Hamiltonian dynamics for Einstein's action in G$\rightarrow$0 limit

Alberto Escalante

Instituto de Física Luis Rivera Terrazas, Benemérita Universidad Autónoma de Puebla, (IFUAP).

Apartado postal J-48 72570 Puebla. Pue., México

LUTh, Observatoire de Paris, Meudon, France,

(aescalan@sirio.ifuap.buap.mx, alberto.escalante@obspm.fr)

Abstract

The Hamiltonian analysis for the Einstein's action in $G \rightarrow 0$ limit is performed. Considering the original configuration space without involve the usual ADM variables we show that the version $G \rightarrow 0$ for Einstein's action is devoid of physical degrees of freedom. In addition, we will identify the relevant symmetries of the theory such as the extended action, the extended Hamiltonian, the gauge transformations and the algebra of the constraints. As complement part of this work, we develop the covariant canonical formalism where will be constructed a closed and gauge invariant symplectic form. In particular, using the geometric form we will obtain by means of other way the same symmetries that we found using the Hamiltonian analysis.

I. INTRODUCTION

Hamiltonian analysis for Einstein's theory of gravity has been great topic of study in the last years. As we know, the history begins with the work reported by Arnowitt-Deser-Misner (ADM) where the $3+1$ split of the space time allows us to study the Hamiltonian dynamics, the constraints and the symmetries of general relativity theory. In the ADM work, the fundamental variables to preform the Hamiltonian analysis are considered the 3-metric and its respectively conjugate momenta [1]. However, when we try to make progress in the quantization of the theory this program presents

difficulties, because the no linearly of the gravitational field is manifested in the constraints. In this manner, at quantum level to work with these variables (ADM variables) presents several problems. In the 80's, the panorama becomes to be clarified thanks to the greats works developed by Ashtekar introducing a kind of new variables for studying the Hamiltonian dynamics for the gravitational field [2, 3, 4]. The use of these new variables leads to a important simplification of the equations of the theory. In this program, both the constraints and the evolution equations of the canonical general relativity become simple polinomials of the field variables. Nevertheless, the price to pay for these simplifications is that the Astekar's variables are complex, and therefore Ashtekar canonical formulation describes complex general relativity. In order to obtain the real physical degrees of freedom one needs to append a posteriori appropiate reality conditions [5, 6]. After the Asthekar's works, the study of canonical gravity in its classical or quantum form has been of great interest in the literature [7, 8, 9, 10, 11, 12, 13], especially in the loop quantum gravity context [14, 15].
On the other hand, in recently works has been proposed to study using the Ashtekar formulation the $G \to 0$ limit of Euclidean or complexified general relativity, where the quantization of the theory in the loop representation is obtained and infinite dimensional space of exact solutions to the constraints are found [16]. The study of Einstein's theory in this limit becomes to be relevant because we could make progress to study a different approach to perturbation theory at quantum level. As we know, the standard way for studying this important part in gravity is making the perturbation around a classical background metric, but in the process the relevant symmetries of Einstein's theory are lost, namely the background independence and diffeomorphisms. However, the model reported in [16] marks a big difference respect to the standard treatment because in the limit the symmetries of general relativity are not lost. Thus, we could have now a new starting point to analyze in the mentioned limit a full diffeomorphism invariant and background independent theory.
On the other side, in this same context we find in [17] other different proposal, where setting the $G \to 0$ limit for general relativity written in the first order formalism and under a change of variables, the theory becomes to be a copy of abelian BF topological field theory. Furthermore, using a kind of (ADM) variables the Hamiltonian analysis for the theory is performed, allows us to find a connection with parametrized field theory [17, 18]. It is important to observe that the models purposed in [16] and [17] are quite different. In the first one model, the Astekar's variables has been

used and the relevant results reported are that Euclidean general relativity in the $G \to 0$ limit is not a free theory because the model has two degrees of freedom. In the second one model, we find that in $G \to 0$ limit general relativity expressed in the first oder formalism becomes to be a free field theory.

With all these antecedents, the purpose of this paper is to report the Hamiltonian analysis for the model presented in [17] without involve the ADM variables. The reason to do this is simple, we wish to report the symmetries and the constraints of the theory from other point of view. This is, in this work we report the Dirac's analysis using only the dynamical variables implicated in the action. In this way, we are showing that is possible to obtain the same physical information for the theory without resort to ADM variables. We finish our analysis developing the covariant canonical formalism for the theory under study, and we obtain by means of a different way the symmetries found using the Hamiltonian method. Therefore, in this work we are establishing the bases to quantize the theory in forthcoming works.

The paper is organized as follows. In Section II, we present a pure Dirac analysis for general relativity in $G \to 0$ limit. As important part that we will find in this section are the extended action, the extended Hamiltonian and the identification of the first and second class constraints. In addition, with the complete classification of the constraints we carry out the counting of the physical degrees of freedom and we present the Dirac bracket for the theory. In Section II.I, using Catellani's algorithm we will find the gauge symmetries for the theory. In particular we we prove that the theory under study is invariant under diffeomorphisms. In Section III, using basic concepts of symplectic geometry we construct a closed and gauge invariant symplectic form on the covariant phase space, which turns represent a complete covariant canonical description of the theory. Using the present geometric form, we reproduce the results found with the Hamiltonian method. In Section IV, we give some conclusions and prospects .

II. Hamiltonian analysis

As we know, the Einstein's action for gravity written in the first order formalism is expressed by

[14, 16]

$$S[e,\omega] = \frac{1}{4}\int_M \epsilon^{IJKL} e_I \wedge e_J \wedge R_{KL}[\omega], \tag{1}$$

where $e^I = e^I{}_\mu dx^\mu$ is the one-form tetrad field, $R^{IJ}[\omega] = \frac{1}{2} R^{IJ}{}_{\mu\nu} dx^\mu \wedge dx^\nu$ is the curvature of the $SO(3,1)$ 1-form connection $\omega_\nu{}^{IJ}$ with $R^{IJ}{}_{\mu\nu} = \partial_\mu \omega_\nu{}^{IJ} - \partial_\nu \omega_\mu{}^{IJ} + G(\omega_\mu{}^{IK}\omega_{\nu K}{}^J - \omega_\nu{}^{IK}\omega_{\mu K}{}^J)$. Here, G is the gravitational coupling constant, ϵ^{IJKL} is the completely antisymmetric object with $\epsilon^{0123} = 1$, $\mu,\nu = 0,1,..,3$ are spacetime indices, x^μ are the coordinates that label the points fo the 4-dimensional manifold M and $I, J = 0, 1.., 3$ are internal indices that can be raised and lowered by the internal Lorentzian metric $\eta_{IJ} = (-1, 1, 1, 1)$.

Setting the $G \to 0$ limit , the above action becomes to be

$$S[e,\omega] = \frac{1}{8}\int_M \epsilon^{\alpha\beta\mu\nu}\epsilon^{IJKL} e_{I\alpha} e_{J\beta}(\partial_\mu \omega_\nu{}^{IJ} - \partial_\nu \omega_\mu{}^{IJ}) dx^4. \tag{2}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the volume 4-form. Calculating the variation of the action (2) we find the next equations of motion

$$\epsilon^{\alpha\beta\mu\nu}\partial_{[\mu} e_{\nu]I} = 0, \tag{3}$$

and

$$\epsilon^{\alpha\beta\mu\nu}\partial_\mu B_{I\alpha\beta} = 0, \tag{4}$$

here, the two-forms $B^I{}_{\alpha\beta}$ are defined by $B^I{}_{\alpha\beta} = -\frac{1}{2}\epsilon^{IJKL} e_{[\alpha J}\omega_{\beta]KL}$, provided that the tetrad is non-degenerate, B^I has inverse $\omega_{\alpha IJ} = \frac{1}{2}\epsilon_{IJKL} e^{\beta K}\left(B^L{}_{\alpha\beta} - \frac{1}{2} e^{\gamma L} e_{\alpha N} B^N{}_{\beta\gamma}\right)$. We can see that equation (3) implies that $e_{\alpha I} = \partial_\alpha f_I$, so $g_{\mu\nu} = \eta_{IJ}\partial_\mu f^I \partial_\nu f^J$. Which corresponds to (locally) Minkowski spacetime [17].

With all these preliminar results, using the variable B and integrating by parts we can rewrite the action (2) in the next form

$$S[B,e] = \frac{1}{2}\int_M \epsilon^{\alpha\beta\mu\nu} B^I{}_{\alpha\beta}(\partial_\mu e_{\nu I} - \partial_\nu e_{\mu I}) dx^4. \tag{5}$$

Thus, we can obtain from (5) the same equations of motion given in (3) and (4) considering to B and e as our new dynamical variables. It is remarkable to note that the action (1) which has an $SO(3,1)$ connection $\omega_\nu{}^{IJ}$, in the $G \to 0$ limit (2) becomes to be a collection of six $U(1)$ connections and the tetrad field $e^I{}_\mu$ is a collection of four gauge invariant vector fields, we will prove this point performing the Hamiltonian analysis in the next lines.

The starting point of this work is the action (5), but to difference of the paper reported in [17] we will not involve a kind of ADM variables for performing the Hamiltonian analysis, in spite of in the canonical gravity context the standard way for developing the Hamiltonian dynamics is using these variables. The reason to do this is because in this work we aim to report the Dirac's method working with the full configuration space, this is, we will develop the Dirac analysis using only the configuration variables involved in the action (5), namely B, e. In this way, we can know the constrains in his complete form without fix any gauge, the symmetries, the extended action and the extended Hamiltonian for the theory. Of course, if we wish we can obtain the results reported by Nuno *et. al* [17] as particular case of this paper considering the second class constraints as strong equations. Thus, with this letter we are establishing the basis to quantize the theory described by (5) which will be reported in forthcoming works.

By performing the 3+1 decomposition in the action (5) we find

$$S[B,e] = \int \left[\eta^{abc} B_{Iab} \dot{e}^I{}_c + \frac{1}{2}\eta^{abc} B_{I0a}(\partial_b e^I{}_c - \partial_c e^I{}_b) - (\eta^{abc} B_{Iab})\partial_c e^I{}_0 \right] dx^4, \tag{6}$$

where $\eta^{abc} = \epsilon^{0abc}$, $a, b, c = 1, 2, 3$. From (6), we can identify the Lagrangian density given by

$$\mathcal{L} = \eta^{abc} B_{Iab} \dot{e}^I{}_c + \frac{1}{2}\eta^{abc} B_{I0a}(\partial_b e^I{}_c - \partial_c e^I{}_b) - (\eta^{abc} B_{Iab})\partial_c e^I{}_0. \tag{7}$$

Dirac's method calls for the definition of the momenta $(\Pi_I{}^{\alpha\beta}, \Pi_I{}^{\alpha})$ canonically conjugate to $(B^I{}_{\alpha\beta}, e^I{}_\mu)$ [19]

$$\Pi_I{}^{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta \dot{B}^I{}_{\alpha\beta}}, \qquad \Pi_I{}^{\alpha} = \frac{\delta\mathcal{L}}{\delta \dot{e}^I{}_{\mu}}, \tag{8}$$

on the other hand, the matrix elements of the Hessian

$$\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu B^I{}_{\alpha\beta})\partial(\partial_\mu B^J{}_{\rho\sigma})}, \quad \frac{\partial^2\mathcal{L}}{\partial(\partial_\mu e^I{}_{\alpha})\partial(\partial_\mu B^J{}_{\rho\sigma})}, \quad \frac{\partial^2\mathcal{L}}{\partial(\partial_\mu e^I{}_{\alpha})\partial(\partial_\mu e^J{}_{\beta})}, \tag{9}$$

are identically zero, the rank of the Hessian is zero. Thus, we expect 40 primary constraints. From the definition of the momenta (8) we identify the next 40 primary constraints

$$\begin{aligned}
\phi_I{}^0 &:= \Pi_I{}^0 \approx 0, \\
\phi_I{}^a &:= \Pi_I{}^a - \eta^{abc} B_{Ibc} \approx 0, \\
\phi_I{}^{0a} &:= \Pi_I{}^{0a} \approx 0, \\
\phi_I{}^{ab} &:= \Pi_I{}^{ab} \approx 0.
\end{aligned} \tag{10}$$

The canonical Hamiltonian density for this system has the next form

$$\begin{aligned} \mathcal{H}_c &= \dot{e}^{\mu}{}_I \Pi_I{}^{\mu} + \dot{B}^I{}_{0a}\Pi_I{}^{0a} + \dot{B}^I{}_{ab}\Pi_I{}^{ab} - \mathcal{L} \\ &= -\frac{1}{2}\eta^{abc}B_{I0a}(\partial_b e^I{}_c - \partial_c e^I{}_b) + \partial_a e^I{}_0 \Pi_I{}^a. \end{aligned} \tag{11}$$

Integrating by parts and neglecting boundary terms at infinity, the canonical Hamiltonian becomes

$$H_c = \int dx^3 \left[-\frac{1}{2}\eta^{abc}B_{I0a}(\partial_b e^I{}_c - \partial_c e^I{}_b) - \partial_a \Pi_I{}^a e^I{}_0 \right]. \tag{12}$$

Following with the method, adding to H_c the 40 primary constraints (10) we identify the primary Hamiltonian

$$H_P = H_c + \int dx^3 \left[\lambda^I{}_0 \phi_I{}^0 + \lambda^I{}_a \phi_I{}^a + \lambda^I{}_{0a}\phi_I{}^{0a} + \lambda^I{}_{ab}\phi_I{}^{ab} \right], \tag{13}$$

where $\lambda^I{}_0, \lambda^I{}_a, \lambda^I{}_{0a}, \lambda^I{}_{ab}$ are Lagrange multipliers enforcing the constraints. For this theory, the non-vanishing fundamental Poisson brackets are given by

$$\begin{aligned} \{e^I{}_\alpha(x), \Pi_J{}^\mu(y)\} &= \delta^\mu_\alpha \delta^I_J \delta^3(x-y), \\ \{B^I{}_{\mu\nu}(x), \Pi_J{}^{\alpha\beta}(y)\} &= \frac{1}{2}\delta^I_J \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu\right)\delta^3(x-y). \end{aligned} \tag{14}$$

The 40×40 matrix whose entries are the Posson brackets among the constraints (10) given by

$$\begin{aligned} \{\phi_I{}^0(x), \phi_J{}^0(y)\} &= 0, \qquad \{\phi_I{}^0(x), \phi_J{}^a(y)\} = 0 \\ \{\phi_I{}^0(x), \phi_I{}^{0a}(y)\} &= 0, \qquad \{\phi_I{}^0(x), \phi_I{}^{ab}(y)\} = 0, \\ \{\phi_I{}^a(x), \phi_J{}^b(y)\} &= 0, \qquad \{\phi_I{}^a(x), \phi_J{}^{0b}(y)\} = 0, \\ \{\phi_I{}^{0a}(x), \phi_J{}^{0b}(y)\} &= 0, \qquad \{\phi_I{}^a(x), \phi_J{}^{cd}(y)\} = -\eta^{acd}\eta_{IJ}\delta^3(x-y) \\ \{\phi_I{}^{0a}(x), \phi_J{}^{cd}(y)\} &= 0, \qquad \{\phi_I{}^{ab}(x), \phi_J{}^{cd}(y)\} = 0 \end{aligned} \tag{15}$$

has rank 24 and 16 linearly independent null-vectors. Thus, the null vectors and consistency conditions yields to the next 16 secondary constraints [19]

$$\begin{aligned} \dot{\phi}_I{}^0 &= \{\phi_I{}^0, \mathcal{H}_P\} \approx 0 \quad \Rightarrow \quad \psi_I := \partial_a \Pi_I{}^a \approx 0, \\ \dot{\phi}_I{}^{0a} &= \{\phi_I{}^{0a}, \mathcal{H}_P\} \approx 0 \quad \Rightarrow \quad \psi_I{}^a := \frac{1}{2}\eta^{abc}(\partial_b e_{Ic} - \partial_c e_{Ib}) \approx 0, \end{aligned} \tag{16}$$

and the next values for the Lagrange multipliers

$$\begin{aligned}
\dot{\phi}_I{}^a &= \{\phi_I{}^a, \mathcal{H}_T\} \approx 0 \quad \Rightarrow \quad \lambda^I{}_{ab} = \frac{1}{2}(\partial_a B^I{}_{0b} - \partial_b B^I{}_{0a}), \\
\dot{\phi}_I{}^{ab} &= \{\phi_I{}^{ab}, \mathcal{H}_T\} \approx 0 \quad \Rightarrow \quad \lambda^I{}_a = 0,
\end{aligned} \tag{17}$$

for the theory under study there are no, third constraints. At this point, we need to separate all the primary and secondary constraints in first and second class constraints. For this step, we need calculate the 56×56 matrix whose entries will be the Poisson brackets between primary and secondary constraints (9) , (14), this is

$$\begin{aligned}
\{\phi_I{}^0(x), \phi_J{}^0(y)\} &= 0, \qquad \{\phi_I{}^0(x), \phi_J{}^a(y)\} = 0, \\
\{\phi_I{}^0(x), \phi_I{}^{0a}(y)\} &= 0, \qquad \{\phi_I{}^0(x), \phi_I{}^{ab}(y)\} = 0, \\
\{\phi_I{}^0(x), \psi_J(y)\} &= 0, \qquad \{\phi_I{}^0(x), \psi_J{}^a(y)\} = 0, \\
\{\phi_I{}^a(x), \phi_J{}^b(y)\} &= 0, \qquad \{\phi_I{}^a(x), \phi_J{}^{0b}(y)\} = 0, \\
\{\phi_I{}^a(x), \psi_J(y)\} &= 0, \qquad \{\phi_I{}^a(x), \phi_J{}^{cd}(y)\} = -\eta^{acd}\eta_{IJ}\delta^3(x-y), \\
\{\phi_I{}^a(x), \psi_J(y)\} &= 0, \qquad \{\phi_I{}^a(x), \psi_J{}^b(y)\} = -\eta^{abc}\eta_{IJ}\partial_c\delta^3(x-y), \\
\{\phi_I{}^{0a}(x), \phi_J{}^{0b}(y)\} &= 0, \qquad \{\phi_I{}^{0a}(x), \phi_J{}^{cd}(y)\} = 0, \\
\{\phi_I{}^{0a}(x), \psi_J(y)\} &= 0, \qquad \{\phi_I{}^{0a}(x), \psi_J{}^b(y)\} = 0, \\
\{\phi_I{}^{ab}(x), \phi_J{}^{cd}(y)\} &= 0, \qquad \{\phi_I{}^{ab}(x), \psi_J(y)\} = 0, \\
\{\phi_I{}^{ab}(x), \psi_J{}^c(y)\} &= 0, \qquad \{\psi_I(x), \psi_J(y)\} = 0, \\
\{\psi_I(x), \psi_J{}^a(y)\} &= 0, \qquad \{\psi_I{}^a(x), \psi_J{}^b(y)\} = 0,
\end{aligned} \tag{18}$$

this matrix has rank 24 and 32 null-vectors. Thus, we expect 24 second class constraints and 32 first class constraints. From the null-vectors we identify the next 32 first class constraints

$$\begin{aligned}
\gamma_I{}^0 &:= \Pi_I{}^0 \approx 0 \\
\gamma_I{}^{0a} &:= \Pi_I{}^{0a} \approx 0, \\
\gamma_I &:= \partial_a \Pi_I{}^a \approx 0, \\
\gamma_I{}^a &:= \frac{1}{2}\eta^{abc}(\partial_b e_{Ic} - \partial_c e_{Ib}) - \partial_b \Pi_I{}^{ab} \approx 0,
\end{aligned} \tag{19}$$

and the rank yields to the next 24 second class constraints

$$
\begin{aligned}
\chi_I{}^{a} &:= \Pi_I{}^{a} - \eta^{abc} B_{Ibc} \approx 0, \\
\chi_I{}^{ab} &:= \Pi_I{}^{ab} \approx 0. \qquad (20)
\end{aligned}
$$

It is important to remark that the constraint $\gamma_I{}^a$ given in (19) is fixed by means of the null vectors (see equation (16)) and become to be a first class constraint. In this way, the method itself allows us to find from the rank and the null vectors of the matrix (18) all the right first and second class constraints for the theory [19]. This is the advantage that we find in Dirac's method when we apply it to the original configuration space, in this case given by $B^I{}_{\alpha\beta}$ and $e^I{}_\alpha$. In general we can apply the analysis presented in this work to every theory. However, the calculation of the rank and the null vectors of the matrixes (15) and (18) usually is not straightforward to perform [19].

Furthermore, the 32 first class constraints given in (19) are not independent because there are 4 reducibility conditions given by $\partial_a \gamma_I{}^a = \partial_a \partial_b \chi_I^{ab} = 0$, this reducibility condition is the equivalent one that we find in the literature in the 4-dimentional BF theories [20] or in topological invariants context [21]. In this manner, the counting of degrees of freedom is a follows. There are 80 canonical variables $(e^I{}_\mu, B^I{}_{\alpha\beta}, \Pi_I{}^\alpha, \Pi_I{}^{\alpha\beta})$, $[32-4]=28$ independent first class constraints $(\gamma_I{}^0, \gamma_I{}^{0a}, \gamma_I, \gamma_I{}^a)$ and 24 independent second class constraints $(\chi_I{}^a, \chi_I{}^{ab})$, thus, we can conclude that theory is devoid of physical degrees of freedom. In others words, the theory defined by the action (5) is only sensitive to external degrees of freedom for example, if we add to (5) matter degrees of freedom the theory will not be topological anymore, just as was claimed in [17]. In addition, the action (5) does not depend explicit of the spacetime metric, so, in this other sense the action becomes to be topological as well [20].

With all these results at hand, we can use the values for the Lagrange multipliers (15), the first class constraints (19), the second class constraints (20) and identify the extended action for the theory expressed by

$$
\begin{aligned}
S_E \quad & \left[e^I{}_\mu, \Pi_I{}^\mu, B^I{}_{\mu\nu}, \Pi_I{}^{\mu\nu}, u_0{}^I, u^I, u_{0a}{}^I, u_a{}^I, v_a{}^I, v^I{}_{ab}\right] = \int \Big\{ \dot{e}^I{}_\mu \Pi_I{}^\mu + \dot{B}^I{}_{0a} \Pi_I{}^{0a} + \dot{B}^I{}_{ab} \Pi_I{}^{ab} \\
- \quad & H - u_0{}^I \gamma_I{}^0 - u^I \gamma_I - u^I{}_a \gamma_I{}^a - u_{0a}{}^I \gamma_I{}^{0a} - v_a{}^I \chi_I{}^a - v^I{}_{ab} \chi_I{}^{ab} \Big\} dx^4, \qquad (21)
\end{aligned}
$$

where H is only combination of first class constraints

$$H = -B^I{}_{0a}\left[\frac{1}{2}\eta^{abc}(\partial_b e_{Ic} - \partial_c e_{Ib}) - \partial_b \Pi_I{}^{ab}\right] - \partial_a \Pi_I{}^a e^I{}_0, \tag{22}$$

and $u_0{}^I, u^I, u_{0a}{}^I, u_a{}^I, v_a{}^I, v^I{}_{ab}$ are the Lagrange multipliers enforcing the first and second class constraints.

From the extended action we can identify the extended Hamiltonian which is given by

$$H_E = H - u_0{}^I \gamma_I{}^0 - u^I \gamma_I - u^I{}_a \gamma_I{}^a - u_{0a}{}^I \gamma_I{}^{0a}. \tag{23}$$

As we know, the equations of motion obtained by means of the extended Hamiltonian in general are quite different with the Euler-Lagrande equations, but the difference is unphysical [19].

In oder to complete our analysis, we can find the equations of motion obtained from the extended action which yields to

$$\begin{aligned}
\delta e^I{}_0 : \dot{\Pi}_I{}^0 &= -\partial_a \Pi_I{}^a, \\
\delta \Pi_I{}^0 : \dot{e}^I{}_0 &= u^I{}_0, \\
\delta e^I{}_a : \dot{\Pi}_I{}^a &= -\frac{1}{2}\eta^{abc}\left(\partial_b B_{I0c} - \partial_c B_{I0b}\right) - \frac{1}{2}\eta^{abc}\left(\partial_b u_{Ia} - \partial_c u_{Ib}\right), \\
\delta \Pi_I{}^a : \dot{e}^I{}_a &= v_a{}^I - \partial_a e^I{}_0 - \partial_a u^I{}_a, \\
\delta B^I{}_{0a} : \dot{\Pi}_I{}^{0a} &= \frac{1}{2}\eta^{abc}(\partial_b e_{Ic} - \partial_c e_{Ib}) - \partial_b \Pi_I{}^{ab}, \\
\delta \Pi_I{}^{0a} : \dot{B}^I{}_{0a} &= u^I{}_{0a}, \\
\delta B^I{}_{ab} : \dot{\Pi}_I{}^{ab} &= \eta^{abc} v_{Ic}, \\
\delta \Pi_I{}^{ab} : \dot{B}^I{}_{ab} &= v^I{}_{ab} + \frac{1}{2}(\partial_b B^I{}_{0b} - \partial_c B^I{}_{0a}) - \frac{1}{2}(\partial_b u^I{}_b - \partial_c u^I{}_a) \\
\delta u_0{}^I : \gamma_I{}^0 &= 0, \\
\delta u_a{}^I : \gamma_I{}^a &= 0, \\
\delta u^I : \gamma_I &= 0, \\
\delta u_{0a} : \gamma_I{}^{0a} &= 0, \\
\delta v_a{}^I : \chi_I{}^a &= 0, \\
\delta v^I{}_{ab} : \chi_I{}^{ab} &= 0.
\end{aligned} \tag{24}$$

On the other hand, we will calculate the constraint algebra which takes the form

$$
\begin{aligned}
\{\gamma_I{}^0(x),\gamma_J{}^0(y)\} &= 0, \qquad \{\gamma_I{}^0(x),\chi_J{}^a(y)\}=0,\\
\{\gamma_I{}^0(x),\gamma_I{}^{0a}(y)\} &= 0, \qquad \{\gamma_I{}^0(x),\chi_I{}^{ab}(y)\}=0,\\
\{\gamma_I{}^0(x),\gamma_J(y)\} &= 0, \qquad \{\gamma_I{}^0(x),\gamma_J{}^a(y)\}=0,\\
\{\chi_I{}^a(x),\gamma_J{}^b(y)\} &= 0, \qquad \{\chi_I{}^a(x),\gamma_J{}^{0b}(y)\}=0,\\
\{\chi_I{}^a(x),\gamma_J(y)\} &= 0, \qquad \{\chi_I{}^a(x),\chi_J{}^{cd}(y)\}=-\eta^{acd}\eta_{IJ}\delta^3(x-y),\\
\{\chi_I{}^a(x),\gamma_J(y)\} &= 0, \qquad \{\chi_I{}^a(x),\gamma_J{}^b(y)\}=0,\\
\{\gamma_I{}^{0a}(x),\gamma_J{}^{0b}(y)\} &= 0, \qquad \{\gamma_I{}^{0a}(x),\gamma_J{}^{cd}(y)\}=0,\\
\{\gamma_I{}^{0a}(x),\gamma_J(y)\} &= 0, \qquad \{\gamma_I{}^{0a}(x),\gamma_J{}^b(y)\}=0,\\
\{\chi_I{}^{ab}(x),\chi_J{}^{cd}(y)\} &= 0, \qquad \{\chi_I{}^{ab}(x),\gamma_J(y)\}=0,\\
\{\chi_I{}^{ab}(x),\gamma_J{}^c(y)\} &= 0, \qquad \{\gamma_I(x),\gamma_J(y)\}=0,\\
\{\gamma_I(x),\gamma_J{}^a(y)\} &= 0, \qquad \{\gamma_I{}^a(x),\gamma_J{}^b(y)\}=0,
\end{aligned}
\tag{25}
$$

where we can see that the constraint algebra is closed.

We will finish this section identify the Dirac bracket for the theory. From the constraint algebra, we can observe that the matrix whose elements are only the Poisson brackets between the second class constraints is given by

$$
C_{\alpha\beta}=\begin{pmatrix} 0 & -\eta^{acd}\eta_{IJ}\delta^3(x-y) \\ \eta^{acd}\eta_{IJ}\delta^3(x-y) & 0 \\ & \end{pmatrix}. \tag{26}
$$

In this manner, we have that the Dirac bracket between two functionals A, B is expressed by

$$
\{A(x),B(y)\}_D=\{A(x),B(y)\}_P+\int dudv\{A(x),\zeta^\alpha(u)\}C^{-1}_{\alpha\beta}(u,v)\{\zeta^\beta(v),B(y)\}, \tag{27}
$$

where $\{A(x),B(y)\}_P$ is the usual Poisson bracket between the functionals A,B, $\zeta^\alpha(u)=(\chi_I{}^a,\chi_I{}^{ab})$ with $C^{-1}_{\alpha\beta}(u,v)$ as the inverse of (26) which has a trivial form. As we know, the Dirac bracket (27) will be useful to make progress in the quantization of the theory.

II.I Gauge generator

Following with the method, in this part we will find the gauge transformations for the theory described by (5). For our purposes, we apply the Castellani's algorithm [22] to construct the gauge generator using the first class constraints (19), this is

$$G = \int_{\Sigma} \left[\partial_0 \varepsilon^I{}_0 \Pi_I{}^0 + \partial_0 \varepsilon^I{}_{0a} \Pi_I{}^{0a} + \varepsilon^I \partial_a \Pi_I{}^a + \varepsilon^I{}_a \left(\frac{1}{2} \eta^{abc} (\partial_b e_{Ic} - \partial_c e_{Ib}) - \partial_b \Pi_I{}^{ab} \right) \right], \tag{28}$$

thus, we find the following gauge transformations on the phase space,

$$\begin{aligned}
\delta_0 e^I{}_0 &= \partial_0 \varepsilon^I{}_0, \\
\delta_0 e^I{}_a &= -\partial_a \varepsilon^I, \\
\delta_0 B^I{}_{0a} &= \partial_0 \varepsilon^I{}_{0a}, \\
\delta_0 B^I{}_{ab} &= -\frac{1}{2}(\partial_a \varepsilon^I{}_b - \partial_b \varepsilon^I{}_a), \\
\delta_0 \Pi_I{}^0 &= 0, \\
\delta_0 \Pi_I{}^a &= -\frac{1}{2}\eta^{abc}(\partial_b \varepsilon_{Ic} - \partial_c \varepsilon_{Ib}), \\
\delta_0 \Pi_I{}^{0a} &= 0, \\
\delta_0 \Pi_I{}^{ab} &= 0.
\end{aligned} \tag{29}$$

In particular, we can choose the parameters to be $\varepsilon^I{}_0 = -\varepsilon^I = -\Lambda^I$, $\varepsilon^I{}_a = -2\varepsilon^I{}_{0a} = \Lambda^I{}_a$ and considering the equations (29) we find

$$\begin{aligned}
e^I{}_\mu &\rightarrow e^I{}_\mu - \partial_\mu \Lambda^I, \\
B^I{}_{\mu\nu} &\rightarrow B^I{}_{\mu\nu} - \frac{1}{2}\left(\partial_\mu \Lambda^I{}_\nu - \partial_\nu \Lambda^I{}_\mu\right),
\end{aligned} \tag{30}$$

where we can see that $e^I{}_\mu$ becomes to be a collection of 4 four gauge invariant vector fields. We can prove by means of easy calculations that the action (5), the equations of motion (3) and (4) are invariant under these gauge transformations. The nature of the gauge transformations and the form of the theory described in (5) which corresponds to BF type, allows us to formulate the next question; What about diffeomorphisms transformations?. Apparently diffeomorphisms symmetry is not present in the theory, but that is not true at all. We can find the answer such as is developed in 2+1 gravity and Chern-Simons theory [22, 24] introducing a new set of gauge parameters

$$
\begin{aligned}
\Lambda^I &= -\xi^\rho e^I{}_\rho, \\
\Lambda^I{}_\mu &= -2\xi^\rho B^I{}_{\rho\mu},
\end{aligned} \tag{31}
$$

obtaining

$$
\begin{aligned}
e^I{}_\mu &\rightarrow e^I{}_\mu + \mathcal{L}_\xi e^I{}_\mu + \xi^\rho \left[\partial_\mu e^I{}_\rho - \partial_\rho e^I{}_\mu\right], \\
B^I{}_{\mu\nu} &\rightarrow B^I{}_{\mu\nu} + \mathcal{L}_\xi B^I{}_{\mu\nu} + \xi^\rho \left[\partial_\mu B^I{}_{\rho\nu} - \partial_\nu B^I{}_{\rho\mu} - \partial_\rho B^I{}_{\mu\nu}\right].
\end{aligned} \tag{32}
$$

Therefore, diffeomorphisms corresponds to an internal symmetries of the theory just as complete general relativity theory.

As conclusion for this section, we can see that it is possible to obtain all the physical information reported in [17] without resort to ADM variables. Of course, we can obtain the results obtained in [17] considering the second class constraints given in (20) as strong equations. However, the spirit of this paper is make progress for futures works where we will investigate the advantage at quantum level between the ADM formulation and the formulation presented in this work.

III Covariant canonical formalism

In order to extend our analysis, in this section we will perform the covariant canonical formalism for the theory described by the action (5). In particular with this method we will establish the necessary elements for study the quantization aspects of the theory in future works, where we will use the symplectic method or the Hamiltonian method developed above. As important results reported in this section, we will find by other way the symmetries found using the Hamiltonian method.

We start calculating the variation of the action, obtaining

$$
\delta S[B,e] = \int_M dx^4 \left[\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}(\partial_\mu e_{\nu I} - \partial_\nu e_{\mu I})\delta B^I{}_{\alpha\beta} - \epsilon^{\alpha\beta\mu\nu}\partial_\mu B^I{}_{\alpha\beta}\delta e^I{}_\nu + \partial_\mu(\epsilon^{\alpha\beta\mu\nu} B_{I\alpha\beta}\delta e^I{}_\nu)\right], \tag{33}
$$

where we can identify the equations of motion (3), (4) and we identify from the pure divergence term the symplectic potential for the theory [23]

$$\Psi^{\mu} = \epsilon^{\mu\nu\alpha\beta} B_{I\alpha\beta} \delta e^{I}{}_{\nu}, \tag{34}$$

which does not contribute locally to the dynamics, but generates the symplectic form on the phase space.

From the equations of motion (3) and (4) we define the fundamental concept in the studio of the covariant canonical formalism of the theory: the covariant phase space for the theory described by (5) is the space space of solutions of Eqs (3), (4), and we will call it Z.

As we known, we can obtain the integral kernel of the geometric structure for the theory by means of the variation (exterior derivative on Z see [23]) of the symplectic potential (34), this is

$$\omega = \int_{\Sigma} J^{\mu} d\Sigma_{\mu} = \int_{\Sigma} \delta \Psi^{\mu} d\Sigma_{\mu} = \int_{\Sigma} \epsilon^{\mu\nu\alpha\beta} \delta B_{I\alpha\beta} \wedge \delta e^{I}{}_{\nu} d\Sigma_{\mu}. \tag{35}$$

where Σ is a Cauchy hypersurface.

In addition, we will prove that our symplectic form is closed and gauge invariant. Moreover, the integral kernel of the geometric form J^{μ} is conserved ($\partial_{\mu} J^{\mu} = 0$), which guarantees that ω is independent of Σ.

To prove that J^{μ} defined in (35) is conserved we need calculate the linearized equations of motion. For this, we replace in (3), (4) $e^{I}{}_{\nu} \rightarrow e^{I}{}_{\nu} + \delta e^{I}{}_{\nu}$ and $B_{I\alpha\beta} \rightarrow B_{I\alpha\beta} + \delta B_{I\alpha\beta}$, keeping to first order in δ we find the linearized equations given by

$$\begin{aligned} \epsilon^{\alpha\beta\mu\nu} \partial_{[\mu} \delta e_{\nu]I} &= 0, \\ \epsilon^{\alpha\beta\mu\nu} \partial_{\mu} \delta B_{I\alpha\beta} &= 0. \end{aligned} \tag{36}$$

In this manner, using the linearized equations we have

$$\partial_{\mu} J^{\mu} = \partial_{\mu} \delta \Psi^{\mu} = \epsilon^{\mu\nu\alpha\beta} \partial_{\mu} \delta B_{I\alpha\beta} \wedge \delta e^{I}{}_{\nu} + \epsilon^{\mu\nu\alpha\beta} \delta B_{I\alpha\beta} \wedge \partial_{[\mu} \delta e^{I}{}_{\nu]} = 0, \tag{37}$$

showing that ω is independent of Σ.

On the other hand, we need to remember that the closeness of ω in this covariant canonical formalism is equivalent one to the Jacobi identity that Poisson brackets satisfy, in the usual Hamiltonian scheme. To prove the closeness of ω, we can observe that $\delta^{2} e^{I}{}_{\nu} = 0$, $\delta^{2} B_{I\alpha\beta} = 0$ because $e^{I}{}_{\nu}$ and $B_{I\alpha\beta}$ are independent 0-forms on the covariant phase space Z and δ is nilpotent, so using this fact in ω we find

$$\delta\omega = \int_\Sigma \delta^2\Psi^\mu d\Sigma_\mu = \int_\Sigma \left[\epsilon^{\mu\nu\alpha\beta}\delta^2 B_{I\alpha\beta}\wedge\delta e^I{}_\nu - \epsilon^{\mu\nu\alpha\beta}\delta B_{I\alpha\beta}\wedge\delta^2 e^I{}_\nu\right] d\Sigma_\mu = 0, \tag{38}$$

this prove that ω is closed.

What about the gauge transformations found above?. For this aim, we consider that upon picking Σ to be the standard initial value surface $t=0$, (35) takes the standard form

$$\omega = \int_\Sigma \delta\Pi_I{}^a\wedge\delta e^I{}_a, \tag{39}$$

where $\Pi_I{}^a \equiv \eta^{abc}B_{Ibc}$.

For two 0-forms f, g defined on Z, the Hamiltonian vector field defined by the symplectic structure (39) is given by [25]

$$X_f = \int_\Sigma \frac{\delta f}{\delta\Pi_I{}^a}\frac{\delta}{\delta e^I{}_a} - \frac{\delta f}{\delta e^I{}_a}\frac{\delta}{\delta\Pi_I{}^a}, \tag{40}$$

and the Poisson bracket $\{f,g\} := -X_f(g)$ is given by

$$\{f,g\} = \int_\Sigma \frac{\delta f}{\delta e^I{}_a}\frac{\delta g}{\delta\Pi_I{}^a} - \frac{\delta f}{\delta\Pi_I{}^a}\frac{\delta g}{\delta e^I{}_a}. \tag{41}$$

On the other hand, we rewrite the first class constraints found in (19) with the test fields $D^I, D^I{}_a, C^I$ and $C^I{}_a$ on Σ in the next form

$$\begin{aligned}
\gamma_I{}^0[D^I] &:= \int_\Sigma D^I\left(\Pi_I{}^0\right),\\
\gamma_I{}^{0a}[D^I{}_a] &:= \int_\Sigma D^I\left(A^I{}_a\Pi_I{}^{0a}\right),\\
\gamma_I[C^I] &:= \int_\Sigma C^I\left(\partial_a\Pi_I{}^a\right),\\
\gamma_I{}^a[C^I{}_a] &:= \int_\Sigma C^I{}_a\left(\frac{1}{2}\eta^{abc}(\partial_b e_{Ic} - \partial_c e_{Ib}) - \partial_b\Pi_I{}^{ab}\right).
\end{aligned} \tag{42}$$

By inspection, the functional derivatives different to zero are given by

$$\begin{aligned}
\frac{\delta\gamma_I[C^I]}{\delta\Pi_I{}^a} &= -\partial_a C^I, \qquad \frac{\delta\gamma_I[C^I]}{\delta e^I{}_a} = 0,\\
\frac{\delta\gamma_I{}^a[C^I{}_a]}{\delta\Pi_I{}^a} &= 0, \qquad \frac{\delta\gamma_I{}^a[C^I{}_a]}{\delta e^I{}_a} = \frac{1}{2}\eta^{abc}\left(\partial_b C_{Ic} - \partial_c C_{Ib}\right).
\end{aligned} \tag{43}$$

Thus, the motion on Z generated by $\gamma_I[C^I]$ is given by

$$\begin{aligned} e^I{}_a &\mapsto e^I{}_a - \epsilon\partial_a C^I + O(\epsilon^2) \\ \Pi_I{}^a &\mapsto \Pi_I{}^a, \end{aligned} \tag{44}$$

and the motion on Z generated by $\gamma_I{}^a[C^I{}_a]$ is given by

$$\begin{aligned} e^I{}_a &\mapsto e^I{}_a \\ \Pi_I{}^a &\mapsto \Pi_I{}^a - \epsilon\frac{1}{2}\eta^{abc}\left(\partial_b C_{Ic} - \partial_c C_{Ib}\right) + O(\epsilon^2). \end{aligned} \tag{45}$$

where ϵ is an infinitesimal parameter [25]. We can see that the gauge transformation (44) and (45) corresponds to those found using Dirac's method (see eq. (30)).

Now, we will show that ω has not components tangent to the gauge directions, which are specified by equation (30) or (44) and (45).

$$\begin{aligned} \delta e'^I{}_\mu &= \delta e^I{}_\mu - \partial_\mu\Lambda^I, \\ \delta B'^I{}_{\mu\nu} &= \delta B^I{}_{\mu\nu} - \frac{1}{2}\left(\partial_\mu\Lambda^I{}_\nu - \partial_\nu\Lambda^I{}_\mu\right), \end{aligned} \tag{46}$$

where in this context Λ^I , $\Lambda^I{}_\mu$ corresponds to be 1-forms on Z. Using this fact, we find that ω will undergo the transformation as

$$\omega' = \int_\Sigma \epsilon^{\mu\nu\alpha\beta}\delta B'_{I\alpha\beta}\wedge\delta e'^I{}_\nu d\Sigma_\mu = \omega - \int_\Sigma \partial_\nu\left[\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\alpha\Lambda_{I\beta} - \partial_\beta\Lambda_{I\alpha}\right)\wedge\Lambda^I\right]d\Sigma_\mu, \tag{47}$$

where the equations (36) has been used, thus, for fields with compact support ω is a gauge invariant geometric form.

Therefore, as a conclusion of this section, we have constructed a closed and gauge invariant symplectic form on Z which in turns represent a complete Hamiltonian description of the covariant phase space for the theory and will allow us to analyze the quantum treatment in forthcoming works.

V. Conclusions and prospects

In this paper, Dirac and the symplectic methods for the Einstein's action in the $G \to 0$ limit has been performed. Within the Dirac's method we developed the analysis working with the complete configuration space and without involve the typical ADM variables as is reported in [17]. As important results obtained using the Hamiltonian method, were the identification of the extended Hamiltonian, the extended action and the separation of the constraints in first and second class. The correct identification of the constraints allowed us to find the relevant symmetries such as the diffeomorphisms and could carry out the counting of the physical degrees of freedom, which the analysis allow one to conclude that the system is a topological field theory. It is important to remark that the present analysis can be useful to understand the $G \to 0$ limit of general relativity, because we have present a background independent and full diffeomorphism invariant free field theory. This fact becomes to be important because in the analysis we have not broken the important symmetries that characterize to Eintein's theory of gravity. In addition, we extended our work constructing a closed and gauge invariant symplectic structure which contains all the relevant Hamiltonian description of the covariant phase space. In particular using the geometric form, we could find the same symmetries that we found using the Hamiltonian method. With the results presented in this paper, we have all the necessary elements to make progress in the quantization of the theory by means of the Dirac's method or covariant canonical formalism which is absent in the literature and will be reported in forthcoming works.

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