bolthausen-Sznitman]** case, their asymptotic behaviours are strictly distinct. The same holds for $(|\hat{\Pi}^{\times}_{s,t}(j)|(\omega), t \in (s,\infty)\cap\mathbb{Q})_{j\geq 1}$. Since $\hat{\Pi}^{\times}_{s-,t} = Coag(\hat{\Pi}^{\times}_{s,t}, \hat{\Pi}^{\times}_{s-,s})$, the equations of *ii)* imply that the partition $\hat{\Pi}^{\times}_{s-,s}$ has a unique non-singleton block I. □

We are now able to end the proof of the theorem.

Proof (Theorem 1-Uniqueness) Using Lemma 5.1, we deduce that on Ω^*, for all $s \in \mathbb{R}$ and every $i \geq 1$

$$\begin{aligned}(|\hat{\Pi}^{\mathcal{P}}_{s-,t}(i)|(\omega), t \in (s,\infty)\cap\mathbb{Q}) &= (|\hat{\Pi}^{\mathcal{M}}_{s-,t}(i)|(\omega), t \in (s,\infty)\cap\mathbb{Q})\\ (|\hat{\Pi}^{\mathcal{P}}_{s,t}(i)|(\omega), t \in (s,\infty)\cap\mathbb{Q}) &= (|\hat{\Pi}^{\mathcal{M}}_{s,t}(i)|(\omega), t \in (s,\infty)\cap\mathbb{Q})\end{aligned}$$

These identities together with Lemma 5.2 ensure that on Ω^*, for all $s \in \mathbb{R}$

$$\hat{\Pi}^{\mathcal{P}}_{s-,s} = \hat{\Pi}^{\mathcal{M}}_{s-,s}$$

Therefore, a.s. $\mathcal{P} = \mathcal{M}$. This concludes the proof of our theorem. □

5.3 A remark on the lookdown ordering

Our pathwise lookdown construction from a stochastic flow of bridges is an infinite dimensional extension of the lookdown construction from the Moran model (see [13], where Model I is a Moran model whereas Model II is a lookdown process). Fix $n \in \mathbb{N}$, and consider a Moran model $(Y_t(1), \ldots, Y_t(n), t \geq 0)$ started from a sequence of n i.i.d. uniform$[0,1]$ (here we use the notation of [13], one should not confuse them with the definitions introduced in this paper). Donnelly and Kurtz introduce a lookdown process $(X_t(1), \ldots, X_t(n), t \geq 0)$ on $[0,\infty)$ by defining a random permutation θ of $[n]$ such that

$$(Y_t(1), \ldots, Y_t(n)) = (X_t(\theta(1)), \ldots, X_t(\theta(n)))$$

Here we do not consider any location/type motion, hence θ does not depend on t (compare with the first Equation of Section 2 in [13]). A careful reading of the proof of Theorem 1.1 in [13] shows that θ^{-1} is the random permutation of $[n]$ such that $(Y_0(\theta^{-1}(i)))_{i\in[n]}$ is ordered by persistence, that is $Y_0(\theta^{-1}(1))$ is the type that fixes, then $Y_0(\theta^{-1}(2))$ is the last type that becomes extinct and so on.
Since a stochastic flow of bridges is somehow an infinite dimensional extension of the Moran model, one should compare Theorem 1.1 of [13] and our lookdown construction (in the **[(CDI) and $\mathbb{P}(E) = 0$]** case). Indeed, both rely on the reordering of the ancestral types by decreasing persistence. Moreover, in both cases, this random reordering is independent of the past of the underlying process (Moran model or flow of bridges). Hence, the random reordering depends on the future of the underlying process and as time passes, the evolution of this process allows one to determine the reordering from the highest levels to the lowest. Thus, the lookdown process should be seen as a future-dependent reordering of an underlying process.

We end this section with a general result about the ordering of the ancestral types induced by the lookdown process. The following proposition is a generalization of a result of Delmas, Dhersin and Siri-Jegousse in [11] on the oldest families of the Fleming-Viot process. Indeed for each $i \geq 1$, $\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)})$ should be understood as the size of the i-th oldest family of the population alive at time t.

Proposition 5.3 *For each $t > 0$, the distribution of $(\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)}))_{i\geq 1}$ conditionally on $\tilde{\rho}_{0,t}$ is a size-biased reordering of the ancestral types masses at time t.*

Proof Fix $t > 0$. We have to check that $(\tilde{\rho}_{0,t}(\mathbf{e}_0^{(i)}))_{i\geq 1}$ is a size-biased reordering of the $(\beta^i)_{i\geq 1}$, where $(U^i, \beta^i)_{i\geq 1}$ is the sequence of jumps of $\tilde{F}_{0,t}$. Recall that $(\mathbf{e}_t^{(i)})_{i\geq 1}$ are i.i.d uniform$[0,1]$, independent of $\tilde{F}_{0,t}$. Thus for each $i, j \geq 1$, we have

$$\mathbb{P}(\tilde{F}_{0,t}^{-1}(\mathbf{e}_t^{(i)}) = U^j) = \beta^j$$

Since the $(\mathbf{e}_0^{(j)})_{j\geq 1}$ are the distinct values taken by the sequence $(\tilde{F}_{0,t}^{-1}(\mathbf{e}_t^{(i)}))_{i\geq 1}$ (see Proposition 4.7), we deduce the assertion of the Proposition. □

6 Encoding of the Beta Fleming-Viot process

In this section, we reformulate the results of Berestycki et al. in [1, 2] on the encoding of the lookdown process associated with the Beta$(2-\alpha, \alpha)$ Fleming-Viot into an α-stable height process, with $\alpha \in (1, 2]$, in terms of the flow of partitions and the ancestral types process. We fix $\alpha \in (1, 2]$ and consider the α-stable branching mechanism $\Psi(q) = q^\alpha$ when $\alpha \in (1, 2)$ and $\Psi(q) = 2q^2$ when $\alpha = 2$. Recall that the notation Beta$(2-\alpha, \alpha)$ refers to the measure given by Equation (10) when $\alpha \in (1, 2)$ while it denotes the measure $\delta_0(dx)$ when $\alpha = 2$.

Let $\mathscr{E}_+$ be the set of positive continuous excursions away from 0. Denote by H an α-stable height process and let ν^{exc} bet its excursion measure on $\mathscr{E}_+$. Proposition 1.3.3 in [14] ensures the existence of a jointly measurable modification $(L(t, x), t \geq 0, x \geq 0)$ of the local-time accumulated by H at level $x \geq 0$ until time $t \geq 0$, which is continuous in t and verifies

$$\lim_{\delta \downarrow 0} \sup_{x\geq 0} \mathbb{E}[\sup_{s\leq t} |\frac{1}{\delta}\int_0^s \mathbf{1}_{[x,x+\delta)}(H_s)ds - L(s,x)|] = 0$$

Set $T_x^r := \inf\{t \geq 0 : L(t, x) > r\}$ for all $x \geq 0$ and $r \geq 0$, that is, the first time at which the local-time of H at level x is greater than r. It is well known [5, 9, 14] that

$$x \mapsto Z_x^r := L(T_0^r, x) \tag{61}$$

is a continuous state branching process with branching mechanism Ψ, started from r. In the sequel, we will consider the process H stopped at T_0^1, thus for all $s \geq T_0^1$, $H_s = 0$. For simplicity, we will omit the superscript r when it is equal to 1.

Let us introduce for each $t \geq 0$

$$U(t) := \begin{cases} \inf\{s > 0 : \int_0^s \frac{\alpha(\alpha-1)\Gamma(\alpha)}{Z_x^{\alpha-1}}dx > t\} & \text{if } \alpha \in (1,2) \\ \inf\{s > 0 : \int_0^s \frac{4}{Z_x}dx > t\} & \text{if } \alpha = 2 \end{cases} \tag{62}$$

Then U is an increasing bijective map from $[0, \infty)$ to $[0, S)$, where $S := \sup_{s\geq 0}(H_s)$.

For each $x \in [0, S)$ conditional on Z_x, the excursions of H above level x are distributed according to a Poisson point process on $[0, Z_x] \times \mathscr{E}_+$ with intensity measure $dl \otimes \nu^{exc}$. We denote by $(z_x^i, \epsilon_x^i)_{i\geq 1}$ the set of points of this point process ordered by decreasing height of the excursions, that is, ϵ_x^1 is the highest excursion, ϵ_x^2 is the second highest and so on, and z_x^i is the local-time accumulated by H at level x until the beginning of the excursion ϵ_x^i, for each $i \geq 1$. Remark that for all $x < y \in [0, S)$, for each $i \in \mathbb{N}$, there exists a unique $k \in \mathbb{N}$ such that the excursion ϵ_y^i is embedded into the excursion ϵ_x^k. Thus, we define the random partition $\hat{\Pi}_{s,t}$, where $s := U^{-1}(x)$ and $t := U^{-1}(y)$ as follows

$$i \overset{\hat{\Pi}_{s,t}}{\sim} j \Leftrightarrow \epsilon_y^i \text{ and } \epsilon_y^j \text{ belong to the same excursion } \epsilon_x^k, \text{ with } k \in \mathbb{N} \tag{63}$$

For each $s \geq 0$ define

$$\tilde{\tau}_s^x := \inf\{t : \int_0^t \mathbf{1}_{\{H_r \leq x\}} dr > s\} \tag{64}$$

$$\bar{\tau}_s^x := \inf\{t : \int_0^t \mathbf{1}_{\{H_r > x\}} dr > s\} \tag{65}$$

Introduce the processes $(\tilde{H}_s^x, s \geq 0)$ and $(\bar{H}_s^x, s \geq 0)$ by setting $\tilde{H}_s^x := H_{\tilde{\tau}_s^x}$ and $\bar{H}_s^x := H_{\bar{\tau}_s^x} - x$. We define the filtration $(\mathcal{F}_x)_{x \in \mathbb{R}_+}$ as follows

$$\mathcal{F}_x := \sigma\{\tilde{H}_s^x, s \geq 0\} \tag{66}$$

Roughly speaking, $\mathcal{F}_x$ contains all the information about the trajectory of H under level x.

Proposition 6.1 *The process* $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < \infty)$ *is a stochastic flow of partitions associated with the measure Beta*$(2 - \alpha, \alpha)(dx)$.

Proof For all $0 \leq r \leq s \leq t$, the identity $\hat{\Pi}_{r,t} = Coag(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$ is an immediate consequence of the definition of the partitions. Moreover we deduce from Proposition 2.1 in [1] and Theorem 1 in [2] that $(\hat{\Pi}_{t-t',t}, t' \in [0, t])$ is a Beta$(2 - \alpha, \alpha)$ coalescent restricted to $[0, t]$ (recall that $\alpha = 2$ corresponds to the Kingman coalescent). Thus, the law of $\hat{\Pi}_{s,t}$ only depends on $t - s$ and $\hat{\Pi}_{t-r,t} \to 0_{[\infty]}$ as $r \downarrow 0$ in probability. Furthermore, we stress that for any $0 \leq s < t \leq t_1 < \ldots < t_n$, $\hat{\Pi}_{s,t}$ is independent of the partitions $(\hat{\Pi}_{t_i,t_{i+1}})_{i \in [n-1]}$. Indeed, $\hat{\Pi}_{s,t}$ only depends on $\mathcal{F}_{U(t)}$ and on the $(z^i_{U(t)})_{i \geq 1}$. Thus, conditional on $Z_{U(t)}$, it is independent of the $(\epsilon^i_{U(t)})_{i \geq 1}$. It is easy to remark that the partitions $(\hat{\Pi}_{t_i,t_{i+1}})_{i \in [n-1]}$ only depend on the $(\epsilon^i_{U(t)})_{i \geq 1}$. The independence property then follows from the fact that the laws of the latter partitions do not depend on $Z_{U(t)}$. □

We now introduce a notation useful in the sequel. For all $y > 0$, all $x \in (0, S)$ and each $i \geq 1$, let $l_y(\epsilon_x^i)$ be the total local-time accumulated by the excursion ϵ_x^i at level y. Finally, set for all $s \in \mathbb{R}$ and $i \in \mathbb{N}$

$$\mathbf{e}_s^{(i)} := \frac{z^i_{U(s)}}{Z_{U(s)}} \tag{67}$$

Proposition 6.2 *For each* $s \in \mathbb{R}$, $(\mathbf{e}_s^{(i)})_{i \geq 1}$ *is a sequence of i.i.d. uniform*$[0, 1]$, *independent of* $\mathcal{F}_{U(s)}$.

Proof Fix $s \in \mathbb{R}$. From Lemma 17 in [2], we know that $\tilde{H}^{U(s)}$ and $\bar{H}^{U(s)}$ are independent conditional on $Z_{U(s)}$. Since $(\mathbf{e}_s^{(i)})_{i\geq 1}$ only depends on $Z_{U(s)}$ and $\bar{H}^{U(s)}$, it suffices to prove that it is a sequence of i.i.d. uniform$[0,1]$ independent of $Z_{U(s)}$. From Itô's excursion theory, we know that $(z^i_{U(s)}, \epsilon^i_{U(s)})_{i\geq 1}$ are distributed according to a Poisson point process on $[0, Z_{U(s)}] \times \mathscr{E}_+$ with intensity measure $dl \otimes \nu^{exc}$ ordered by decreasing height of the excursions. We deduce that $(z^i_{U(s)})_{i\geq 1}$ are i.i.d. uniform$[0, Z_{U(s)}]$. Renormalizing by $Z_{U(s)}$, we obtain that $(\mathbf{e}_s^{(i)})_{i\geq 1}$ are i.i.d. uniform$[0,1]$ conditional on $Z_{U(s)}$. Since a mixture of i.i.d. uniform$[0,1]$ is still i.i.d. uniform$[0,1]$, it follows that $(\mathbf{e}_s^{(i)})_{i\geq 1}$ are i.i.d. uniform$[0,1]$. This implies that the sequence $(\mathbf{e}_s^{(i)})_{i\geq 1}$ is independent of $Z_{U(s)}$. The independence from $\mathcal{F}_{U(s)}$ follows.
□

Define for all $0 \leq s \leq t$

$$\rho_{s,t}(.) = \sum_{i\geq 1} \frac{l_{U(t)-U(s)}(\epsilon^i_{U(s)})}{Z_{U(t)}} \delta_{\mathbf{e}_s^{(i)}}(.) \tag{68}$$

and let $(\xi_{s,t}(i), t \in [s,\infty))_{i\geq 1} := \mathscr{L}_s(\tilde{\hat{\Pi}}, (\mathbf{e}_s^{(i)})_{i\geq 1})$ and $(\Xi_{s,t}(i), t \in [s,\infty))_{i\geq 1} := \mathscr{E}_s(\tilde{\hat{\Pi}}, (\mathbf{e}_s^{(i)})_{i\geq 1})$, where $\tilde{\hat{\Pi}}$ is the regularized modification of $\hat{\Pi}$, see Subsection 3.2.

Proposition 6.3 *For all $s \geq 0$, $(\Xi_{s,t}, t \in [s,\infty))$ is a càdlàg modification of the process $(\rho_{s,t}, t \in [s,\infty))$. Thus, the latter is a Beta$(2-\alpha,\alpha)$ Fleming-Viot process.*

Proof Fix $s \geq 0$. Since $\hat{\Pi}$ is a stochastic flow of partitions associated with the measure Beta$(2-\alpha,\alpha)(dx)$, we deduce that $(\Xi_{s,t}, t \in [s,\infty))$ is a Beta$(2-\alpha,\alpha)$ Fleming-Viot process.
Remark that for all $t \geq s$, a.s. $\Xi_{s,t}(\mathbf{e}_s^{(i)})$ is the proportion of excursions above level $U(t)$ embedded into the excursion $\epsilon^i_{U(s)}$, and since the excursions above level $U(t)$ are distributed according to a Poisson point process on $[0, Z_{U(t)}] \times \mathscr{E}_+$ with intensity measure $dl \otimes \nu^{exc}$, it is immediate to check that for each $i \geq 1$ and $t \geq s$, a.s.

$$\Xi_{s,t}(\mathbf{e}_s^{(i)}) = \frac{l_{U(t)-U(s)}(\epsilon^i_{U(s)})}{Z_{U(t)}}$$

Thus $\Xi_{s,t} = \rho_{s,t}$ a.s. for every $t \in [s,\infty)$ and $\rho_{s,.}$ is a Beta$(2-\alpha,\alpha)$ Fleming-Viot process. □

7 Appendix

7.1 Proof of Proposition 4.3

To alleviate notation, we prove the proposition for $s = 0$. Let $\mathscr{M}_f(a)$ be the set of finite measures on $[0,a]$, equipped with its weak topology. Consider a $\mathscr{M}_f(1)$-valued process $(m_t, t \geq 0)$ associated with Neveu continuous state branching process (CSBP in short) as defined in [9]. Therefore $(Z_t(x), t \geq 0) := (m_t([0,x]), t \geq 0)$ is a Neveu CSBP started from x, for each $x \in [0,1]$. We define for all $t \geq 0$

$$\rho_t(dx) := \frac{m_t(dx)}{m_t([0,1])}$$

$(\rho_t, t \geq 0)$ is a Λ Fleming-Viot process where $\Lambda(dx) = dx$ (see [9] for a proof of this result). Hence, we will prove the proposition by considering the process $(m_t, t \geq 0)$.
For all $t > 0$, the distribution function $(S_t(x), x \in [0,1])$ of m_t is a subordinator without drift whose Lévy measure has an infinite mass (see Section 3 in [5]). Denote by $\mathcal{U}$ the set of ancestral types of this measure-valued process, that is

$$\mathcal{U} := \{x \in [0,1] : \exists t > 0, m_t(x) > 0\} \tag{69}$$

Lemma 7.1 *The set $\mathcal{U}$ is a countable subset of $[0,1]$. For each $u \in \mathcal{U}$, for all $t > 0$, $m_t(u) > 0$.*

Proof Since the distribution function of m_t has no drift part, the set of atoms of m_{t+s} is included in the set of atoms of m_t for all $t, s > 0$. Moreover, if $m_t(u) > 0$ for a given time $t > 0$ and a given point $u \in [0,1]$, then the process $(m_{t+s}(u), s \geq 0)$ is a Neveu CSBP started from $m_t(u)$, which is independent of $(m_{t+s}([0,1]\backslash\{u\}), s \geq 0)$. Since a Neveu CSBP does not get extinct in finite time almost surely, we deduce that an ancestral type has a positive mass at any time. □

Fix $T > 0$ and condition on $\sigma\{m_s; s \in [0,T]\}$. $(m_{t+T}(u), t \geq 0)_{u \in \mathcal{U}}$ is a collection of independent Neveu's CSBP started with initial population sizes $(m_T(u))_{u \in \mathcal{U}}$. Introduce for each $t \geq 0$, the finite measure $m'_t(.) := m_{T+t} \circ S_T^{-1}(.)$. One can easily remark that $(m'_t, t \geq 0)$ is a $\mathscr{M}_f(Z_T(1))$-valued process associated with Neveu CSBP, with initial population size $Z_T(1)$.
Fix $i \in \mathbb{N}$ and $\vec{u} := (u_1, \ldots, u_i) \in \mathcal{U}^i$ all distinct. The idea is to consider the restriction of this process to $[0, Z_T(1)]\backslash \underset{1 \leq j \leq i}{\cup} (Z_T(u_j-), Z_T(u_j)]$, which is a measure-valued process associated with Neveu CSBP, with initial population size $Z_T(1) - \sum_{j=1}^i m_T(u_j)$. Set

$$\begin{aligned} f_{\vec{u}} : [0, Z_T(1)]\backslash \underset{1 \leq j \leq i}{\cup} (Z_T(u_j-), Z_T(u_j)] \quad &\longrightarrow \quad [0,1] \\ x \quad &\longmapsto \quad \frac{x - \sum_{j=1}^i \mathbf{1}_{\{x > Z_T(u_j)\}} m_T(u_j)}{Z_T(1) - \sum_{j=1}^i m_T(u_j)} \end{aligned}$$

Using the map $f_{\vec{u}}$ to rescale $[0, Z_T(1)]\backslash \underset{1 \leq j \leq i}{\cup} (Z_T(u_j-), Z_T(u_j)]$ onto $[0,1]$, and dividing by the total mass of the process allows one to assert that $(\frac{m'_t(f_{\vec{u}}^{-1}(.))}{m_{T+t}([0,1]\backslash\{\vec{u}\})}, t \geq 0)$ is a Λ Fleming-Viot process, with $\Lambda(dx) = dx$.
Thus one can consider the Eve $e(\vec{u}) \in [0,1]$ of this Λ Fleming-Viot, and set $\eta(\vec{u}) := S_T^{-1} \circ f_{\vec{u}}^{-1}(e(\vec{u}))$. Clearly $\eta(\vec{u}) \in \mathcal{U}$.

Lemma 7.2 *There exists a unique reordering $(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots)$ of $\mathcal{U}$ such that*

$$\eta(\vec{u}) = \begin{cases} \mathbf{e}^{(1)} \text{ if } \vec{u} = (u_1) \text{ and } u_1 \in \mathcal{U}\backslash\{\mathbf{e}^{(1)}\} \\ \mathbf{e}^{(j)} \text{ if } \vec{u} = (\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(j-1)}) \\ \mathbf{e}^{(j)} \text{ if } \vec{u} = (\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(j-1)}, u_j) \text{ and } u_j \in \mathcal{U}\backslash \underset{1 \leq k \leq j}{\cup} \{\mathbf{e}^{(k)}\} \end{cases} \tag{70}$$

Proof This is an easy consequence of the Eve's property of Bertoin and Le Gall. $\square$

Remark that $\mathbf{e}^{(1)}$ is the Eve of $(\rho_t, t \geq 0)$ and that for any $i > 1$

$$\begin{aligned}\mathbf{e}^{(i)} &= \inf\{y \in [0,1] : \lim_{t\to\infty} \frac{F_t(y) - \sum_{1\leq j\leq i-1} \mathbf{1}_{\{y\geq \mathbf{e}_0^{(j)}\}}\rho_t(\mathbf{e}_0^{(j)})}{1 - \sum_{1\leq j\leq i-1} \rho_t(\mathbf{e}_0^{(j)})} = 1\} \\ &= \sup\{y \in [0,1] : \lim_{t\to\infty} \frac{F_t(y) - \sum_{1\leq j\leq i-1} \mathbf{1}_{\{y\geq \mathbf{e}_0^{(j)}\}}\rho_t(\mathbf{e}_0^{(j)})}{1 - \sum_{1\leq j\leq i-1} \rho_t(\mathbf{e}_0^{(j)})} = 0\}\end{aligned}$$

We have proven the proposition.

7.2 Proof of Lemma 4.9

Suppose *a)*. Then we know that $B_t \overset{(d)}{\to} Id$ as $t \downarrow 0$ from the Continuity Lemma 1, in [6]. Since the limit is a continuous function, we have for every $0 \leq x \leq 1$, $B_t(x) \overset{(d)}{\to} x$. The limit being deterministic, the convergence also holds in probability $B_t(x) \overset{(\mathbb{P})}{\to} x$.
Fix $\epsilon > 0$. Denote by $\lfloor x \rfloor$ the integer part of any real x. There exists $t_0 > 0$ such that for every $t \in (0, t_0)$ and $k \in [\lfloor 1/\epsilon \rfloor]$

$$\mathbb{P}(|B_t(k\epsilon) - k\epsilon| > \epsilon) < \frac{\epsilon}{2^k}$$

From the monotonicity of B_t, we get $\mathbb{P}(\|B_t - Id\|_\infty > 2\epsilon) < 2\epsilon$. Hence, $B_t \overset{(\mathbb{P})}{\to} Id$.
Suppose *b)*. Fix $n \geq 1$ and $\epsilon > 0$, we will prove there exists $t_0 > 0$ such that for all $t \in (0, t_0)$

$$\mathbb{P}(d_{\mathscr{P}}(\pi(B_t), \pi(Id)) < 2^{-n}) > 1 - 2\epsilon$$

There exists $p \in \mathbb{N}$ such that

$$\mathbb{P}(\{\exists i, j \text{ s.t. } 1 \leq i < j \leq n \text{ and } |V_i - V_j| < \frac{2}{p}\}) < \epsilon \tag{71}$$

Moreover, there exists $t_0 > 0$ such that for all $t \in (0, t_0)$

$$\mathbb{P}(\bigcap_{0\leq k\leq p}\{B_t(\frac{k}{p}) \in [\frac{k}{p} - \frac{1}{3p}, \frac{k}{p} + \frac{1}{3p}[\}) > 1 - \epsilon \tag{72}$$

The monotonicity of B_t^{-1}, and the two previous equations ensure that

$$\mathbb{P}(\{(B_t^{-1}(V_1), \ldots, B_t^{-1}(V_n)) \text{ are all distinct}\}) > 1 - 2\epsilon$$

Thus, we obtain a) $\square$

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