On the enumeration of three-rowed standard Young tableaux of skew shape in terms of Motzkin numbers

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Abstract

The enumeration of standard Young tableaux (SYTs) of shape λ can be easily computed by the hook-length formula. In 1981, Amitai Regev proved that the number of SYTs having at most three rows with n entries equals the nth Motzkin number M_n. In 2006, Regev conjectured that the total number of SYTs of skew shape $\lambda/(2,1)$ over all partitions λ having at most three parts with n entries is the difference of two Motzkin numbers, $M_{n-1} - M_{n-3}$. Ekhad and Zeilberger proved Regev's conjecture using a computer program. In his paper [2], S.-P. Eu found a bijection between Motzkin paths and SYTs of skew shape with at most three rows to prove Regev's conjecture, and Eu also indirectly showed that for the fixed $\mu = (\mu_1, \mu_2)$ the number of SYTs of skew shape λ/μ over all partitions λ having at most three parts can be expressed as a linear combination of the Motzkin numbers. In this paper, we will find an explicit formula for the generating function for the general case: for each partition μ having at most three parts the generating function gives a formula for the coefficients of the linear combination of Motzkin numbers. We will also show that these generating functions are unexpectedly related to the Chebyshev polynomials of the second kind.

1 Introduction

A *partition* λ of a nonnegative integer n is a weakly decreasing sequence

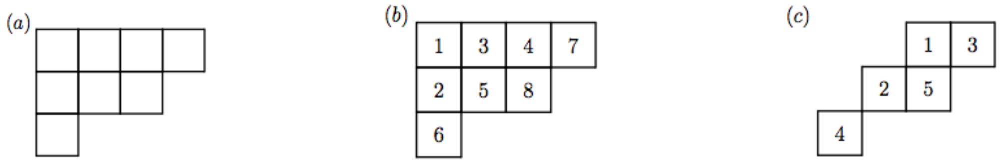
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\sum_{i=1}^{l} \lambda_i = n$ and $\lambda_l > 0$. The integers λ_i are called *parts*. The number of parts of λ is the *length* of λ, denoted $l(\lambda)$. We denote $\mathrm{Par}(n)$ to be the set of all partitions of n, with $\mathrm{Par}(0)$ consisting of the empty partition $\emptyset$, and we let

$$\mathrm{Par} := \cup_{n\geq 0} \mathrm{Par}(n).$$

If $\lambda \in \mathrm{Par}(n)$, then we also write $\lambda \vdash n$ or $|\lambda| = n$. Any partition λ can be identified with its *Young diagram*, which is a collection of boxes arranged in left-justified rows with the ith row containing λ_i boxes for $1 \le i \le l(\lambda)$. If λ and μ are partitions such that $\mu_i \le \lambda_i$ for all i, we write $\mu \subset \lambda$. Let us assume $|\lambda| = n$ and $\mu \subset \lambda$ throughout this paper.

A *skew shape* λ/μ is a pair of partitions (λ, μ) such that the Young diagram of λ contains the Young diagram of μ. If $\mu = \emptyset$, then we assume that $\lambda/\mu = \lambda$. A *skew Young diagram* of shape λ/μ is a Young diagram of λ with a Young diagram of μ removed from it. A *standard Young tableau* (SYT) of skew shape λ/μ is obtained by taking a skew Young diagram of shape λ/μ and writing numbers $1, 2, \ldots, n$ in the n boxes of this diagram such that the numbers increase from left to right in each row and from top to bottom down in each column. A Young tableau is called *semistandard* (SSYT) if the entries weakly increase from left to right in each row and strictly increase from top to bottom down in each column. The size of a SSYT is the number of its entries. As shown in Figure below, (a) is a Young diagram of shape $(4, 3, 1)$, (b) is a SYT of shape $(4, 3, 1)$, and (c) is a SYT of skew shape $(4, 3, 1)/(2, 1)$, respectively.



Now let us review several definitions for symmetric functions [12]. Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates. For $n \in \mathbb{N}$, a *homogeneous symmetric function of degree* n over the complex number field $\mathbb{C}$ is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n, $c_\alpha \in \mathbb{C}$, x^α stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, and $f(x_{\omega(1)}, x_{\omega(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation ω of the positive integers $\mathbb{P}$. Note that a symmetric function of degree zero is just a complex number.

The *monomial symmetric functions* m_λ for $\lambda \in \mathrm{Par}$ are defined by

$$m_\lambda := \sum_{\alpha} x^\alpha,$$

where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the entries of the vector $\lambda = (\alpha_1, \alpha_2, \ldots)$.

The *complete homogeneous symmetric functions* h_λ for $\lambda \in \mathrm{Par}$ are defined by the formulas

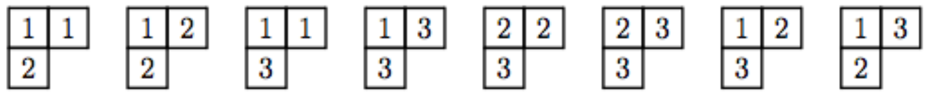
$$h_n := \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \text{ for } n \ge 1 \qquad (\text{with } h_0 = 1)$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \qquad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots).$$

Let λ/μ be a skew shape. The *skew Schur function* $s_{\lambda/\mu} = s_{\lambda/\mu}(x)$ *of* λ/μ in the variables $x = (x_1, x_2, \ldots)$ is the sum of monomials

$$s_{\lambda/\mu}(x) := \sum_T x^T = \sum_T x_1^{t_1} x_2^{t_2} \cdots,$$

summed over all SSYTs T of skew shape λ/μ where each t_i counts the occurrences of the number i in the SSYT T. If $\mu = \emptyset$, (that is, $\lambda/\mu = \lambda$), then we call $s_\lambda(x)$ the Schur function of shape λ. For example, the SSYTs of shape $(2, 1)$ with largest part at most three are given by



So, we have

$$\begin{aligned} S_{(2,1)}(x_1, x_2, x_3) &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 \\ &= m_{(2,1)}(x_1, x_2, x_3) + 2m_{(1,1,1)}(x_1, x_2, x_3). \end{aligned}$$

2 About standard Young tableaux having at most three rows

In their paper [5], Gordon and Houten showed that the sum of Schur functions of skew shape λ/μ, for fixed μ over all partitions λ with at most a fixed number of rows, can be expressed as a Pfaffian of a matrix of complete homogeneous symmetric functions. Before we state their results, let us define the functions h, g_i, and f_j used in them.

For $i \geq 0$ and $j \geq 1$, let us define h, g_i, and f_j by

$$\begin{aligned} h &:= \sum_{n=0}^{\infty} h_n = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} m_\lambda, \\ g_i &:= \sum_{n=0}^{\infty} h_n h_{n+i} = \sum_{n=0}^{\infty} \sum_{\lambda_1 \vdash n} m_{\lambda_1} \sum_{\lambda_2 \vdash n+i} m_{\lambda_2}, \end{aligned}$$

and

$$f_j := g_0 + 2(g_1 + \cdots + g_{j-1}) + g_j.$$

Gordon and Houten proved that

$$\sum_{l(\lambda) \leq 2m} s_{\lambda/\mu} = \mathrm{Pf}(D_{2m}),$$

where Pf denotes the Pfaffian and $D_{2m} = (f_{\lambda_i - \mu_j + j - i})_{1 \leq i,j \leq 2m}$, and

$$\sum_{l(\lambda) \leq 2m+1} s_{\lambda/\mu} = \mathrm{Pf}\begin{pmatrix} 0 & H \\ -H^t & D_{2m} \end{pmatrix}, \tag{1}$$

where the matrix is obtained by bordering D_{2m} with a row H of h's, a column $-H^t$ (transpose of $-H$) of $-h$'s, and a zero.

For $\mu = \emptyset$, Gordon [4] showed that

$$\sum_{l(\lambda)\le 2m} s_\lambda = \det(g_{i-j} + g_{i+j-1})_{1\le i,j\le m}, \tag{2}$$

and

$$\sum_{l(\lambda)\le 2m+1} s_\lambda = h\det(g_{i-j} - g_{i+j})_{1\le i,j\le m}. \tag{3}$$

For the case $m = 1$, identity (2) reduces to

$$\sum_{l(\lambda)\le 2} s_\lambda = g_0 + g_1. \tag{4}$$

We can directly prove identity (4) by considering the two cases of partitions of even or odd numbers and then by applying Pieri's rule to each case. For $m = 1$, identity (3) reduces to

$$\sum_{l(\lambda)\le 3} s_\lambda = h(g_0 - g_2),$$

which can also be proved by applying Pieri's rule to expand hg_0 and hg_2.

To count SYTs of skew shape λ/μ, we need to find the coefficient of $x_1x_2\cdots x_{n-|\mu|}$ in $s_{\lambda/\mu}$. Now let us transform symmetric functions into formal power series by applying the map θ from the algebra of symmetric functions to formal power series in x, which is defined by

$$\theta(m_\lambda) = \begin{cases} x^r/r!, & \text{if } \lambda = (1^r) \text{ for some } r \\ 0, & \text{otherwise} \end{cases}$$

and extended by linearity. Then we can easily show that the map θ is a homomorphism with the property

$$\theta(h_n) = \frac{x^n}{n!}.$$

Let us apply the linear map θ to g_i. For $i \ge 0$, we have

$$\begin{aligned}\theta(g_i) &= \theta\Big(\sum_{n=0}^{\infty} h_n h_{n+i}\Big) \\ &= \sum_{n=0}^{\infty} \theta(h_n)\theta(h_{n+i}) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{x^{n+i}}{(n+i)!} \\ &= \sum_{n=0}^{\infty} \binom{2n+i}{n}\frac{x^{2n+i}}{(2n+i)!}.\end{aligned}$$

Now we want to convert the generating function $\theta(g_i)$ to an ordinary generating function. To do this, we need the following lemma.

Lemma 2.1. *Define L to be the linear map from the algebra of formal power series $\mathbb{C}[[x]]$ to itself defined by $\frac{x^n}{n!} \mapsto x^n$, extended by linearity. Then we have*

$$L(e^x f(x)) = \frac{1}{1-x} F\Big(\frac{x}{1-x}\Big), \text{ where } F(x) = L\big(f(x)\big).$$

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. Then $F(x) = L\big(f(x)\big) = \sum_{n=0}^{\infty} a_n x^n$ and

$$\begin{aligned}
L(e^x f(x)) &= \sum_{n=0}^{\infty} a_n L\Big(e^x \frac{x^n}{n!}\Big) \quad \text{by linearity} \\
&= \sum_{n=0}^{\infty} a_n L\Big(\sum_{k=0}^{\infty} \binom{n+k}{k} \frac{x^{n+k}}{(n+k)!}\Big) \\
&= \sum_{n=0}^{\infty} a_n x^n \sum_{k=0}^{\infty} \binom{n+k}{k} x^k \\
&= \sum_{n=0}^{\infty} a_n \frac{x^n}{(1-x)^{n+1}} \\
&= \frac{1}{1-x} F\Big(\frac{x}{1-x}\Big).
\end{aligned}$$

□

Let $c(x)$ be the Catalan number generating function

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x},$$

and $m(x)$ be the Motzkin number generating function

$$m(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2},$$

where $M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} n!/\big(k!(k+1)!(n-2k)!\big)$ is the nth Motzkin number. From the well-known fact

$$\sum_{n=0}^{\infty} \binom{2n+s}{n} x^n = \frac{c(x)^s}{\sqrt{1-4x}}, \tag{5}$$

we can express the Motzkin number generating function in terms of the Catalan number generating function:

$$\frac{1}{1-x} c\left(\frac{x^2}{(1-x)^2}\right) = m(x).$$

Let $\Psi := L \circ \theta$. Then Ψ is a linear map from the algebra of symmetric functions to the algebra of formal power series $\mathbb{C}[[x]]$.

Now let us compute $\Psi(g_i)$ for $i \geq 0$.

$$\begin{aligned}\Psi(g_i) &= L(\theta(g_i)) \\ &= L\left(\sum_{n=0}^{\infty}\binom{2n+i}{n}\frac{x^{2n+i}}{(2n+i)!}\right) \\ &= \sum_{n=0}^{\infty}\binom{2n+i}{n}x^{2n+i} \\ &= \frac{x^i c(x^2)^i}{\sqrt{1-4x^2}},\end{aligned} \tag{6}$$

where the last equation follows from identity (5).

For $j, k \geq 1$, let us consider a partition $\mu = (\mu_1, \mu_2)$ having $\mu_1 = j+k-2$ and $\mu_2 = j-1$ in equation (1). Then equation (1) reduces to

$$\sum_{l(\lambda)\leq 3} s_{\lambda/\mu} = h(f_j + f_k - f_{j+k}). \tag{7}$$

Let $G_3(\mu_1, \mu_2)$ be the sum (7). That is,

$$G_3(j+k-2, j-1) = h(f_j + f_k - f_{j+k}). \tag{8}$$

Now let us first consider the case $\mu = (k-1, 0)$, that is, $\mu = (k-1)$. Letting $j = 1$ in equation (8), we can find the number of SYTs of skew shape $\lambda/(k-1)$ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-k+1$. Since

$$f_n = g_0 + 2(g_1 + \cdots + g_{n-1}) + g_n \text{ for } n > 0,$$

we have

$$G_3(k-1, 0) = h(g_0 + g_1 - g_k - g_{k+1}).$$

By equation (6) and Lemma 2.1, we have

$$\begin{aligned}\Psi(hg_k) &= L\big(\theta(hg_k)\big) \\ &= L\big(\theta(h)\theta(g_k)\big) \\ &= L\big(e^x\theta(g_k)\big) \\ &= \Big(\frac{x}{1-x}\Big)^k c\Big(\frac{x^2}{(1-x)^2}\Big)^k \frac{1}{\sqrt{1-2x-3x^2}} \\ &= \frac{x^k m(x)^k}{\sqrt{1-2x-3x^2}}.\end{aligned} \tag{9}$$

To simplify the term that we computed above, let us define α_k by

$$\alpha_k := \Psi(hg_k), \text{ for all } k \geq 0.$$

Then we have $\alpha_k = x^k m(x)^k/\sqrt{1-2x-3x^2}$.

To find the number of SYTs of skew shape $\lambda/(k-1)$ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-k+1$, let us apply Ψ to $G_3(k-1, 0)$. Then by linearity we deduce

$$\begin{aligned}
\Psi\big(G_3(k-1,0)\big) &= \alpha_0 + \alpha_1 - \alpha_k - \alpha_{k+1} \\
&= \frac{1 + xm(x) - x^k m(x)^k - x^{k+1} m(x)^{k+1}}{\sqrt{1-2x-3x^2}} \\
&= \frac{\big(1+xm(x)\big)\big(1 - x^k m(x)^k\big)}{\sqrt{1-2x-3x^2}} \\
&= \frac{m(x)}{1 - x^2 m(x)^2}\big(1+xm(x)\big)\big(1 - x^k m(x)^k\big) \\
&= \frac{m(x)}{1 - xm(x)}(1 - x^k m(x)^k). \qquad (10)
\end{aligned}$$

For example, consider the case $k = 1$ in (10), that is, $\mu = \emptyset$. Then we know that the number of SYTs of shape λ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n$, is equal to the nth Motzkin number since

$$\Psi\big(G_3(0,0)\big) = m(x).$$

Consider the case $k = 2$ in (10), that is, $\mu = (1)$. Since the Motzkin number generating function $m(x)$ satisfies the functional identity

$$m(x) = 1 + xm(x) + x^2 m(x),$$

we deduce

$$\begin{aligned}
\Psi\big(G_3(1,0)\big) &= m(x)\big(1 + xm(x)\big) \\
&= \frac{m(x) - 1}{x}.
\end{aligned}$$

This tells us that the number of SYTs of skew shape $\lambda/(1)$ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-1$ is equal to the nth Motzkin number for $n \geq 1$.

Now let us find the generating function for $\Psi\big(G_3(k-1,0)\big)$ for $k \geq 1$. From (10), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} (\alpha_0 + \alpha_1 - \alpha_k - \alpha_{k+1}) y^k &= \sum_{k=1}^{\infty} \frac{m(x)}{1 - xm(x)}(1 - x^k m(x)^k) y^k \\
&= \frac{m(x)}{1 - xm(x)}\left(\frac{1}{1-y} - \frac{1}{1 - xym(x)}\right) \\
&= -\frac{y^2}{(1-y)(x + xy - y + xy^2)} \\
&\quad + \frac{xy}{(1-y)(x + xy - y + xy^2)} m(x), \qquad (11)
\end{aligned}$$

where the last equation follows by rationalizing the denominator. By factoring the denominator of the first expression in the right side of equation (11), we have

$$-\frac{y^2}{(1-y)(x+xy-y+xy^2)} = -\frac{y^2}{x(1-y)(1+y-y/x+y^2)}, \tag{12}$$

so when expanded in powers of y, the function (12) has only negative powers of x. However, the left side of equation (11) has no negative powers of x. So we can conclude that the function (12) must be cancelled with negative powers of x from the second expression in the right side of equation (11). Therefore, we know that the generating function for $\Psi\big(G_3(k-1)\big)$ is the part of

$$\frac{xy}{(1-y)(x+xy-y+xy^2)}m(x),$$

consisting of nonnegative powers of x.

Let

$$R(x,y) := \frac{xy}{(1-y)(x+xy-y+xy^2)}. \tag{13}$$

By factoring

$$\frac{xy}{(x+xy-y+xy^2)} = \frac{y}{1+(1-x^{-1})y+y^2},$$

we can see that the coefficient of y^n in (13) is a polynomial in $1/x$. So there are polynomials $r_k(x) = \sum_i r_{k,i}x^i$ such that the function (13) can be written as

$$\frac{y}{(1-y)\left(1+(1-x^{-1})y+y^2\right)} = \sum_{k=1}^{\infty} r_k\left(\frac{1}{x}\right)y^k. \tag{14}$$

Now we need to compute the coefficient of $x^{n-(k-1)}y^k$ in the expression $R(x,y)m(x)$ to find the number of SYTs of skew shape $\lambda/(k-1)$ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-(k-1)$. By (14), we have

$$\begin{aligned}
[x^{n-k+1}y^k]R(x,y)m(x) &= [x^{n-k+1}]r_k\left(\frac{1}{x}\right)m(x)\\
&= [x^{n-k+1}]\sum_{i=0}^{\infty} M_i x^i r_k\left(\frac{1}{x}\right)\\
&= [x^{n-k+1}]\sum_{i,j=0}^{\infty} M_i x^i r_{k,j}x^{-j}\\
&= \sum_i M_i r_{k,i-n+k-1}\\
&= \sum_i M_{i+n-k+1} r_{k,i}.
\end{aligned}$$

So, we summarize with the following theorem.

$k \setminus i$	0	1	2	3	4	5	6	7
0	0							
1	1							
2	0	1						
3	0	−1	1					
4	1	0	−2	1				
5	0	2	1	−3	1			
6	0	−2	3	3	−4	1		
7	1	0	−6	3	6	−5	1	
8	0	3	3	−12	1	10	−6	1

Table 1: The values of $r_{k,i}$

Theorem 2.1. *For $k \geq 1$, the number of standard Young tableaux of skew shape $\lambda/(k-1)$ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-k+1$ can be written as the linear combination of Motzkin numbers*

$$\sum_i r_{k,i} M_{i+n-k+1},$$

where the number M_n is the nth Motzkin number and the coefficients $r_{k,i}$ are defined by

$$\sum_{k=1}^{\infty} r_k(x) y^k = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} r_{k,i} x^i y^k = \frac{y}{(1-y)\left(1+(1-x)y+y^2\right)}.$$

Table 1 shows the coefficients $r_{k,i}$ for k from 1 to 8.

2.1 Generalization of Theorem 2.1

Now we want to consider a generalization of Theorem 2.1; for a partition μ having at most three rows, we want to express the number of SYTs of skew shape λ/μ over all partitions λ of n having at most three parts, filled with the numbers $1, 2, \ldots, n-|\mu|$, as a linear combination of Motzkin numbers. It is enough to consider partitions μ having at most two rows.

First let us compute $\Psi(hf_k)$ for $k > 0$. By equation (9) we have

$$\begin{aligned}
\Psi(hf_k) &= L\Big(\theta(h\big(g_0 + 2(g_1 + \cdots + g_{k-1}) + g_k\big)\Big) \\
&= \frac{1}{\sqrt{1-2x-3x^2}}\Big(1 + 2\big(xm(x) + \cdots + x^{k-1}m(x)^{k-1}\big) + x^k m(x)^k\Big) \\
&= \frac{1}{\sqrt{1-2x-3x^2}} \frac{1 - x^{k+1}m(x)^{k+1} + xm(x) - x^k m(x)^k}{1 - xm(x)} \\
&= \frac{m(x)}{1 - x^2 m(x)^2} \frac{\big(1 + xm(x)\big)\big(1 - x^k m(x)^k\big)}{1 - xm(x)} \\
&= \frac{m(x)}{\big(1 - xm(x)\big)^2}\big(1 - x^k m(x)^k\big). \qquad (15)
\end{aligned}$$

Let $\beta_k := \Psi(hf_k)$ for $k > 0$. Then by (15), we can compute the generating function $B(y)$ for β_k as the following:

$$\begin{aligned}
B(y) &= \sum_{k=1}^{\infty} \beta_k y^k = \sum_{k=1}^{\infty} \frac{m(x)}{\big(1 - xm(x)\big)^2} \big(1 - x^k m(x)^k\big) y^k \\
&= \frac{m(x)}{\big(1 - xm(x)\big)^2} \Big(\frac{1}{1-y} - \frac{1}{1 - xm(x)y}\Big) \\
&= \frac{-y(x^2 - x^2 y)m(x)}{(1-y)(1-3x)(x + xy - y + xy^2)} + \frac{y(x - y + 2xy)}{(1-y)(1-3x)(x + xy - y + xy^2)}, \qquad (16)
\end{aligned}$$

where the last equation follows by rationalizing the denominator. Then the generating function for $\Psi(hf_j) + \Psi(hf_k) - \Psi(hf_{j+k})$ is

$$\begin{aligned}
&\sum_{j,k=1}^{\infty} \big(\Psi(hf_j) + \Psi(hf_k) - \Psi(hf_{j+k})\big) y^j z^k \\
&\qquad = \sum_{j,k=1}^{\infty} \big(\beta_j + \beta_k - \beta_{j+k}\big) y^j z^k \\
&\qquad = \frac{z}{1-z} B(y) + \frac{y}{1-y} B(z) - \sum_{j,k=1}^{\infty} \beta_{j+k} y^j z^k \\
&\qquad = \frac{z}{1-z} B(y) + \frac{y}{1-y} B(z) - \sum_{n=1}^{\infty} \beta_n \frac{y^n z - y z^n}{y - z} \\
&\qquad = \frac{z}{1-z} B(y) + \frac{y}{1-y} B(z) - \frac{1}{y-z} \Big(zB(y) - yB(z)\Big). \qquad (17)
\end{aligned}$$

So, plugging equation (16) into (17) gives the generating function for $\Psi(hf_j) + \Psi(hf_k) - \Psi(hf_{j+k})$, which is equal to

$$\begin{aligned}
&\frac{yz(1 - yz)m(x)}{(1-y)\big(1 + (1 - 1/x)y + y^2\big)(1-z)\big(1 + (1 - 1/x)z + z^2\big)} \\
&\qquad - \frac{yz(xyz + xy + xz - yz)}{(1-y)\,(x + xy - y + xy^2)\,(1-z)\,(x + xz - z + xz^2)}. \qquad (18)
\end{aligned}$$

Let

$$T(x, y, z) := \frac{yz(1 - yz)}{(1-y)\big(1 + (1 - 1/x)y + y^2\big)(1-z)\big(1 + (1 - 1/x)z + z^2\big)}, \qquad (19)$$

and let

$$S(x, y) := \frac{y}{(1-y)\big(1 + (1-x)y + y^2\big)}.$$

Then by equation (14) we have

$$S(x, y) = \sum_{k=1}^{\infty} r_k(x) y^k.$$

We know that when expanded in powers of y, the second term in (18) has only negative powers of x. However the left side of equation in (17) has no negative powers of x. So the first term in (18) must be cancelled with negative powers of x from the second term in (18). Also, the expression (19) has only negative powers of x. Then we simplify the function

$$\begin{aligned}T(1/x,y,z) &= \frac{yz(1-yz)}{(1-y)\big(1+(1-x)y+y^2\big)(1-z)\big(1+(1-x)z+z^2\big)}\\ &= yz(1-yz)S(x,y)S(x,z)\\ &= yz(1-yz)\sum_{j=1}^{\infty} r_j(x)y^j \sum_{k=1}^{\infty} r_k(x)z^k\\ &= \sum_{j,k=1}^{\infty} \left(r_j(x)r_k(x) - r_{j-1}(x)r_{k-1}(x)\right) y^j z^k, \end{aligned} \tag{20}$$

where $r_0(x)=0$.

Define the polynomial $r_{j,k}(x)$ by

$$T(1/x,y,z) = \sum_{j,k} r_{j,k}(x) y^j z^k. \tag{21}$$

Then by (20) and (21) we have

$$r_{j,k}(x) = r_j(x)r_k(x) - r_{j-1}(x)r_{k-1}(x).$$

Next, let us compute the coefficient of $x^{n-(|\mu_1|+|\mu_2|)}y^j z^k$ in the expression $T(x,y,z)m(x)$. By (21), we have

$$\begin{aligned}[x^{n-(|\mu_1|+|\mu_2|)}y^j z^k]\, T(x,y,z)m(x) &= [x^{n-(|\mu_1|+|\mu_2|)}]\, r_{j,k}\left(\frac{1}{x}\right) m(x)\\ &= [x^{n-(|\mu_1|+|\mu_2|)}] \sum_{i=0}^{\infty} M_i x^i r_{j,k}\left(\frac{1}{x}\right)\\ &= [x^{n-(|\mu_1|+|\mu_2|)}] \sum_{i,l=0}^{\infty} M_i x^i r_{i,j,k} x^{-l}\\ &= \sum_i M_i r_{i-n+(|\mu_1|+|\mu_2|),j,k}\\ &= \sum_i M_{i+n-(|\mu_1|+|\mu_2|)} r_{i,j,k}.\end{aligned}$$

So, we summarize with the following theorem.

Theorem 2.2. *For positive integers μ_1 and μ_2 with $\mu_1 \geq \mu_2$, the number of standard Young tableaux of skew shape $\lambda/(\mu_1,\mu_2)$ over all partitions λ of n having at most three parts, filled with the numbers $1,2,\ldots,n-(\mu_1+\mu_2)$, can be written as the linear combination of Motzkin numbers*

$$\sum_i M_{i+n-(|\mu_1|+|\mu_2|)} r_{i,j,k}$$

where the number M_n *is the* n*th Motzkin number,* $j = \mu_2 + 1$*,* $k = \mu_1 - \mu_2 + 1$*, and* $r_{i,j,k}$ *is the coefficient of* x^i *in* $r_{j,k}(x) = r_j(x)r_k(x) - r_{j-1}(x)r_{k-1}(x)$ *defined in (21).*

2.2 Focus on the polynomial $r_n(x)$

Now we study the polynomial $r_n(x)$. From (14), we have

$$\sum_{n=0}^{\infty} r_n(x)y^n = \frac{y}{(1-y)\left(1+(1-x)y+y^2\right)}. \tag{22}$$

Define polynomials $q_n(x)$ by

$$\sum_{n=0}^{\infty} q_n(x)z^n = \frac{1}{1-xz+z^2}. \tag{23}$$

Then $q_n(x) = U_n(x/2)$ where $U_n(x)$ for all $n \geq 0$ is the Chebyshev polynomial of the second kind (A093614), which can be defined by the generating function

$$\frac{1}{1-2xz+z^2} = \sum_{n=0}^{\infty} U_n(x)z^n.$$

Now we are trying to find a formula for $r_n(x)$ in terms of the Chebyshev polynomial of the second kind. Then by (22) we have

$$\sum_{n=0}^{\infty} r_n(x+1)y^n = \frac{y}{(1-y)(1-xy+y^2)}. \tag{24}$$

So, from (23) and (24), we deduce

$$r_n(x+1) = \sum_{i=0}^{n-1} U_i(x/2).$$

Then we can rewrite (24) as the sum of an even function and an odd function of x:

$$\frac{y}{(1-y)(1-xy+y^2)} = \frac{y(1+y^2)}{(1-y)(1-xy+y^2)(1+xy+y^2)} + \frac{xy^2}{(1-y)(1-xy+y^2)(1+xy+y^2)}.$$

Let

$$P_e(x,y) := \frac{y(1+y^2)}{(1-y)(1-xy+y^2)(1+xy+y^2)}$$

and

$$P_o(x,y) := \frac{xy^2}{(1-y)(1-xy+y^2)(1+xy+y^2)}.$$

Then the following lemma shows that we can express $P_e(x,y)$ and $P_o(x,y)$ as the Chebyshev polynomial of the second kind.

Lemma 2.2. *The even function $P_e(x,y)$ and the odd function $P_o(x,y)$ in $y/(1-y)(1-xy+y^2)$ can be expressed as the Chebyshev polynomial of the second kind:*

$$P_e(x,y) = y(1+y)\sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right)^2 y^{2n}, \tag{25}$$

and

$$P_o(x,y) = (1+y)\sum_{n=1}^{\infty} U_n\left(\frac{x}{2}\right) U_{n-1}\left(\frac{x}{2}\right) y^{2n}. \tag{26}$$

Proof. Let $i = \sqrt{-1}$. Substituting iy^2 for x and $-xi$ for a and b in equation (7) in [10] gives us that

$$\begin{aligned}\frac{P_e(x,y)}{y(1+y)} &= \frac{1}{1-xy^2+y^4} * \frac{1}{1-xy^2+y^4}\\ &= \sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right) y^{2n} * \sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right) y^{2n}\\ &= \sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right)^2 y^{2n},\end{aligned}$$

where $*$ denotes the Hadamard product in y. So we have equation (25).

Similarly, the function $P_o(x,y)/(1+y)$ can be written as

$$\begin{aligned}\frac{P_o(x,y)}{(1+y)} &= \frac{1}{1-xy^2+y^4} * \frac{y^2}{1-xy^2+y^4}\\ &= \sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right) y^{2n} * \sum_{n=0}^{\infty} U_n\left(\frac{x}{2}\right) y^{2n+2}\\ &= \sum_{n=1}^{\infty} U_n\left(\frac{x}{2}\right) U_{n-1}\left(\frac{x}{2}\right) y^{2n},\end{aligned}$$

where $*$ denotes the Hadamard product in y. □

By Lemma 2.2 and $q_n(x) = U_n(x/2)$, the even function P_e of x satisfies that

$$\begin{aligned}P_e(x,y) &= y(1+y)\sum_{n=0}^{\infty} q_n(x)^2 y^{2n}\\ &= q_0(x)^2 y + q_0(x)^2 y^2 + q_1(x)^2 y^3 + q_1(x)^2 y^4 + \cdots\\ &= \sum_{n=1}^{\infty} q_{\lfloor (n-1)/2\rfloor}(x)^2 y^n,\end{aligned}$$

where $q_n(x)$ is the polynomial in (23) for all $n \geq 0$. Also the odd function can be expressed as

$$P_o(x,y) = \sum_{n=2}^{\infty} q_{\lfloor n/2\rfloor}(x) q_{\lfloor n/2\rfloor -1}(x) y^n.$$

So the expression (24) can be written as

$$\frac{y}{(1-y)(1-xy+y^2)} = q_0(x)^2 y + \sum_{n=1}^{\infty} \big(q_{\lfloor (n-1)/2 \rfloor}(x)^2 + q_{\lfloor n/2 \rfloor}(x) q_{\lfloor n/2 \rfloor - 1}(x)\big) y^n. \tag{27}$$

Therefore, equating coefficients in (27) gives $r_1(x+1) = q_0(x)^2$ and

$$r_n(x+1) = q_{\lfloor (n-1)/2 \rfloor}(x)^2 + q_{\lfloor n/2 \rfloor}(x) q_{\lfloor n/2 \rfloor - 1}(x), \text{ when } n \geq 2.$$

That is, for all $m > 0$,

$$r_{2m}(x+1) = q_{m-1}(x)\big(q_{m-1}(x) + q_m(x)\big),$$

and

$$r_{2m+1}(x+1) = q_m(x)\big(q_{m-1}(x) + q_m(x)\big).$$

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