Partial Actions on Categories[1]

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Abstract

In this paper we introduce the definition of partial action on small k-categories generalizing the similar well known notion of partial actions on algebras. The point of view of partial action which we use in this paper is the one which was introduced by Exel in his work on C^*-algebras, see [8]. Various generalizations were done afterward, see [3, 5, 6, 7]. Also we define the notion partial skew category. We prove similar results to the ones in [4]. Finally we show a result given conditions for a partial action to have a globalization.

1 Introduction

The point of view of partial group actions which we consider here was introduced in the context of operator algebras by R. Exel in [8]. Partial group actions are natural to be consider from distinct points of view. A different way of looking at it, of the one considered here, appeared earlier in [10]. From the point of view considered in this paper a purely algebraic treatment was given recently in [5], [6]. In particular, several aspects of Galois theory can be generalized to partial group actions, see [7] (at least

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under the additional assumption that the associated ideals are generated by central idempotents). Recently, Caeneepel and Janssen in [3] developed the theory of partial (co)-action of a Hopf algebra and generalized the Hopf Galois theory to this situation. Moreover, other authors worked with partial actions of Hopf algebras, see [2].

In this paper we will extend the notion of partial actions to categories. Recall that a category C is said to be small if the objects of C is a set C_0, called the set of objects. Actually we will need the following notion.

Definition 1.1. *Let k be a commutative ring. A small weak not necessarily associative k-category D (WNNA, for short) consists of:*

(1) A set D_0, called the set of objects of D;

(2) For each pair (x, y) of objects of D a k-module ${}_yD_x$, called the set of morphisms from x to y;

(3) For each x, y, z in D_0, a k-bilinear map $\circ : {}_zD_y \times {}_yD_x \to {}_zD_x$ called a composition, where $\circ(f, g)$ will be denoted by $f \circ g$ ($\circ$ is not necessarily associative).

If the composition is associative, then D is called a weak category. If for every $x \in D_0$ there exists $1_x \in {}_xD_x$ such that $f \circ 1_x = f$ and $1_x \circ g = g$, for every $f \in {}_yD_x$ and $g \in {}_xD_y$, then we say that D is WNNA category with identities or, more precisely, a not necessarily associative category (i.e, a NNA category). A k-category is a WNNA category with identities and associative composition.

The notions of subcategory, functors, full, faithful and so on, can be defined for WNNA categories in a similar way that are defined for categories.

Throughout this paper k will always denote a commutative ring and C a small weak k-category. Actually the notion of weak categories and not necessarily categories are not used strongly on the paper. We consider weak categories because as in the algebras situation the objects in our ideals do not need to have identities, but once more the important case is when this happens. This is the important case when we can show that there is an envelopping action. So one very important case which we consider is the case where all the ideals have local identities and in this case we can restrict ourselves to k-categories.

Actions of groups on a small k-category C were extensively studied by several authors, see [4], [9] and the references quoted therein. The following question arises: is it possible to generalize group actions on k-categories to the partial situation? The main purpose of this paper is to give a positive answer to this question.

Let G be a group. Recall that an action of G on a small k-category C is an action of G on the set C_0 of objects of C and a family of k-module isomorphisms $s : {}_yC_x \to {}_{sy}C_{sx}$, for each $s \in G$ and for each couple of objects x and y in C_0 and we have that $s(gf) = (sg)(sf)$ in case g and f are morphisms which can be composed in the category. Moreover, for elements $t, s \in G$ and a morphism f we have $(ts)f = t(sf)$ and $ef = f$, where e is the identity of G. A category C together with an action of G on C is called a G-category. We remark here that this notion can be defined for WNNA categories in the same way.

Recalling the definition of an ideal in a category we give the following

Definition 1.2. *An ideal in a WNNA category C is a collection I of morphisms such that if f is in I then $(gf)h$ and $g(fh)$ are in I whenever $(gf)h$ and $g(fh)$ are defined. Moreover, if C is a WNNA k-category, for I to be an ideal we require in addition that ${}_aI_b$ is a k-submodule of the k-module ${}_aC_b$, where ${}_aI_b$ denotes the set of all morphisms in ${}_aC_b$ which belong to I.*

Every ideal I in a WNNA k-category C can be looked as a WNNA subcategory, also denoted by I. In this case $I_0 = C_0$ and ${}_aI_b = {}_aC_b \cap I$, for any a, b in I_0.

In Section 2 we recall the definition of a partial action of a group on a set X and on a k-algebra A, and the partial skew group algebra introduced by Exel and Dokuchaev [6].

We also introduce the notion of partial orbit and show that the family of partial orbits of a set X form a partition of it, which is a generalization of what happens in the case of global actions.

In Section 3 we introduce the notion of a partial action of a group on a weak k-categories and define the partial skew category. We prove a coherence result between our approach and the ring-theoretical approach, in the case the weak category has a finite number of objects. Moreover, we show that the partial skew category is equivalent to the full subcategory of the partial skew category formed by taking one element in each equivalent class.

In Section 4 we give conditions for a partial action of a group on a small k-category C to have an enveloping action.

Definition 1.3. *Let C be a k-WNNA category, x an object of C and I an ideal of C. A morphism e in ${}_xI_x$ is called a local identity if, e is an idempotent, $ef = f$ for all $f \in {}_xI_y$, and $fe = f$ for all $f \in {}_yI_x$. Morever the local identity is called central if $fe = ef$ for all $f \in {}_xC_x$.*

It is convenient to point out that if C is a k-category and I is an ideal of C such that for each object $x \in C$ there is a morphism $e \in {}_xI_x$ that is a local identity, then I is itself a small k-category.

2 Partial skew group algebras

Let G be a group and X a set. A *partial action* α of G on X is a collection of subsets S_g, $g \in G$, of X and bijections $\alpha_g : S_{g^{-1}} \to S_g$ such that:

(i) $S_1 = X$ and α_1 is the identity mapping of X;

(ii) $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$;

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for any $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

Remark:

1. The property (ii) is equivalent to $\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh}$, for all $g, h \in G$.

2. We also have that $\alpha_{g^{-1}} = \alpha_g^{-1}$, for every $g \in G$.

Following [1], given a partial action α of G on X, a globalization of α, also called an enveloping action, is a pair (Y, β) such that X can be considered as a subset of Y, β is a global action of G on Y, $Y = \cup_{g\in G}\beta_g(X)$, $S_g = X \cap \beta_g(X)$ and $\alpha_g : S_{g^{-1}} \to S_g$ is equal to $\beta_g|_{S_{g^{-1}}}$, $g \in G$. In other words, α is the restriction of β to X.

If we have an additional structure some conditions can be imposed to the subsets S_g and the maps α_g. For example, for partial actions on topological spaces all the S_g are open subspaces of X and all the mappings α_g are homeomorphisms of topological spaces. In Theorem 1.1 of [1] the author proved that globalization does exist for partial actions on topological spaces, which clearly implies the result for partial actions on sets.

For the definition of a partial action of a group G on a k-algebra the authors considered that any S_g is an ideal of R and that every α_g is an isomorphism of algebras, $g \in G$.

Let α be a partial action of G on the k-algebra R. The partial skew group algebra $R\star_\alpha G$, see [6], is defined as the set of all finite formal sums $\sum_{g\in G} a_g u_g$, $a_g \in S_g$ for every $g \in G$, where the addition is defined in the usual way and the multiplication is determined by $(a_g u_g)(b_h u_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)u_{gh}$.

For the sake of completeness we recall now some facts and give a proposition.

Let A be a k-algebra with identity element and $\{e_i\}_{i=1}^n$ a complete set of orthogonal idempotents, i.e., a set of orthogonal idempotents whose sum is the identity of A. Then we define a k-category with a finite number of objects, denoted either by $(C_A, \{e_i\}_{i=0}^n)$ or C_A, as follows:

$Obj(C_A) = \{e_i : 1 \leq i \leq n\}$ and $Hom(e_i, e_j) = e_i A e_j$, for all $1 \leq i, j \leq n$. Composition is defined in a natural way via the product $(e_i A e_j)(e_j A e_k) \subset e_i A e_k$.

Conversely, given a k-category C which has a finite number of objects $\{e_i : 1 \leq i \leq n\}$, we define the, so called, k-algebra of homomorphism $a(C)$ in the following way: $a(C)$, as a set, is equal to $\oplus_{i,j} Hom(e_i, e_j)$. Note that the elements of $a(C)$ can be seen as matrices. So the addition and multiplication in $a(C)$ is defined as for matrices.

The following proposition is probably known nevertheless we give a proof here.

Proposition 2.1. *(i) Let A be a k-algebra with identity element and $\{e_i\}_{i=1}^n$ a complete set of orthogonal idempotents of A. Then the associated k-algebra $a(C_A)$ of C_A is isomorphic to A.*

(ii) Let C be a k-category with a finite number of objects. Then the categories C and $C_{a(C)}$ are equivalent.

Proof. (i) We define $\varphi : A \to a(C_A)$ by $\varphi(r) = (e_i r e_j)_{i,j}$ and $\psi : a(C_A) \to A$ by $\psi((e_i r_{i,j} e_j)_{i,j}) = \sum_{i,j} e_i r_{i,j} e_j$. It is easy to see that φ is a homomorphism of algebras, $\varphi \circ \psi = id_{a(C_A)}$ and $\psi \circ \varphi = id_A$.

(ii) Denote by $\{a_1, ..., a_n\}$ the objects of the category C. By definition $a(C) = \oplus_{i,j} Hom(a_i, a_j)$ with matrix operations. We define the category $C_{a(C)}$ by taking the complete set of orthogonal idempotents $\{_{a_1}1_{a_1}, ..., _{a_n}1_{a_n}\}$ in $a(C)$. Let T be the functor from C into $C_{a(C)}$ defined by $T(a_i) = {}_{a_i}1_{a_i}$, for any $1 \leq i \leq n$, and for each $f : a_i \to a_j$, $T(f) = {}_{a_i}1_{a_i} f {}_{a_j}1_{a_j}$. It can easily be seen that T is an equivalence of categories. ■

Definition 2.2. *Let α be a partial action of G on a set X. For each $a, b \in X$ we say that a, b are α-equivalent, $a \sim_\alpha b$, if there exists $g \in G$ such that $a \in S_{g^{-1}}$ and $b = \alpha_g(a)$. Briefly $a \sim_\alpha b$ will be denoted by $a \sim b$.*

Lemma 2.3. *Let α be a partial action of G on a set X. Then the relation $\sim$ is an equivalence relation.*

Proof. Straightforward. ■

Let α be a partial action of a group G on a set X. For each $x \in X$ the partial orbit of x is $H^\alpha(x) = \{\alpha_g(x) : x \in S_{g^{-1}}\}$. By Lemma 2.3 the set of all partial orbits form a partition of X.

Example 2.4. Consider $X = \{e_1, e_2, e_3, e_4\}$ and denote by G the cyclic group generated by σ of order 5. Let us take the subsets $S_1 = X, S_\sigma = \{e_2\}, S_{\sigma^2} = \{e_4\}, S_{\sigma^3} = \{e_3\}, S_{\sigma^4} = \{e_1\}$ and define α by $\alpha_1 = id_X$, $\alpha_\sigma(e_1) = e_2$, $\alpha_{\sigma^2}(e_3) = e_4$, $\alpha_{\sigma^3}(e_4) = e_3$, $\alpha_{\sigma^4}(e_2) = e_1$. It is easy to see that α is a partial action of G on X, $H^\alpha(e_1) = \{e_1, e_2\} = H^\alpha(e_2)$ and $H^\alpha(e_3) = \{e_3, e_4\} = H^\alpha(e_4)$.

Assume that α is a partial action of G on X and let (Y, β) be the globalization of α. Thus for any $g \in G$, β_g is a bijection of Y. Hence β is an action on Y and so it defines an equivalence relation denoted by $\approx$: for $u, v \in Y$, $u \approx v$ if there exists $g \in G$ such that $\beta_g(u) = v$. We denote the orbit of $y \in Y$ by $H^\beta(y)$.

We have the following

Lemma 2.5. *Let α be a partial action of G on X and let (Y, β) be its enveloping action. Then the equivalence relation defined by α on X is the restriction of the equivalence relation defined by β on Y. In particular, for any $x \in X$ $H^\alpha(x) = H^\beta(x) \cap X$.*

Proof. Straightforward. ■

The following remark will be useful.

Remark 2.6. *(See [4]) Let C be a k-category with a finite number of objects $\{f_i : 1 \leq i \leq n\}$. For each $i \in \{1, ..., n\}$ we associate the projective C-module $C_{fi}(f_j) = {}_{f_j}C_{f_i}$. Note that $\sum_{1\leq i\leq n} \oplus C_{f_i} \simeq a(C)$. Hence $End_C(\coprod_{i=1}^n C_{f_i}) = End(a(C)) \simeq a(C)$.*

Theorem 2.7. *Let G be a finite group and α a partial action of G on a k-algebra with identity A. Suppose that $S_g = Ae_g$, where $(e_g)_{g\in G}$ is a set of orthogonal idempotents whose sum is 1 such that $Ae_g = e_gA$, for all $g \in G$. Assume that $\{H_1, ..., H_n\}$ is the family of partial orbits of the set $\{e_g : g \in G\}$. Let $\{f_1, \cdots f_n\}$ be the idempotents, chosen one for each orbit. Then $End_{A*_\alpha G}(\coprod_{i=1}^n A *_\alpha Gf_i)$ is Morita equivalent to $A *_\alpha G$. In particular, if the idempotents are all in the orbit of a fix idempotent e then $End_{A*_\alpha G}(A *_\alpha Ge)$ is Morita equivalent to $A *_\alpha G$.*

Proof. Note that the ideals S_g are idempotents and by ([6], Theorem 3.1), the partial skew group algebra $A *_\alpha G$ is an associative algebra. By Proposition 2.1 $A *_\alpha G$ is isomorphic to $a(C_{A*_\alpha G})$ and $C_{A*_\alpha G}$ is isomorphic to the full subcategory formed by the objects $\{f_1, ..., f_n\}$, where $f_i \in H_i$. Hence, $a(C_{A*_\alpha G})$ is Morita equivalent to $a(C\{f_i\}_{i=1}^n)$. So, by Remark 2.6 $\coprod_{i,j=1}^n f_jA *_\alpha Gf_i$ is isomorphic to $End_{A*_\alpha G}(\coprod_{i=1}^n A *_\alpha Gf_i)$. ■

Example 2.8. *As in Example 2.4, take $S = \bigoplus_{i=1}^4 Re_i$, where R is a commutative ring, with the same partial action as before. Using the above theorem we get that $T = S *_\alpha G$ is Morita equivalent to $End(e_1T + e_2T)$*

3 Partial actions of groups on weak categories

In the remaining of the paper C will denote a small weak k-category, where k is commutative ring and C_0 the set of objects of C.

In the next definition we will give first a partial action α_0 of a group G on the set C_0. So for any $g \in G$ a subset C_0^g is given and $\alpha_0^g : C_0^{g^{-1}} \to C_0^g$ is a bijection. If $x \in C_0^{g^{-1}}$, $\alpha_0^g(x)$ will be denoted by gx.

Definition 3.1. *Let G be a group. We say that $\alpha = \{\alpha^g | g \in G\}$ is a partial action of G on C if the following conditions hold:*

(i) G acts partially on the set of objects C_0 of C. This partial action will be denoted by α_0 and the subsets associated to this partial action by C_0^g, $g \in G$;

(ii) For each $g \in G$ there exists an ideal $\mathcal{I}^g$ of C such that ${}_a\mathcal{I}_b^g = 0$ if one of the elements a or b are not in C_0^g;

(iii) There are equivalence of weak categories $\alpha^g : \mathcal{I}^{g^{-1}} \to \mathcal{I}^g$, for any $g \in G$, such that for $f \in {}_y\mathcal{I}_x^{g^{-1}}$, $\alpha^g(f) \in {}_{gy}\mathcal{I}_{gx}^g$, where x, y are in $C_0^{g^{-1}}$;

(iv) $\mathcal{I}^e = C$ and $\alpha^e = Id$;

(v) For any pair of objects $(x, y) \in C_0 \times C_0$ we have that

$$\alpha^{h^{-1}}({}_y\mathcal{I}_x^h \cap {}_y\mathcal{I}_x^{g^{-1}}) \subseteq {}_{h^{-1}y}\mathcal{I}_{h^{-1}x}^{(gh)^{-1}}, \text{ if } x, y \in C_0^h;$$

(vi) If $x, y \in C_0^h \cap C_0^{g^{-1}}$ and $f \in \alpha^{h^{-1}}({}_y\mathcal{I}_x^h \cap {}_y\mathcal{I}_x^{g^{-1}})$, then $\alpha^g(\alpha^h(f)) = \alpha^{gh}(f)$.

Note that the conditions (v) and (vi) above fit with the conditions (ii) and (iii) of the definition of partial actions of groups on algebras given in [6]. Also, as in [6], it can easily be checked that condition (iv) can be replaced by the condition $\alpha^{h^{-1}}({}_y\mathcal{I}_x^h \cap {}_y\mathcal{I}_x^{g^{-1}}) = {}_{h^{-1}y}\mathcal{I}_{h^{-1}x}^{(gh)^{-1}} \cap {}_{h^{-1}y}\mathcal{I}_{h^{-1}x}^{h^{-1}}$.

Now we give natural examples of partial actions on small weak k-categories.

Example 3.2. Assume that C is a small weak k-category and β is a global action of G on C. Let D be an ideal of C. We define a partial action α of G on D by restriction of β to D as follows:

The partial action is a global action on $D_0 = C_0$

The ideals $\mathcal{I}^g$ of D are defined by ${}_y\mathcal{I}_x^g = {}_yD_x \cap \beta_g({}_{g^{-1}y}D_{g^{-1}x})$, where, as above, $\beta_{g^{-1}}(x)$ is denoted simply by $g^{-1}x$.

Let us show first that $\mathcal{I}^g$ is an ideal of D. Let $f \in {}_y\mathcal{I}_x^g$ and l, m morphisms such that $l \in {}_zD_y$ and $m \in {}_xD_u$, with u and z in $C_0 = D_0$. Then there are

$$t \in {}_{g^{-1}y}D_{g^{-1}x}, \ \tilde{l} \in {}_{g^{-1}z}C_{g^{-1}y}, \ \tilde{m} \in {}_{g^{-1}x}C_{g^{-1}u}$$

with $f = \beta_g(t)$, $l = \beta_g(\tilde{l})$ and $m = \beta_g(\tilde{m})$. So $lfm = \beta_g(\tilde{l})\beta_g(t)\beta_g(\tilde{m}) = \beta_g(\tilde{l}t\tilde{m}) \in {}_zD_u \cap \beta_g({}_{g^{-1}z}D_{g^{-1}u}) = {}_z\mathcal{I}_u^g$.

It is easy to check that $\alpha^g : {}_y\mathcal{I}_x^{g^{-1}} \to {}_{gy}\mathcal{I}_{gx}^g$ is a bijection and an isomorphism of k-modules, for any $x, y \in D_0^g$.

Now let $f \in \alpha^{h^{-1}}({}_y\mathcal{I}_x^h \cap {}_y\mathcal{I}_x^{g^{-1}})$, i.e.,

$$\alpha^h(f) \in {}_y\mathcal{I}_x^h \cap {}_y\mathcal{I}_x^{g^{-1}} = {}_yD_x \cap \beta_h({}_{h^{-1}y}D_{h^{-1}x}) \cap \beta_{g^{-1}}({}_{gy}D_{gx}).$$

Hence $f \in \beta_{h^{-1}}(\beta_{g^{-1}}({}_{gy}D_{gx})) = \beta_{(gh)^{-1}}({}_{gy}D_{gx})$ and consequently

$$f \in {}_{h^{-1}y}D_{h^{-1}x} \cap \beta_{(gh)^{-1}}({}_{gy}D_{gx}) = {}_{h^{-1}y}\mathcal{I}_{h^{-1}x}^{(gh)^{-1}}.$$

Thus condition (iv) of the definition of partial actions is satisfied. Finally, condition (v) also holds since α^g is defined as restriction of β_g. ■

Remark 3.3. Note that we can give a more general example if we change slightly the definition of ideal of a weak category. In the following an ideal D of C will be a subset D_0 of C_0 together with a collection of morphisms ${}_yD_x$, for every $x, y \in D_0$, satisfying the same conditions of the Definition 1.1.

Example 3.4. Assume that C is a small weak k-category and β is a global action of G on C. Let D be an ideal of C in the sense of the above remark. For any $g \in G$ we put $D_0^g = D_0 \cap \beta_g(D_0)$. The partial action α_0 on D_0 is defined as the restriction of β, i.e. $\alpha_0^g : D_0^{g^{-1}} \to D_0^g$ is equal to $\beta|_{D_0^{g^{-1}}}$. The rest is defined as in the above example. The ideals $\mathcal{I}^g$ of D are defined by ${}_y\mathcal{I}_x^g = {}_yD_x \cap \beta_g({}_{g^{-1}y}D_{g^{-1}x})$ and α^g is the restriction of β_g to $\mathcal{I}^{g^{-1}}$ It is easy to show, as above, that this gives a partial action of G on D which not global in D_0.

Now we introduce a partial version of skew category.

Definition 3.5. *Let α be partial action of a group G on a small weak k-category C. We define the skew WNNA category $C *_\alpha G$ as follows:*

(i) $(C *_\alpha G)_0 = C_0$.

(ii) For each $x, y \in C_0$, ${}_y(C *_\alpha G)_x = \oplus_{g \in G}\ {}_y\mathcal{I}_{gx}^g$.

For each $f \in {}_z\mathcal{I}_{ty}^t$, $l \in {}_y\mathcal{I}_{gx}^g$ we define the composition by the following rule: $fl = \alpha^t(\alpha^{t^{-1}}(f) \circ l) \in {}_z\mathcal{I}_{(tg)x}^{tg}$.

Example 3.6. *Let G be a group and α a partial action on a small k-category C. If for any $g \in G$ the ideals $\mathcal{I}^g = C$, then for all $x, y \in C_0$, $\alpha_g : {}_xC_y \to {}_{gx}C_{gy}$*

is an isomorphism of k-modules. Note that the action of G on C_0 is global in this case. Thus, α is a global action of G on C and $C *_\alpha G$ defined above is the ordinary skew category $C[G]$, see [4].

We know that the partial skew group algebra introduced by Dokuchaev and Exel is not necessarily associative (Example 3.5 of [6]). Similarly the composition map in $C *_\alpha G$ is not necessarily associative, in general. The case where it is associative is of special interest, because of this we give the next definition.

Definition 3.7. *Let G be a group and α a partial action of G on a small k-category C. We say that the partial action α is associative if the composition of maps in $C *_\alpha G$ is associative.*

Remark 3.8. *As a consequence of the definition above, if C is small k-category and the partial action α is associative, then $C *_\alpha G$ is a category and we call it the partial skew category.*

In the next theorem we assume that α is a partial action of G on a small k-category C with finite number of objects such that α is not associative. In this case, $a(C *_\alpha G)$ is not necessary associative k-algebra.

Theorem 3.9. *Let α be a partial action of a group G on a small k-category C with a finite number of objects. Then G acts partially on $a(C) = \oplus_{x,y\in C_0}\ {}_yC_x$ and $a(C *_\alpha G)$ is isomorphic to $a(C) *_\alpha G$.*

Proof. For each $g \in G$ let $a(C)_g = \oplus_{x,y\in C_0}\ {}_y\mathcal{I}^g_x$ is an ideal of $a(C)$ and $\alpha_g : a(C)_{g^{-1}} \to a(C)_g$, defined by $\alpha_g|_{{}_y\mathcal{I}^{g^{-1}}_x} = \alpha^g|_{{}_y\mathcal{I}^{g^{-1}}_x}$, for all $x, y \in C_0^{g^{-1}}$, is an isomorphism of ideals.

Now we show that α is a partial action of G on $a(C)$. The first condition of the definition of partial actions is obvious. For the second suppose that $f \in \alpha_{h^{-1}}(a(C)_h \cap a(C)_{g^{-1}})$. We can assume that $f \in {}_yC_x$, so $\alpha_h(f) \in {}_{hy}C_{hx}$ and consequently $f \in a(C)_{(gh)^{-1}}$. Thus the second condition of the definition of partial actions is fulfilled. Finally the condition (iii) of Definition 1.1 in [6] follows immediately from condition (vi) of Definition 3.1.

We define $\varphi : a(C *_\alpha G) \to a(C) *_\alpha G$ by $\varphi(f_g) = f_g u_g$, where f_g is an elementary morphism in ${}_y\mathcal{I}^g_{gx} \subseteq {}_y(C *_\alpha G)_x$ and $\{u_g \mid g \in G\}$ denotes the canonical generators of $a(C) \star_\alpha G$. We clearly have that φ is a well defined homomorphism of k-algebras. Finally $\Psi : a(C) *_\alpha G \to a(C *_\alpha G)$ defined by $\Psi(f_g u_g) = f_g$, for any $f_g \in a(C)_g$, is clearly an inverse of φ. ∎

Recall that an algebra A is strongly associative if for any partial action α of a group G on A is always associative. A semiprime algebra is strongly associative ([6], Corollary 3.4)

The following is immediate from Theorem 3.7.

Corollary 3.10. *Let G be a group, A a strongly associative k-algebra and C_A the category with a single object $\{A\}$ and endomorphism k-algebra A. Suppose that G acts partially on C_A. Then $a(C_A *_\alpha G) \simeq A *_\alpha G$ and so α is associative. In particular, if A is semiprime then the associated skew NNA category is associative for any partial action α of a group G on A .*

Let α be a partial action of G on a small k-category C. We define the following relation: $x \sim y$ if there exists $g \in G$ such that ${}_x1_x \in {}_x\mathcal{I}_x^{g^{-1}}$ and $y = \alpha^g(x)$, where $1_x \in {}_x\mathcal{I}_x^{g^{-1}}$ denotes the identity morphism from x to x.

The proof of following lemma is standard.

Lemma 3.11. *The relation $\sim$ is an equivalence relation.*

The next proposition has a similar proof as Lemma 2.5, of [4]. For the sake of completeness we give a proof here, adapted to our case.

Proposition 3.12. *Let α be a partial action of a group G on a small k-category C such that α is associative. If $x \sim y$, then the objects x and y are isomorphic in $C *_\alpha G$.*

Proof. Suppose that x, y are equivalent, i.e, there exists $h \in G$ such that $y = hx$ and ${}_x1_x \in {}_x\mathcal{I}_x^{g^{-1}}$. Since ${}_y1_y \in {}_y\mathcal{I}_y^h$, we have that ${}_y1_y = {}_y1_{hx} \in {}_y\mathcal{I}_{hx}^h \subseteq {}_y(C *_\alpha G)_x$. On the other hand, ${}_x1_x = {}_x1_{h^{-1}y} \in {}_x\mathcal{I}_{h^{-1}y}^{h^{-1}} \subset {}_x(C *_\alpha G)_y$. We claim that ${}_x1_x = {}_x1_{h^{-1}y} \circ {}_y1_{hx}$. In fact, ${}_x1_{h^{-1}y} \circ {}_y1_{hx} = \alpha^{h^{-1}}(\alpha^h({}_x1_{h^{-1}y}) \circ {}_y1_{hx}) = \alpha^{h^{-1}}({}_{hx}1_y \circ {}_y1_{hx}) = \alpha^{h^{-1}}({}_{hx}1_{hx}) = {}_x1_x$. Using similar methods we can show that ${}_y1_y = {}_y1_{hx} \circ {}_x1_{h^{-1}y}$. ∎

The next corollary is a direct consequence of the above result.

Corollary 3.13. *Let α be a partial action of a group G on a small k-category C such that α is associative and S a representative set of the equivalence relation defined before (that is, there is in S exactly one element of each equivalence class) and C^* be the full subcategory of $C *_\alpha G$ whose objects are the elements of S. Then C^* is equivalent to $C *_\alpha G$.*

4 Globalization of partial actions

In this section we consider always small k-categories, unless otherwise stated.

Examples 3.2 and 3.4 are natural examples of partial actions of groups on small k-categories which can be obtained by restriction of global actions to ideals. Moreover, if a partial action α is obtained in that way, then $C *_\alpha G$, defined as before, is an associative category. Thus it is natural to ask when a partial action can be obtained by restriction of a global action. This question has been considered in [1] for partial actions on topological spaces and in [6] for partial action on algebras with identity element.

Definition 4.1. *Let B and D be two WNNA categories. We say that $T : B \to D$ is a quasi-functor if the following conditions are satisfied:*

(i) For each object b of B, $T(b)$ is an object of D;

(ii) For each morphism f in B, $T(f)$ is a morphism in D;

(iii) Given two morphims f,g in B such that $\exists f \circ g$ in B we have that $T(f) \circ T(g)$ does exist in D and $T(f \circ g) = T(f) \circ T(g)$.

Note that if B and D are categories, then a functor $T : B \to D$ is a quasi-functor such that $T({}_x1_x) = {}_{T(x)}1_{T(x)}$, for all $x \in B_0$.

Induced by the definition given in [6], Section 4, we give the following.

Definition 4.2. *Let (C, α) be a small k-category C together with a partial action α of G on C. We say that a pair (D, β), where D is a k-category and β is a global action of G on D, is an enveloping (also called a globalization) of (C, α) if the following conditions are satisfied:*

(i) There is a faithful quasi-functor $j : C \to D$;

(ii) For each $f \in {}_yj(C)_x$, $g \in {}_zD_y$ and $h \in {}_xD_v$, where $x, y, z, v \in j(C_0)$, we have $gfh \in {}_zj(C)_v$;

(iii) $j({}_y\mathcal{I}_x^g) = j({}_yC_x) \cap \beta_g(j({}_{g^{-1}y}C_{g^{-1}x}))$, *for all* $x, y \in C_0$;

(iv) $j \circ \alpha^g(f) = \beta_g \circ j(f)$, *for any* $f \in {}_y\mathcal{I}_x^{g^{-1}}$;

(v) ${}_yD_x = \sum_{g\in G} \beta_g(j({}_{g^{-1}y}C_{g^{-1}x}))$, *for any* $x, y \in D_0$ *such that* $g^{-1}x, g^{-1}y \in C_0$.

It is convenient to remark that when (C, α) *has an enveloping action* (D, β) *we have that* $C *_\alpha G$ *is a subcategory of the skew category* $D[G]$.

Definition 4.3. *Given small* k*-categories* D *and* D', *we say that global actions* (D, β) *and* (D', β') *of* G *on* D *and* D', *respectively, are equivalent if there exists an equivalence of categories* $\Phi : D \to D'$ *such that for any* $g \in G$ *we have* $\beta_g \circ \Phi = \Phi \circ \beta'_g$.

Lemma 4.4. *Assume that* α *is a partial action of a group* G *on a small* k*-category* C *which has a globalization* (D, β) *and let* $j : C \to D$ *the canonical faithful functor. Then for any* $x \in C_0$ *and* $g_1, ..., g_n \in G$ *the submodule* $\sum_{1\le i\le n} \beta_{g_i}(j({}_{g_i^{-1}x}C_{g_i^{-1}x}))$ *of* ${}_xD_x$ *has an identity element with respect to composition.*

Proof. By induction it is enough to prove the result for $n = 2$. Put $N = \beta_g(j({}_{g^{-1}x}C_{g^{-1}x})) + \beta_h(j({}_{h^{-1}x}C_{h^{-1}x}))$. Since $\beta_g(j({}_{g^{-1}x}1_{g^{-1}x}))$ is an identity for $\beta_g(j({}_{g^{-1}x}C_{g^{-1}x}))$ and $\beta_h(j({}_{h^{-1}x}1_{h^{-1}x}))$ an identity for $\beta_h(j({}_{h^{-1}x}C_{h^{-1}x}))$, it is easy to see that

$$\beta_g(j({}_{g^{-1}x}1_{g^{-1}x})) + \beta_h(j({}_{h^{-1}x}1_{h^{-1}x})) - \beta_g(j({}_{g^{-1}x}1_{g^{-1}x}))\beta_h(j({}_{h^{-1}x}1_{h^{-1}x}))$$

is an identity for N. ■

In the next result we will assume that the k-subspace ${}_x\mathcal{I}_x^g$ contains a local identity, for any $g \in G$ and $x \in C_0$. We should point out that this local identity is not necessarily the identity in ${}_xC_x$. Now we prove the main theorem of this section.

Theorem 4.5. *Let* α *be a partial action of a group* G *on a small* k*-category* C *such that* α_0 *is global on* C_0. *Then there exists an enveloping action of* (C, α) *if and only if all the* k*-spaces* ${}_x\mathcal{I}_x^g$ *contains a local identity element, for any* $x \in C_0$ *and* $g \in G$. *Moreover, the enveloping action, if does exists, it is unique up to equivalence.*

Proof. Assume that (D, β) is an enveloping action of (C, α) and denote by $j : C \to D$ the functor of Definition 4.2. Note that, in this case, since α_0 is global in C_0 we can assume that $C_0 = D_0$ and so $j(C)$ is an ideal of D. Now ${}_x\mathcal{I}_x^g$ has an identity $j^{-1}(j({}_x1_x)\beta_g(j({}_{g^{-1}x}1_{g^{-1}x})))$, for any $g \in G$ and $x \in C_0$, where ${}_x1_x$ denotes the identity of ${}_xC_x$.

Conversely, in the rest of the proof we assume that ${}_x\mathcal{I}_x^g$ contains an identity ${}_x1_x^g$, for any $x \in C_0$ and $g \in G$, that we shortly denote by 1^g when there is no possibility of misunderstanding. Define the category B as follows: $B_0 = C_0$ and for any $x, y \in B_0$ the k-module of morphisms ${}_yB_x$ is defined as the k-module $F(G, {}_yC_x)$ of all the maps σ from G to the direct product $\prod_{g\in G} {}_{gy}C_{gx}$ such that $\sigma(g) \in {}_{g^{-1}y}C_{g^{-1}x}$, for any $g \in G$.

As in [6] we write $\sigma\mid_g$ to denote $\sigma(g)$. The composition of the morphisms $\sigma \in {}_yB_x$ and $\tau \in {}_zB_y$ is defined by $\tau \circ \sigma \mid_h = \tau \mid_h \circ \sigma \mid_h$.

We define a global action β of G on B as follows: if $\sigma \in {}_yB_x$ and $h \in G$, then we put $\beta_h(\sigma) \mid_g = \sigma \mid_{h^{-1}g}$. Since $\sigma \mid_{h^{-1}g} \in {}_{g^{-1}hy}C_{g^{-1}hx}$ it follows that $\beta_h(\sigma) \in {}_{hy}B_{hx}$ and so β is well-defined. It is easy to see that β is an action of G on B.

Now we define a functor $j : C \to B$. As map from C_0 to B_0, j is the identity. If $\sigma \in {}_yC_x$ we put $j(\sigma) \mid_h = \alpha^{h^{-1}}(\sigma 1^h) \in {}_{h^{-1}y}C_{h^{-1}x}$. Thus $j(\sigma) \in {}_yB_x$ and consequently j is well-defined. We see that j is a faithful functor from C into B. Suppose that $\mu, \eta \in C$ and $j(\mu) = j(\eta)$. Then for any $h \in G$ we have $j(\mu) \mid_h = j(\eta) \mid_h$ and so $\alpha^{h^{-1}}(\mu 1^h) = \alpha^{h^{-1}}(\eta 1^h)$. Taking $h = e$, the identity of G, we obtain $\mu = \eta$.

Now assume that $\mu \in {}_yC_x$ and $\eta \in {}_zC_y$. Hence $\eta\mu \in {}_zC_x$ and $j(\eta\mu) \mid_h = \alpha^{h^{-1}}(\eta\mu 1^h)$. On the other hand

$$j(\eta)j(\mu) \mid_h = j(\eta) \mid_h \circ j(\mu) \mid_h = \alpha^{h^{-1}}(\eta 1^h) \circ \alpha^{h^{-1}}(\mu 1^h) = \alpha^{h^{-1}}(\eta 1^h \mu 1^h) = \alpha^{h^{-1}}(\eta\mu 1^h).$$

Therefore $j(\eta\mu) = j(\eta)j(\mu)$.

Let D be the subcategory of B defined as follows: the set of objects D_0 of D is equal to C_0 and the set of morphisms from x to y is given by ${}_yD_x = \sum_{g\in G} \beta_g(j({}_{g^{-1}y}C_{g^{-1}x}))$. It is clear that D is a small k-subcategory of B, $j : C \to D$ is a faithful functor and β is a global action of G on D. Also condition (iv) of the definition of enveloping action is fulfilled.

Recall that $\alpha^{h^{-1}}({}_y\mathcal{I}^h_x \cap {}_y\mathcal{I}^{g^{-1}}_x) = {}_{h^{-1}y}\mathcal{I}^{(gh)^{-1}}_{h^{-1}x} \cap {}_{h^{-1}y}\mathcal{I}^{h^{-1}}_{h^{-1}x}$. Using this and taking $h^{-1} = g$ we easily obtain $\alpha^g(1^{g^{-1}}1^h) = 1^g 1^{gh}$, for any $g, h \in G$. To simplify notation we will write $\mathcal{I}^g$ instead of ${}_x\mathcal{I}^g_y$, for $x, y \in C_0$.

For any $\eta \in \mathcal{I}^{g^{-1}}$ we have $\alpha^g(\eta 1^{g^{-1}h}) \in \alpha^g(\mathcal{I}^{g^{-1}} \cap \mathcal{I}^{g^{-1}h}) = \mathcal{I}^g \cap \mathcal{I}^h$. Thus using conditions (v) and (vi) of Definition 3.1 we obtain

$$\alpha^{h^{-1}g}(\eta 1^{g^{-1}h}) = \alpha^{h^{-1}}(\alpha^g(\eta 1^{g^{-1}h})) = \alpha^{h^{-1}}(\alpha^g(\eta)\alpha^g(1^{g^{-1}}1^{g^{-1}h})) =$$

$$\alpha^{h^{-1}}(\alpha^g(\eta)1^g 1^h) = \alpha^{h^{-1}}(\alpha^g(\eta)1^h).$$

Now we show condition (iv) of Definition 4.2. In fact, let $\eta \in \mathcal{I}^{g^{-1}}$. Then $\beta_g(j(\eta)) \mid_h = j(\eta) \mid_{g^{-1}h} = \alpha^{h^{-1}g}(\eta 1^{g^{-1}h}) = \alpha^{h^{-1}}(\alpha^g(\eta)1^h) = j(\alpha^g(\eta)) \mid_h$. Consequently $\beta_g(j(\eta)) = j(\alpha^g(\eta))$, for any $\eta \in \mathcal{I}^{g^{-1}}$ and (iv) holds.

Let see now that condition (iii) of Definition 4.2 holds. Let $\sigma \in j({}_yC_x) \cap \beta_g(j({}_{g^{-1}y}C_{g^{-1}x}))$. Then there are $\eta \in {}_yC_x$ and $\mu \in {}_{g^{-1}y}C_{g^{-1}x}$ such that $\sigma = j(\eta) = \beta_g(j(\mu))$. Thus for any $h \in G$ we have $j(\eta) \mid_h = \beta_g(j(\mu)) \mid_h$. This implies that $\alpha^{h^{-1}}(\eta 1^h) = j(\mu) \mid_{g^{-1}h} = \alpha^{h^{-1}g}(\mu 1^{g^{-1}h})$. Hence taking $h = e$ we obtain $\eta = \alpha^g(\mu 1^{g^{-1}}) \in \mathcal{I}^g$ and so $\sigma \in j(\mathcal{I}^g)$. The argument shows that $j(C) \cap \beta_g(j(C)) \subseteq j(\mathcal{I}^g)$.

On the other hand, if $\nu = j(\eta)$ for $\eta \in \mathcal{I}^g$ we have that $\nu \in j(C)$. Also there exists $\tau \in \mathcal{I}^{g^{-1}}$ with $\alpha_g(\tau) = \eta$. Hence, $\nu = j(\alpha_g(\tau))$ and so for any $h \in G$ we have $\nu \mid_h = j(\alpha_g(\tau)) \mid_h = \beta_g(j(\tau) \mid_h$. So, $\nu \in \beta_g(j(C))$ and the relation (iii) follows.

Finally, we see that also condition (ii) is satisfied. Let $\eta \in {}_{g^{-1}z}C_{g^{-1}y}$ and $\mu \in {}_yC_x$. Then

$$\beta_g(j(\eta)) \mid_h \circ j(\mu) \mid_h = j(\eta) \mid_{g^{-1}h} \circ j(\mu) \mid_h = \alpha^{h^{-1}g}(\eta 1^{g^{-1}h})\alpha^{h^{-1}}(\mu 1^h) =$$

$$\alpha^{h^{-1}}(\alpha^g(\eta 1^{g^{-1}})1^h)\alpha^{h^{-1}}(\mu 1^h) = \alpha^{h^{-1}}(\alpha^g(\eta 1^{g^{-1}})\mu 1^h) = j(\alpha^g(\eta 1^{g^{-1}})\mu) \mid_h .$$

Consequently $\beta_g(j(\eta)) \circ j(\mu) = j(\alpha^g(\eta 1^{g^{-1}})\mu) \in j(C)$. Similarly we prove $j(\mu) \circ \beta_g(j(\eta)) = j(\mu\alpha^g(\eta 1^{g^{-1}})) \in j(C)$. Thus condition (ii) also holds.

It remains to prove the uniqueness. Assume that (D, β) and (E, γ) are two globalizations of α, where $j : C \to D$ and $j' : C \to E$ are the canonical functors of Definition 4.2. Then ${}_yD_x = \sum_{g\in G} \beta_g(j({}_{g^{-1}y}C_{g^{-1}x}))$ and ${}_yE_x = \sum_{g\in G} \gamma_g(j'({}_{g^{-1}y}C_{g^{-1}x}))$, for any $x, y \in C_0 = D_0 = E_0$. We define the mapping $\Theta : {}_yD_x \to {}_yE_x$ by $\Theta(\sum_{1\le i\le n} \beta_{g_i}(j(f_i))) = \sum_{1\le i\le n} \gamma_{g_i}(j'(f_i))$, for $f_i \in {}_{g^{-1}y}C_{g^{-1}x}$, $1 \leq i \leq n$. As in the last part of the proof of Theorem 4.5 of [6] it follows that Θ is well-defined and so it is an isomorphism of k-modules. This completes the proof. ■

We were unable either to prove or to disprove the result corresponding to Theorem 4.5, in general. The implication in one direction always holds. In fact, to end the paper we can easily to obtain the following, repeating the first part of the proof of Theorem 4.5.

Proposition 4.6. *Let α be a partial action of a group G on C and suppose that the partial action α has an enveloping action (D, β). Then for each $x \in C_0$, ${}_x\mathcal{I}_x^g$ has a local identity, for all $g \in G$.*

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