ARTINIAN LEVEL ALGEBRAS OF CODIMENSION 3

JEAMAN AHN[1] AND YONG SU SHIN[2,*]

Abstract. In this paper, we continue the study of which h-vectors $\mathbf{H} = (1, 3, \dots, h_{d-1}, h_d, h_{d+1})$ can be the Hilbert function of a level algebra by investigating Artinian level algebras of codimension 3 with the condition $\beta_{2,d+2}(I^{\text{lex}}) = \beta_{1,d+1}(I^{\text{lex}})$, where I^{lex} is the lex-segment ideal associated with an ideal I. Our approach is to adopt an homological method called *Cancellation Principle*: the minimal free resolution of I is obtained from that of I^{lex} by canceling some adjacent terms of the same shift.

We prove that when $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$, R/I can be an Artinian level k-algebra only if either $h_{d-1} < h_d < h_{d+1}$ or $h_{d-1} = h_d = h_{d+1} = d + 1$ holds. We also show that for $\mathbf{H} = (1, 3, \dots, h_{d-1}, h_d, h_{d+1})$, the Hilbert function of an Artinian algebra of codimension 3 with the condition $h_{d-1} = h_d < h_{d+1}$,

(a) if $h_d \le 3d + 2$, then h-vector $\mathbf{H}$ cannot be level, and

(b) if $h_d \ge 3d + 3$, then there is a level algebra with Hilbert function $\mathbf{H}$ for some value of h_{d+1}.

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be an n-variable polynomial ring over a field k of characteristic zero, and I be a homogeneous ideal of R. The numerical function

$$\mathbf{H}_{R/I}(t) := \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring R/I.

Recall that if n and i are positive integers, then n can be written uniquely in the form

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \ge j \ge 1$ (see Lemma 4.2.6, [9]).

Following [5], we define, for any integers a and b,

$$\left(n_{(i)}\right)^a_b = \binom{n_i + a}{i + b} + \binom{n_{i-1} + a}{i - 1 + b} + \cdots + \binom{n_j + a}{j + b}$$

where $\binom{m}{n} = 0$ for either $m < n$ or $n < 0$.

Let $\mathbf{H} = (h_0, h_1, \dots, h_i, \dots)$ be a sequence of non-negative integers. We say that $\mathbf{H}$ is an O-sequence if $h_0 = 1$ and $h_{i+1} \le ((h_i)_{(i)})^1_1$ for all $i \ge 1$. Given an O-sequence $\mathbf{H} = (h_0, h_1, \dots)$, we define the *first difference* of $\mathbf{H}$ as

$$\Delta\mathbf{H} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \dots).$$

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*Corresponding author.

If $A = R/I$ is an *Artinian k-algebra*, then we associate the graded algebra $A = k \oplus A_1 \oplus \cdots \oplus A_s$, ($A_s \neq 0$) with a vector of nonnegative integers, which is an $(s+1)$-tuple, called the *h-vector* of A and denoted by

$$\mathbf{H}_A := \mathbf{H} = (h_0, h_1, \ldots, h_s),$$

where $h_i = \dim_k A_i$. We call s the *socle degree* of A. The *socle* of A is defined to be the annihilator of the maximal homogeneous ideal, namely

$$\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\} \text{ where } m = \sum_{i=1}^{s} A_i.$$

Let $\mathcal{F}$ be the graded minimal free resolution of an homogeneous ideal $I \subset R$, i.e.,

$$\mathcal{F}: \ 0 \ \to \ \mathcal{F}_{n-1} \ \to \ \mathcal{F}_{n-2} \ \to \ \cdots \ \to \ \mathcal{F}_1 \ \to \ \mathcal{F}_0 \ \to \ I \ \to \ 0,$$

where $\mathcal{F}_i = \bigoplus_{j=1}^{\gamma_i} R^{\beta_{i,j}}(-j)$. The numbers j are called the *shifts* associated to I, and the numbers $\beta_{i,j}$ are called the *graded Betti numbers* of I. When we need to emphasize the ideal I, we shall use $\beta_{i,j}(I)$ for $\beta_{i,j}$.

An algebra A is called an *Artinian level algebra* if the last module $\mathcal{F}_{n-1}$ in the minimal free resolution of A is of the form $R(-s)^a$, where s and a are positive integers. We also say that a numerical sequence $\mathbf{H} = (h_0, h_1, \ldots, h_{s-1}, h_s)$ is a *level O-sequence* if there is an Artinian level algebra A with the Hilbert function $\mathbf{H}$.

As for the level O-sequence, an interesting question is how to determine if a given numerical sequence is a level O-sequence. A great deal of research has been conducted with the aim of answering to this question (see e.g., [1, 2, 3, 5, 6, 7, 8, 11, 14, 16, 18, 28, 30, 34, 35]). In particular, there is an excellent broad overview of level algebras in the memoir [14]. Despite this, it is sometimes distressingly difficult to find ones with specific desired properties, and several interesting problems are still open.

In [2], we proved that an Artinian algebra with Hilbert function $\mathbf{H} = (1, 3, h_2, \ldots, h_{d-1}, h_d, h_{d+1})$ with the condition $h_{d-1} > h_d = h_{d+1}$ cannot be level if $h_d \leq 2d+3$, and proved that if $h_d \geq 2d+4$ then there is a level O- sequence of codimension 3 with Hilbert function $\mathbf{H}$ for some value of h_{d-1}. To prove the result, we used the cancellation principle saying that the minimal free resolution of I is obtained from that of either $\mathrm{Gin}(I)$ or I^{lex} by canceling some adjacent terms of the same shift, where $\mathrm{Gin}(I)$ is the generic initial ideal of I with respect to the reverse lexicographic order and I^{lex} is the lex-segment ideal associated with an ideal I (see [22], [32]).

By the cancellation principle, one knows that $\mathbf{H} = (1, 3, \ldots, h_s)$ cannot be a level O-sequence if $\beta_{1,d+2}(\mathrm{Gin}(I)) < \beta_{2,d+2}(\mathrm{Gin}(I))$ or $\beta_{1,d+2}(I^{\mathrm{lex}}) < \beta_{2,d+2}(I^{\mathrm{lex}})$ for some $d < s$. However, the problem that we wish to solve is to determine whether a given h-vector can be a level O-sequence with the condition $\beta_{1,d+2}(\mathrm{Gin}(I)) = \beta_{2,d+2}(\mathrm{Gin}(I))$ or $\beta_{1,d+2}(I^{\mathrm{lex}}) = \beta_{2,d+2}(I^{\mathrm{lex}})$. In this case, it is known that an Artinian algebra $A = R/I$ of codimension 3 with Hilbert function $\mathbf{H} = (1, 3, \ldots, h_s)$ cannot be a level algebra (Theorem 3.14, [2]) if

(a) $\beta_{1,d+2}(\mathrm{Gin}(I)) = \beta_{2,d+2}(\mathrm{Gin}(I))$ for some $d < s$, or
(b) $\beta_{1,d+2}(I^{\mathrm{lex}}) = \beta_{2,d+2}(I^{\mathrm{lex}})$ with the condition $h_{d-1} > h_d = h_{d+1}$ for some $d < s$.

From this result, we wish to determine what Hilbert functions can be an Artinian level O-sequences with the condition

$$\beta_{1,d+2}(I^{\mathrm{lex}}) = \beta_{2,d+2}(I^{\mathrm{lex}}) \text{ for some } d < s. \tag{1.1}$$

We first prove that R/I can be an Artinian level k-algebra only if either $h_{d-1} < h_d < h_{d+1}$ with $\Delta h_d = \Delta h_{d+1}$, or $h_{d-1} = h_d = h_{d+1} = d+1$ with the condition (1.1) (see Theorem 3.3 and Corollary 3.8). Using these results, we also prove that for $\mathbf{H} = (1, 3, \ldots, h_{d-1}, h_d, h_{d+1})$, the Hilbert function of an Artinian algebra of codimension 3 with the condition $h_{d-1} = h_d < h_{d+1}$,

(a) if $h_d \le 3d+2$, then h-vector $\mathbf{H}$ cannot be level, and
(b) if $h_d \ge 3d+3$, then there is a level algebra with Hilbert function $\mathbf{H}$ for some value of h_{d+1}.

In Section 2, we introduce some preliminary results and background materials which will be used throughout the remaining part of the paper. In Section 3, we make use of *cancellation in resolutions* to study Artinian level algebras of codimension 3 with the condition (1.1). Finally, Section 4 is devoted to investigate Artinian level or non-level algebras with the condition $h_{d-1} = h_d < h_{d+1}$.

We use a computer program CoCoA [33] to build some of examples (e.g., Examples 3.6 and 4.6), with the fact that a differentiable O-sequence can always be a truncation of an Artinian Gorenstein O-sequence (see [14, 15, 16, 17, 19, 20, 23, 24]).

2. Background and Preliminary Results

In this section, we introduce some important results and recall some results of Macaulay, Green, and Stanley.

Theorem 2.1 ([21], Chapter 5 in [29])**.** *Let L be a general linear form in R and we denote by h_d the degree d entry of the Hilbert function of R/I and ℓ_d the degree d entry of the Hilbert function of $R/(I, L)$. Then, we have the following inequalities.*

(a) Macaulay's Theorem: $h_{d+1} \le \left((h_d)_{(d)}\right)^{1}_{1}$.
(b) Green's Hyperplane Restriction Theorem: $\ell_d \le \left((h_d)_{(d)}\right)^{-1}_{0}$.

For any homogeneous ideal I of $R = k[x_1, \ldots, x_n]$, note that the Hilbert function does not change by passing to $\mathrm{Gin}(I)$ or I^{lex}, and we have

$$\beta_{q,i}(I) \le \beta_{q,i}(\mathrm{Gin}(I)) \le \beta_{q,i}(I^{\mathrm{lex}})$$

(see [1, 4, 22, 26, 31]). In particular, if $\beta_{q,i}(\mathrm{Gin}(I)) = 0$ or $\beta_{q,i}(I^{\mathrm{lex}}) = 0$, then $\beta_{q,i}(I) = 0$.

In [25], they introduced the *s-reduction number* $r_s(R/I)$ of R/I and have shown the following lemma.

Lemma 2.2 ([1, 25])**.** *For a homogeneous ideal I of R and for $s \ge \dim(R/I)$, the s-reduction number $r_s(R/I)$ is given by*

$$\begin{aligned} r_s(R/I) &= \min\{\ell \mid \text{Hilbert function of } R/(I+J) \text{ vanishes in degree } \ell+1\} \\ &= \min\{\ell \mid x_{n-s}^{\ell+1} \in \mathrm{Gin}(I)\} \\ &= r_s(R/\mathrm{Gin}(I)) \end{aligned}$$

where J is generated by s general linear forms of R.

Now we continue to introduce some lemmas and theorems that will be used to prove the main results of this paper.

Lemma 2.3 (Lemma 3.2, [2])**.** *Let $A = R/I$ be an Artinian algebra and let L be a general linear form. Suppose that $\dim_k((I : L)/I)_d > (n-1)\dim_k((I : L)/I)_{d+1}$ for some $d > 0$. Then A has a socle element in degree d.*

We denote by $\mathcal{G}(I)$ the set of minimal (monomial) generators of I and $\mathcal{G}(I)_d$ the elements of $\mathcal{G}(I)$ having degree d. For a monomial $T = x_1^{a_1} \cdots x_n^{a_n} \in R$, define

$$m(T) := \max\{j \mid a_j > 0\}.$$

Theorem 2.4 (Eliahou-Kervaire, [12])**.** *Let I be a stable monomial ideal of R. Then we have*

$$\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T)-1}{q}.$$

Lemma 2.5 (Lemma 3.8, [2]). *Let J be a stable ideal of R. Then we have*

$$\dim_k\left((J : x_n)/J\right)_{d-1} = \left|\{T \in \mathcal{G}(J)_d \mid x_n \text{ divides } T\}\right|.$$

We now recall the well known result in [22], from which the generic initial ideal with respect to the degree reverse lexicographic order is extremely well-suited to the quotient by general linear forms.

Proposition 2.6 (Corollary 2.15, [22]). *Consider the degree reverse lexicographic order on the monomials of $R = k[x_1, \dots, x_n]$. Let I be a homogeneous ideal in R and H be a general linear form in R. Then*

$$\operatorname{Gin}(I + (H)/(H)) = (\operatorname{Gin}(I) + (x_n))/(x_n).$$

Remark 2.7. Let I be a homogeneous ideal of $R = k[x_1, \dots, x_n]$ and L be a general linear form in R. Using Proposition 2.6 and the exact sequence

$$0 \;\to\; R/(I : L)(-1) \;\overset{\times L}{\to}\; R/I \;\to\; R/(I, L) \;\to\; 0,$$

we have

$$\begin{array}{rcl} \dim_k(I : L)_t &=& \dim_k\left(\operatorname{Gin}(I) : x_n\right)_t \\ \mathbf{H}(R/(I, L), t) &=& \mathbf{H}(R/(\operatorname{Gin}(I), x_n), t) \\ \dim_k((I : L)/I)_t &=& \dim_k\left((\operatorname{Gin}(I) : x_n)/\operatorname{Gin}(I)\right)_t \end{array}$$

for $t \geq 0$.

Remark 2.8. Let I^{lex} be the lex-segment ideal associated with a homogeneous ideal I in $R = k[x_1, \dots, x_n]$ and L be a general linear form in R. Then, by Theorem 2.4, [13], we have the following equality

$$\mathbf{H}(R/(I^{\text{lex}}, L), d) = \left(\mathbf{H}(R/I^{\text{lex}}, d)_{(d)}\right)^{-1}_{\ 0}.$$

In this case, we may assume that x_n is general with respect to I^{lex}. Indeed, for $d \geq 1$, we have

$$\begin{array}{rcll} \left(\mathbf{H}(R/I, d)_{(d)}\right)^{-1}_{\ 0} &=& \left(\mathbf{H}(R/I^{\text{lex}}, d)_{(d)}\right)^{-1}_{\ 0} & \\ &=& \mathbf{H}(R/(I^{\text{lex}}, L), d) & \text{(by Theorem 2.4, [13])} \\ &=& \mathbf{H}(R/(\operatorname{Gin}(I^{\text{lex}}), x_n), d) & \text{(by Proposition 2.6 and Remark 2.7)} \\ &=& \mathbf{H}(R/(I^{\text{lex}}, x_n), d) & \text{(by Lemma 2.3, [10]).} \end{array}$$

The following lemma shows that we can write some of Betti numbers of the lex-segment ideal associated with a height three ideal I with respect to binomial expansion of the Hilbert function.

Lemma 2.9. *Let $A = R/I$ be an Artinian ring of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$. Suppose that $h_d < \binom{2+d}{2}$. Then, we have*

(a) $\beta_{2,d+2}(I^{\text{lex}}) = h_{d-1} - h_d + ((h_d)_{(d)})^{-1}_{\ 0}$.

(b) $\beta_{1,d+2}(I^{\text{lex}}) = ((h_d)_{(d)})^{1}_{1} + h_d - 2h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{\ 0}$.

Proof. (a) From the following exact sequence

$$0 \to ((I^{\text{lex}} : x_3)/I^{\text{lex}})_{d-1} \to (R/I^{\text{lex}})_{d-1} \overset{\times x_3}{\longrightarrow} (R/I^{\text{lex}})_d \to (R/(I^{\text{lex}}, x_3))_d \to 0,$$

we have

$$\begin{array}{rcll} \beta_{2,d+2}(I^{\text{lex}}) &=& \sum_{T \in \mathcal{G}(I^{\text{lex}})_d} \binom{m(T)-1}{2} & \text{(by Theorem 2.4)} \\ &=& \dim((I^{\text{lex}} : x_3)/I^{\text{lex}})_{d-1} & \text{(by Lemma 2.5)} \\ &=& h_{d-1} - h_d + (h_d)^{-1}_{\ 0} & \text{(by Remark 2.8)} \end{array} \tag{2.1}$$

as we wished.

(b) Since I^{lex} is a lex-segment ideal associated with an ideal I of R, we see that

$$\beta_{0,d+1}(I^{\text{lex}}) = ((h_d)_{(d)})^{1}_{1} - h_{d+1}.$$

Let $\mathcal{G}(I^{\text{lex}})_{d+1}$ be the set of minimal generators of I^{lex} in degree $d+1$. Then,

$$\begin{aligned}
\beta_{1,d+2}(I^{\text{lex}}) &= \sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=2}\binom{1}{1} + \sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=3}\binom{2}{1} && \text{(by Theorem 2.4)}\\
&= \sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=2}\binom{1}{1} + 2\Big[\sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=3}\binom{1}{1}\Big]\\
&= \Big[\sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=2}\binom{1}{1} + \sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=3}\binom{1}{1}\Big] +\\
&\quad \sum_{T\in\mathcal{G}(I^{\text{lex}})_{d+1},\, m(T)=3}\binom{1}{1}\\
&= \big|\mathcal{G}(I^{\text{lex}})_{d+1}\big| + \big|\{T\in\mathcal{G}(J)_{d+1} \mid x_3 \text{ divides } T\}\big|\\
&\quad \Big(\text{since } h_d < \binom{2+d}{2},\ x_1^{d+1}\notin \mathcal{G}(I^{\text{lex}})_{d+1}\Big)\\
&= \big|\mathcal{G}(I^{\text{lex}})_{d+1}\big| + \dim_k((I^{\text{lex}}:x_3)/I^{\text{lex}})_d && \text{(by Lemma 2.5)}\\
&= \big(((h_d)_{(d)})^1_1 - h_{d+1}\big) + \beta_{2,d+3}(I^{\text{lex}}) && \text{(by equation (2.1))}\\
&= ((h_d)_{(d)})^1_1 - h_{d+1} + h_d - h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{\ 0} && \text{(by Lemma 2.9 (a))}\\
&= ((h_d)_{(d)})^1_1 - 2h_{d+1} + h_d + ((h_{d+1})_{(d+1)})^{-1}_{\ 0},
\end{aligned}$$

as we wanted to prove. □

3. O-sequences with the Condition on $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$

First, we investigate if some Artinian O-sequence with the condition

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$$

is level.

Lemma 3.1. *Let $A = R/I$ be an Artinian ring of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$. Suppose that for some $d < s$,*

(a) *$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) > 0$, and*

(b) *$\beta_{2,d+3}(I^{\text{lex}}) > 0$.*

Then A is not level.

Proof. Assume that there exists an Artinian level algebra A with Hilbert function $\mathbf{H}$, and let $\bar{I} = (I_{\le d+1})$ and $\bar{A} = R/\bar{I}$. Then we have

$$\begin{aligned}
\beta_{1,d+2}(\bar{I}^{\text{lex}}) &= \beta_{1,d+2}(I^{\text{lex}}),\\
\beta_{2,d+2}(\bar{I}^{\text{lex}}) &= \beta_{2,d+2}(I^{\text{lex}}), \quad \text{and}\\
\beta_{2,d+3}(\bar{I}^{\text{lex}}) &= \beta_{2,d+3}(I^{\text{lex}}).
\end{aligned}$$

Hence, the assumption $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$ and $\beta_{2,d+3}(I^{\text{lex}}) > 0$ implies that

$$\beta_{1,d+2}(\bar{I}^{\text{lex}}) = \beta_{2,d+2}(\bar{I}^{\text{lex}}), \quad \text{and} \tag{3.1}$$

$$\beta_{2,\,d+3}(\bar{I}^{\text{lex}}) = \beta_{2,\,d+3}(I^{\text{lex}}) > 0. \tag{3.2}$$

Since A is level and $I_t = (\bar{I})_t$ for every $t \le d+1$,

$$0 = \beta_{2,d+2}(I) = \dim_k \operatorname{soc}(A)_{d-1} = \dim_k \operatorname{soc}(\bar{A})_{d-1} = \beta_{2,d+2}(\bar{I}). \tag{3.3}$$

Furthermore, using Lemma 2.9 in [2], we have the following equality

$$\beta_{1,d+2}(\bar{I}^{\text{lex}}) - \beta_{1,d+2}(\bar{I}) = [\beta_{0,d+2}(\bar{I}^{\text{lex}}) - \beta_{0,d+2}(\bar{I})] + [\beta_{2,d+2}(\bar{I}^{\text{lex}}) - \beta_{2,d+2}(\bar{I})].$$

Hence it follows from equations (3.1) and (3.3) that

$$-\beta_{1,d+2}(\bar{I}) = \beta_{0,d+2}(\bar{I}^{\text{lex}}) - \beta_{0,d+2}(\bar{I}) \ge 0,$$

which means that $\beta_{0,d+2}(\bar{I}^{\rm lex}) = \beta_{0,d+2}(\bar{I}) = 0$ since $\bar{I}$ is generated in degree $d+1$. This concludes from Theorem 2.4 that

$$\beta_{0,d+2}(\bar{I}^{\rm lex}) = \beta_{1,d+3}(\bar{I}^{\rm lex}) = 0.$$

In other words, any cancellation on shifts is impossible in the last free module of the minimal free resolution of $R/\bar{I}^{\rm lex}$ in degree d, and thus we have that $\beta_{2,\,d+3}(\bar{I}^{\rm lex}) = \beta_{2,\,d+3}(\bar{I}) > 0$. Hence $\bar{A}$ has a socle element in degree d, and so does A, which is a contradiction, as we wanted. □

Example 3.2. Consider an Artinian O-sequence $\mathbf{H} = (1,3,6,10,15,16,18)$. Then the minimal free resolution of $R/I^{\rm lex}$ with Hilbert function is

$$\begin{array}{rlcl} 0 & \to & R^2(-\mathbf{7}) \oplus R(-\mathbf{8}) \oplus R^{18}(-9) & \to & R^6(-6) \oplus R^2(-\mathbf{7}) \oplus R^{39}(-8) \\ & \to & R^5(-5) \oplus R(-6) \oplus R^{21}(-7) & \to & R \ \to \ R/I^{\rm lex} \ \to \ 0. \end{array}$$

Then

$$\beta_{2,7}(I^{\rm lex}) = \beta_{1,7}(I^{\rm lex}) = 2 \quad \text{and} \quad \beta_{2,8}(I^{\rm lex}) = 1.$$

By Lemma 3.1, any Artinian ring with Hilbert function $\mathbf{H}$ cannot be a level algebra.

Theorem 3.3. *Let $A = R/I$ be an Artinian ring of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_{d-1}, h_d, h_{d+1})$. Suppose that*

$$\beta_{1,d+2}(I^{\rm lex}) = \beta_{2,d+2}(I^{\rm lex}) > 0 \text{ for some } d < s.$$

If A is level, then
- (a) $h_{d-1} = h_d = h_{d+1} = d+1$, *or*
- (b) $h_{d-1} < h_d < h_{d+1}$.

Proof. We shall prove this theorem using the contrapositive.

(a) Assume $h_{d-1} = h_d = h_{d+1}$. First if $h_d \le d$, then $((h_d)_{(d)})^{-1}_{\ 0} = 0$ and thus, by Lemma 2.9, we have that

$$0 < \beta_{2,d+2}(I^{\rm lex}) = h_{d-1} - h_d + ((h_d)_{(d)})^{-1}_{\ 0} = 0,$$

which is impossible.

Second, if $h_d \ge d+2$, then $((h_{d+1})_{(d+1)})^{-1}_{\ 0} \ge 1$ and thus, by Lemma 2.9 again,

$$\begin{array}{rcl} \beta_{2,d+3}(I^{\rm lex}) & = & h_d - h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{\ 0} \\ & = & ((h_{d+1})_{(d+1)})^{-1}_{\ 0} > 0. \end{array}$$

Hence, by Lemma 3.1, A is not level.

(b) Now suppose h_{d-1}, h_d and h_{d+1} are not the same, and (b) does not hold. There are five cases to be considered.

Case 1. If $h_{d-1} > h_d = h_{d+1}$, then by Theorem 4.5 in [2], A is not level.

Case 2. If $h_{d-1} \ge h_d > h_{d+1}$, then $h_{d+1} < \binom{2+(d+1)}{2}$ and thus, by Lemma 2.9,

$$\begin{array}{rcl} \beta_{2,d+3}(I^{\rm lex}) & = & h_d - h_{d+1} + ((h_{d+1})^{-1}_{\ 0} \\ & \ge & h_d - h_{d+1} \\ & > & 0. \end{array}$$

Hence, by Lemma 3.1, A is not level.

Case 3. Suppose that $h_{d-1} \ge h_d < h_{d+1}$. For this case, we shall use the reduction number $r_1(A)$.

Assume $r_1(A) < d$. Note that, for a general linear form L in R, it follows from Lemma 2.2 that

$$\mathbf{H}(R/(I,L),t) = 0 \quad \text{for} \quad t \ge d.$$

For such t with the following exact sequence

$$0 \ \to \ ((I:L)/I)_{t-1}(-1) \ \to \ (R/I)_{t-1}(-1) \ \overset{\times L}{\to} \ (R/I)_t \ \to \ 0,$$

we have

$$h_{t-1} = h_t + \dim_k((I : L)/I)_{t-1} \geq h_t.$$

So $h_{d-1} \geq h_d \geq h_{d+1}$, which is not the case. Thus, we now assume that $r_1(A) \geq d$.

Suppose that A is level and let L be a general linear form in $R = k[x_1, x_2, x_3]$. Now consider the exact sequence

$$0 \;\to\; ((I : L)/I)_{d-1}(-1) \;\to\; (R/I)_{d-1}(-1) \;\overset{\times L}{\to}\; (R/I)_d \;\to\; (R/(I, L))_d \;\to\; 0. \qquad (3.4)$$

Since $d \leq r_1(A)$, we see that $\dim(R/(I, L))_d > 0$, and so

$$\begin{array}{rcll} \dim((I : L)/I)_{d-1} & = & h_{d-1} - h_d + \dim(R/(I, L))_d & \text{(by equation (3.4))} \\ & \geq & \dim(R/(I, L))_d > 0 & \text{(since } h_{d-1} \geq h_d\text{)}. \end{array}$$

Moreover, since A is level, we have

$$\begin{array}{rcll} 0 & < & \dim((I : L)/I)_{d-1} & \\ & \leq & 2\dim((I : L)/I)_d & \text{(by Lemma 2.3 (a))} \\ & = & 2\dim((\mathrm{Gin}(I) : x_3)/\mathrm{Gin}(I))_d & \text{(by Remark 2.7)} \\ & = & 2\beta_{2,d+3}(\mathrm{Gin}(I)) & \text{(by Lemma 2.5)} \\ & \leq & 2\beta_{2,d+3}(I^{\mathrm{lex}}) & \text{(by the theorem of BHP in [4, 26, 31])}. \end{array}$$

Thus, by Lemma 3.1, A has a socle element in degree d, which is a contradiction.

Case 4. If $h_{d-1} < h_d > h_{d+1}$, then

$$\beta_{2,d+3}(I^{\mathrm{lex}}) = h_d - h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{\;0} > 0.$$

Hence, by Lemma 3.1, A is not level.

Case 5. Suppose $h_{d-1} < h_d = h_{d+1}$. If $h_d \leq d$ then $((h_d)_{(d)})^{-1}_{\;0} = 0$, and thus

$$0 < \beta_{2,d+2}(I^{\mathrm{lex}}) = h_{d-1} - h_d + ((h_d)_{(d)})^{-1}_{\;0} < 0,$$

which is impossible. Hence $h_d \geq d + 1$.

If $h_d = d + 1$, then $((h_d)_{(d)})^{-1}_{\;0} = 1$, and so

$$\begin{array}{rcll} h_d & > & h_{d-1} & \\ & = & h_d - ((h_d)_{(d)})^{-1}_{\;0} + \beta_{2,d+2}(I^{\mathrm{lex}}) & \text{(by Lemma 2.9 (a))} \\ & = & (d + 1) - 1 + \beta_{2,d+2}(I^{\mathrm{lex}}) & \\ & \geq & d + 1 & \text{(since } \beta_{2,d+2}(I^{\mathrm{lex}}) > 0\text{)} \\ & = & h_d, & \end{array}$$

which is impossible. Thus we have $h_d \geq d + 2$, that is, $((h_{d+1})_{(d+1)})^{-1}_{\;0} \geq 1$. Then, by Lemma 2.9 (a), we obtain

$$\beta_{2,d+3}(I^{\mathrm{lex}}) = h_d - h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{\;0} > 0.$$

Therefore, by Lemma 3.1, A is not level, which completes the proof. □

The following example shows that there exists an Artinian level O-sequence which satisfies the condition $h_{d-1} = h_d = h_{d+1} = d + 1$.

Example 3.4. Let $I = (x_1^2, x_2^3) + (x_1, x_2, x_3)^7$. Then the Hilbert function of R/I is

$$(1, 3, 5, 6, 6, 6, 6)$$

and the reduction number $r_1(R/I)$ is

$$\min\{\ell \mid x_2^{\ell+1} \in I\} = 2.$$

Moreover, it is immediate that

$$\mathrm{soc}(R/I) = (R/I)_6,$$

and so the minimal free resolution of R/I is

$$\begin{array}{llllllll} 0 & \to & R(-9)^6 & \to & R(-5)\oplus R(-8)^{12} & \to & R(-2)\oplus R(-3)\oplus R(-7)^6 \\ & \to & R & \to & R/I & \to & 0. \end{array}$$

Note that the minimal free resolution of R/I^{lex} is

$$\begin{array}{ll} 0 & \to R(-6)\oplus R(-\mathbf{7})\oplus R^6(-9) \\ & \to R(-4)\oplus R^2(-5)\oplus R^2(-6)\oplus R(-\mathbf{7})\oplus R^{12}(-8) \\ & \to R(-2)\oplus R(-3)\oplus R(-4)\oplus R(-5)\oplus R(-6)\oplus R^6(-7) \quad\to\quad R \quad\to\quad R/I^{\text{lex}} \quad\to\quad 0. \end{array}$$

This means that R/I is an Artinian level algebra with the condition $h_4 = h_5 = h_6 = 5+1$, and

$$\beta_{1,5+2}(I^{\text{lex}}) = \beta_{2,5+2}(I^{\text{lex}}) = 1.$$

Remark 3.5. In Example 3.4, we constructed an Artinian level algebra R/I which satisfied the condition $h_{d-1} = h_d = h_{d+1} = d+1$ and $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) > 0$. The following are other examples of Artinian level O-sequences which satisfy the condition $h_{d-1} < h_d < h_{d+1}$ and $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) > 0$.

Example 3.6 (CoCoA)**.** We provide two examples of our results via calculations done by CoCoA.

(a) Consider a differentiable O-sequence $\mathbf{H} = (1,3,6,10,\mathbf{13},\mathbf{15},\mathbf{17},19,20)$ and an Artinian algebra R/I with Hilbert function $\mathbf{H}$. Then the minimal free resolution of R/I^{lex} is

$$\begin{array}{ll} 0 & \to R(-\mathbf{7})\oplus R(-10)\oplus R^{20}(-11) \\ & \to R(-5)\oplus R^2(-6)\oplus R(-\mathbf{7})\oplus R^2(-9)\oplus R^{42}(-10) \\ & \to R^2(-4)\oplus R(-5)\oplus R(-6)\oplus R(-8)\oplus R^{22}(-9) \quad\to\quad R/I^{\text{lex}} \quad\to\quad 0, \end{array}$$

and hence

$$\beta_{2,7}(I^{\text{lex}}) = \beta_{1,7}(I^{\text{lex}}) = 1.$$

Moreover, the sequence $\mathbf{H}$ is a level O-sequence since any differentiable O-sequence can be a truncation of an Artinian Gorenstein O-sequence.

(b) Here is another differentiable O-sequence $\mathbf{H} = (1,3,6,10,\mathbf{12},\mathbf{14},\mathbf{16},18,19,20)$, which is also a level O-sequence by the same argument as in (a). Furthermore, the minimal free resolution of R/I^{lex} is

$$\begin{array}{ll} 0 & \to R(-\mathbf{6})\oplus R(-10)\oplus R(-11)\oplus R^{20}(-12) \\ & \to R^3(-5)\oplus R(-\mathbf{6})\oplus R^2(-9)\oplus R^2(-10)\oplus R^{42}(-11) \\ & \to R^3(-4)\oplus R(-5)\oplus R(-8)\oplus R(-9)\oplus R^{22}(-10) \quad\to\quad R \quad\to\quad R/I^{\text{lex}} \quad\to\quad 0, \end{array}$$

and thus

$$\beta_{2,6}(I^{\text{lex}}) = \beta_{1,6}(I^{\text{lex}}) = 1.$$

Remark 3.7. In Example 3.6, both examples show that $\Delta h_d = \Delta h_{d+1}$. From this observation, we obtain the following result.

Corollary 3.8. *Let $A = R/I$ be an Artinian ring of codimension 3. Suppose that*

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) > 0 \text{ for some } d < s.$$

If A is level and $h_{d-1} < h_d < h_{d+1}$, then $\Delta h_d = \Delta h_{d+1}$.

Proof. Note that it suffices to prove that $\Delta h_d = \Delta h_{d+1}$ for $h_{d-1} < h_d < h_{d+1}$.

Suppose that A is level. Using Lemma 3.1, we see that

$$\beta_{2,d+3}(I^{\text{lex}}) = 0. \tag{3.5}$$

Furthermore, it is a simple consequence of Eliahou-Kervaire (Theorem 2.4) that $\beta_{2,d+2}(I^{\text{lex}}) > 0$ implies $\beta_{0,d}(I^{\text{lex}}) > 0$. Hence we have

$$h_d < \binom{2+d}{2}. \tag{3.6}$$

Since $h_d < h_{d+1}$, one can easily check that $d+1 < h_{d+1}$ and thus $d+1 < h_{d+1} < \binom{2+(d+1)}{2}$. Then the $(d+1)$-*binomial expansion* of h_{d+1} is of the form

$$(h_{d+1})_{(d+1)} := \binom{1+(d+1)}{d+1} + \cdots + \binom{1+(d-(c-2))}{d-(c-2)} + \binom{d-(c-1)}{d-(c-1)} + \cdots + \binom{\delta}{\delta} \tag{3.7}$$

where $\delta \geq 1$. It follows from Lemma 2.9 (a) and (3.5) that

$$\Delta h_{d+1} = ((h_{d+1})_{(d+1)})^{-1}_{0} - \beta_{2,d+3}(I^{\text{lex}}) = ((h_{d+1})_{(d+1)})^{-1}_{0}. \tag{3.8}$$

Now we consider the case $c < d$ only in equation (3.7). Indeed, if $c-1 \leq d \leq c$, then we have

$$(h_{d+1})_{(d+1)} = \begin{cases} \binom{2+d}{d+1} + \cdots + \binom{3}{2} + \binom{2}{1}, & \text{if } d = c-1, \\ \binom{2+d}{d+1} + \cdots + \binom{4}{3} + \binom{3}{2} + \binom{1}{1}, & \text{if } d = c. \end{cases}$$

Using Pascal's identity and equation (3.8) for both cases, we have

$$h_d = \binom{2+d}{d},$$

which contradicts equation (3.6). Hence, by equation (3.8),

$$\begin{aligned} h_d &= h_{d+1} - ((h_{d+1})_{(d+1)})^{-1}_{0} \\ &= \binom{1+d}{d} + \cdots + \binom{1+(d-(c-1))}{d-(c-1)} + \binom{d-(c-1)}{d-(c-1)} + \cdots + \binom{\delta}{\delta}, \\ &= \begin{cases} \binom{1+d}{d} + \cdots + \binom{1+(d-(c-1))}{d-(c-1)} + \binom{1+(d-c)}{d-c}, & \text{if } \delta = 1, \\ \binom{1+d}{d} + \cdots + \binom{1+(d-(c-1))}{d-(c-1)} + \binom{d-c}{d-c} + \cdots + \binom{\delta-1}{\delta-1}, & \text{if } \delta > 1, \end{cases} \end{aligned}$$

i.e.,

$$((h_d)_{(d)})^{1}_{1} = \begin{cases} \binom{1+(d+1)}{(d+1)} + \cdots + \binom{1+(d-(c-2))}{(d-(c-2))} + \binom{1+(d-(c-1))}{(d-(c-1))}, & \text{if } \delta = 1, \\ \binom{1+(d+1)}{(d+1)} + \cdots + \binom{1+(d-(c-2))}{(d-(c-2))} + \binom{d-(c-1)}{d-(c-1)} + \cdots + \binom{\delta}{\delta}, & \text{if } \delta > 1. \end{cases}$$

Thus

$$((h_d)_{(d)})^{1}_{1} - h_{d+1} = \begin{cases} 1, & \text{if } \delta = 1, \\ 0, & \text{if } \delta > 1. \end{cases}$$

Moreover, by Lemma 2.9 (b),

$$\begin{aligned} 0 &< \beta_{1,d+2}(I^{\text{lex}}) \\ &= ((h_d)_{(d)})^{1}_{1} + h_d - 2h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{0} \\ &= ((h_d)_{(d)})^{1}_{1} - h_{d+1} - \Delta h_{d+1} + ((h_{d+1})_{(d+1)})^{-1}_{0} \\ &= ((h_d)_{(d)})^{1}_{1} - h_{d+1} \quad \text{(by equation (3.8))}. \end{aligned}$$

This means that

$$\beta_{1,d+2}(I^{\rm lex}) = ((h_d)_{(d)})^1_1 - h_{d+1} = 1 \quad \text{and} \quad \delta = 1. \tag{3.9}$$

In other words,

$$(h_d)_{(d)} = \binom{1+d}{d} + \cdots + \binom{1+(d-(c-1))}{d-(c-1)} + \binom{1+(d-c)}{(d-c)}.$$

Hence, $((h_{d+1})_{(d+1)})^{-1}_{\ 0} = c$ and $((h_d)_{(d)})^{-1}_{\ 0} = c+1$, and so we obtain

$$\begin{array}{rcll} \Delta h_d & = & ((h_d)_{(d)})^{-1}_{\ 0} - \beta_{2,d+2}(I^{\rm lex}) & \text{(by Lemma 2.9 (a))} \\ & = & c+1-\beta_{1,d+2}(I^{\rm lex}) & \text{(since } \beta_{1,d+2}(I^{\rm lex}) = \beta_{2,d+2}(I^{\rm lex}) > 0) \\ & = & c & \text{(by equation (3.9))} \\ & = & ((h_{d+1})_{(d+1)})^{-1}_{\ 0} & \\ & = & \Delta h_{d+1}, & \text{(by equation (3.8))} \end{array}$$

as we wished. □

Example 3.9. Let R/I be an Artinian ring with Hilbert function $\mathbf{H} = (1,3,6,10,\mathbf{15},\mathbf{16},\mathbf{18},20)$. Then the minimal free resolution of $R/I^{\rm lex}$ is

$$\begin{array}{rlll} 0 & \to & R^{\mathbf{2}}(-\mathbf{7}) \oplus R(-8) \oplus R^{20}(-10) & \to \quad R^6(-6) \oplus R^2(-\mathbf{7}) \oplus R(-8) \oplus R^{42}(-9) \\ & \to & R^5(-5) \oplus R(-6) \oplus R(-7) \oplus R^{22}(-8) & \to \quad R \quad \to \quad R/I^{\rm lex} \quad \to \quad 0. \end{array}$$

Thus

$$\beta_{2,7}(I^{\rm lex}) = \beta_{1,7}(I^{\rm lex}) = 2 \quad \text{and} \quad \Delta h_5 = 1 \neq 2 = \Delta h_6.$$

By Theorem 3.8, any Artinian ring R/I with Hilbert function $\mathbf{H}$ cannot be level.

4. O-sequences with The Condition $h_{d-1} = h_d < h_{d+1}$

In this section, we consider Artinian O-sequences with the condition $h_{d-1} = h_d < h_{d+1}$. To describe an Artinian O-sequence with this condition, we begin with the following lemma.

Lemma 4.1. *Let c and d be positive integers satisfying $d < c < \binom{d+2}{2}$. Then*

$$(c_{(d)})^{-1}_{\ 0} - (c_{(d)})^1_1 + c = 0.$$

Proof. Without loss of generality, we assume that

$$c_{(d)} = \binom{1+d}{d} + \cdots + \binom{1+d-\alpha}{d-\alpha} + \binom{d-(\alpha+1)}{d-(\alpha+1)} + \cdots + \binom{\delta}{\delta}.$$

Then we have

$$\begin{array}{rcl} ((c)_{(d)})^{-1}_{\ 0} & = & \alpha+1, \quad \text{and} \\ ((c)_{(d)})^1_1 - c & = & \alpha+1, \end{array}$$

and thus

$$((c)_{(d)})^{-1}_{\ 0} - ((c)_{(d)})^1_1 + c = 0,$$

as we wished. □

The following result is an useful criterion to determine if A is level.

Proposition 4.2. *Let $A = R/I$ be an Artinian ring of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_s)$. Suppose that $h_{d-1} = h_d < h_{d+1}$ for some $d < s$. Then A is not level if*

$$((h_{d+1})_{(d+1)})^{-1}_{\ 0} \leq 2(\Delta h_{d+1}).$$

Proof. Since $h_{d-1} = h_d$, we get $h_d < \binom{2+d}{2}$. If $h_d \le d$, by Macaulay's Theorem we have $h_{d+1} \le d = h_d$. So we may assume that $d < h_d < \binom{2+d}{2}$. Hence $((h_d)_{(d)})^{-1}_{\ 0} > 0$.

Since $((h_{d+1})_{(d+1)})^{-1}_{\ 0} \le 2(\Delta h_{d+1})$, we obtain that

$$\begin{aligned}
\beta_{2,d+2}(I^{\text{lex}}) &= h_{d-1} - h_d + ((h_d)_{(d)})^{-1}_{\ 0} \quad \text{(by Lemma 2.9 (a))} \\
&= ((h_d)_{(d)})^{-1}_{\ 0} \\
&= ((h_d)_{(d)})^{-1}_{\ 0} + \beta_{1,d+2}(I^{\text{lex}}) - ((h_d)_{(d)})^{1}_{1} - h_d + 2h_{d+1} - ((h_{d+1})_{(d+1)})^{-1}_{\ 0} \\
&\quad \text{(by Lemma 2.9 (b))} \\
&= (((h_d)_{(d)})^{-1}_{\ 0} - ((h_d)_{(d)})^{1}_{1} + h_d) + (2\Delta h_{d+1} - ((h_{d+1})_{(d+1)})^{-1}_{\ 0}) + \beta_{1,d+2}(I^{\text{lex}}) \\
&\ge (((h_d)_{(d)})^{-1}_{\ 0} - ((h_d)_{(d)})^{1}_{1} + h_d) + \beta_{1,d+2}(I^{\text{lex}}) \quad \text{(by Lemma 4.1)} \\
&= \beta_{1,d+2}(I^{\text{lex}}).
\end{aligned}$$

If $\beta_{2,d+2}(I^{\text{lex}}) > \beta_{1,d+2}(I^{\text{lex}})$, then A has a socle element in degree $d-1$, which means A is not level. If

$$\beta_{2,d+2}(I^{\text{lex}}) = \beta_{1,d+2}(I^{\text{lex}}) = ((h_d)_{(d)})^{-1}_{\ 0} > 0,$$

by Theorem 3.3 A is not level, which completes the proof. □

Example 4.3. Consider an O-sequence $\mathbf{H} = (1, 3, 6, 10, \mathbf{15}, \mathbf{15}, \mathbf{16})$. Then

$$2 = ((16)_{(6)})^{-1}_{\ 0} \le 2\Delta h_6 = 2.$$

Therefore, by Proposition 4.2, any Artinian algebra with Hilbert function $\mathbf{H}$ cannot be level.

Before we construct an Artinian level O-sequence with the condition $((h_{d+1})_{(d+1)})^{-1}_{\ 0} > 2(\Delta h_{d+1})$, we introduce the theorem of Iarrobino to obtain a new level O-sequence from the given level-O-sequence. Moreover, let us recall the main facts of the theory of *inverse system*, or *Macaulay duality*, which will be a fundamental tool to build an example. For a complete description, we refer to [22] and [28].

Let $S = k[y_1, \dots, y_n]$ and consider S as a graded $R = k[x_1, \dots, x_n]$-module where the action of x_i on S is partial differentiation with respect to y_i. Then there is a one to one correspondence between graded Artinian algebras R/I and finitely generated graded R-submodules M in S, where $I = \text{Ann}(M)$ is the annihilator of M in R, and conversely $M = I^{-1}$ is the R-submodules of S which is annihilated by I.

Theorem 4.4 (Theorem 4.8A, [27])**.** *Let $\mathbf{H}' = (h_0, h_1, \dots, h_s)$ be the h-vector of a level algebra $A = R/\text{Ann}(M)$. Then, if F is a general form of degree s, the level algebra $B = R/\text{Ann}(\langle M, F\rangle)$ has the h-vector $\mathbf{H} = (H_0, H_1, \dots, H_s)$ where*

$$H_i = \min\left\{h_i + \binom{r-1+s-i}{s-i}, \binom{r-1+i}{i}\right\}$$

for $i = 1, \dots, s$.

The following theorem shows that there is an Artinian level algebra whose Hilbert function satisfies the condition

$$h_{d-1} = h_d < h_{d+1} \quad \text{and} \quad ((h_{d+1})_{(d+1)})^{-1}_{\ 0} > 2(\Delta h_{d+1}).$$

Theorem 4.5. *Let $\mathbf{H} = (1, 3, h_2, \dots, h_{d-1}, h_d, h_{d+1})$ be an O-sequence satisfying*

$$h_{d-1} = h_d < h_{d+1}.$$

(a) *If $h_d \le 3d+2$, then $\mathbf{H}$ is not level.*
(b) *If $h_d \ge 3d+3$, then there exists an Artinian level algebra with the Hilbert function $\mathbf{H}$ for some value of h_{d+1}.*

Proof. (a) ***Case 1.*** Suppose that $h_d < 3d$. Since

$$h_d \le (3d-1)_{(d)} = \binom{1+d}{d} + \binom{d}{d-1} + \binom{d-2}{d-2} + \cdots + \binom{1}{1},$$

and $h_{d+1} \le ((h_d)_{(d)})^1_1$, we see that $((h_{d+1})_{(d+1)})^{-1}_{\ \ 0} \le 2$. Hence,

$$((h_{d+1})_{(d+1)})^{-1}_{\ \ 0} \le 2 \le 2\Delta h_{d+1}.$$

Therefore, by Proposition 4.2, $\mathbf{H}$ cannot be a level O-sequence.

Case 2. Suppose that $3d \le h_d \le 3d+2$. If $h_d = 3d$, then $h_{d-1} = h_d = 3d < \binom{2+d}{d}$. Hence $d \ge 3$ and

$$(h_d)_{(d)} = \binom{1+d}{d} + \binom{d}{d-1} + \binom{d-1}{d-2}.$$

This implies that

$$h_{d+1} \le ((h_d)_{(d)})^1_1 = \binom{2+d}{1+d} + \binom{1+d}{d} + \binom{d}{d-1}, \quad \text{that is,} \quad ((h_{d+1})_{(d+1)})^{-1}_{\ \ 0} \le 3.$$

By the similar argument as above, we obtain

$$((h_{d+1})_{(d+1)})^{-1}_{\ \ 0} \le 3$$

for $h_d = 3d+1$ or $3d+2$ as well.

If $h_{d+1} \ge h_d + 2$, i.e., $\Delta h_{d+1} \ge 2$, then we see that

$$((h_{d+1})_{(d+1)})^{-1}_{\ \ 0} \le 3 \le 4 \le 2\Delta h_{d+1},$$

and thus, by Proposition 4.2, A is not level.

We now assume that $h_{d+1} = h_d + 1$. Then, it follows from Lemma 2.9 that

h_d	$3d$	$3d+1$	$3d+2$
$\beta_{1,d+2}(I^{\text{lex}})$	3	3	4
$\beta_{2,d+2}(I^{\text{lex}})$	3	3	3
$\beta_{2,d+3}(I^{\text{lex}})$	1	1	2

By Lemma 3.1 it is enough to prove that $\mathbf{H}$ is not level for the case where

$$h_d = 3d+2 \quad \text{and} \quad h_{d+1} = 3d+3.$$

Assume that there is an Artinian level algebra $A = R/I$ with Hilbert function $\mathbf{H}$. By Lemma 2.9, the Betti diagram of R/I^{lex} is as follows.

total	1	–	–	–
0	1	.	.	.
…		…		
$d-1$	.	$*$	$*$	3
d	.	2	4	2
$d+1$	.	$*$	$*$	$*$

Let $J := (I_{\le d+1})$. Note that I^{lex} and J^{lex} agree in degree $\le d+1$. We then rewrite the Betti diagram of R/J^{lex} as follows.

total	1	$-$	$-$	$-$
0	1	.	.	.
$\cdots$			$\cdots$	
$d-1$	.	$*$	$*$	3
d	.	2	4	2
$d+1$	.	a	b	$*$

Since R/I is level and $(I_{\le d+1})$ has no generators in degree $d+2$, we have

$$0 \le a \le 1 \quad \text{(by the cancellation principle).}$$

Case 2-1. If $a = 0$, then by the result of Eliahou-Kervaire (Theorem 2.4), we have $b = 0$, which means $R/(I_{\le d+1})$ has a two dimensional socle element in degree d, so does R/I. This is a contradiction.

Case 2-2. If $a = 1$, then J^{lex} has one generator in degree $d+2$. Define

$$h_{d+2} := \mathbf{H}(R/J^{\text{lex}}, d+2).$$

Then we have

$$h_{d+2} = ((h_{d+1})_{(d+1)})^1_1 - 1 = ((3d+3)_{(d+1)})^1_1 - 1 = 3d+5, \text{ i.e.,}$$
$$((h_{d+2})_{(d+2)})^{-1}_0 = ((3d+5)_{(d+2)})^{-1}_0 = 2.$$

Hence, from Lemmas 2.5 and 2.9 we have

$$\begin{aligned} \dim_k((J^{\text{lex}} : x_3)/J^{\text{lex}})_{d+1} &= \left| \left\{ T \in \mathcal{G}(J^{\text{lex}})_{d+2} \,\middle|\, x_3 \text{ divides } T \right\} \right| \\ &= \beta_{2,d+4}(J^{\text{lex}}) \\ &= h_{d+1} - h_{d+2} + ((h_{d+2})_{(d+2)})^{-1}_0 \\ &= 0. \end{aligned} \tag{4.1}$$

Since $x_1^{d+2} \notin \mathcal{G}(J^{\text{lex}})_{d+2}$, by Theorem 2.4 and equation (4.1), we find

$$b = \beta_{1,d+3}(J^{\text{lex}}) = \sum_{T \in \mathcal{G}(J^{\text{lex}})_{d+2}} \binom{m(T)-1}{1} = 1.$$

Using the cancellation principle, we know R/J has at least one socle element in degree d. Since R/I and R/J agree in degree $\le d+1$, R/I has also a socle element in degree d. This is a contradiction.

(b) Applying Theorem 4.4 to a differentiable O-sequence

$$\mathbf{H}' = (1, 3, 6, \ldots, 3(d-1) + (\ell - 3), \overset{d\text{-th}}{3d + (\ell - 3)}, 3(d+1) + (\ell - 3))$$

with $\ell \ge 3$, we obtain an Artinian level O-sequence

$$\begin{aligned} H_{d-1} &= \min\left\{ 3(d-1) + (\ell-3) + \binom{4}{2}, \binom{d+1}{2} \right\} &= 3d + \ell, \\ H_d &= \min\left\{ 3d + (\ell-3) + \binom{3}{1}, \binom{d+2}{2} \right\} &= 3d + \ell, \quad \text{and} \\ H_{d+1} &= \min\left\{ 3(d+1) + (\ell-3) + \binom{2}{0}, \binom{d+3}{2} \right\} &= 3d + (\ell+1), \end{aligned}$$

as we wished. □

Example 4.6. Consider a differentiable O-sequence $\mathbf{H}' = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 58, 61, 64)$, which is an Artinian level O-sequence. By Theorem 4.4, we can construct a new level O-sequence as follows.

$$\mathbf{H} = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \mathbf{64}, \mathbf{64}, \mathbf{65}),$$

which satisfies the following two conditions

$$h_{10} = h_{11} < h_{12} \quad \text{and} \quad 6 = ((65)_{(12)})^{-1}_{\ 0} > 2\Delta h_{12} = 2.$$

The above example 4.6 also shows that there is an Artinian level algebra whose Hilbert function satisfies the conditions

$$h_{10} = h_{11} < h_{12} \quad \text{and} \quad 64 = h_{11} > 3d = 3 \cdot 11 = 33.$$

If we couple our previous work done in [2] with the results of the previous and this sections, we obtain the following result.

Theorem 4.7. *Let R/I be an Artinian ring of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_{d+1})$. Then,*

(a) *if $h_{d-1} > h_d = h_{d+1}$ with $h_d \leq 2d + 3$, then R/I is not level,*
(b) *if $h_{d-1} > h_d = h_{d+1}$ with $h_d \geq 2d + 4$, the R/I is level for some value of h_{d-1},*
(c) *if $h_{d-1} = h_d < h_{d+1}$ with $h_d \leq 3d + 2$, then R/I is not level,*
(d) *if $h_{d-1} = h_d < h_{d+1}$ with $h_d \geq 3d + 3$, then R/I is level for some value of h_{d+1},*
(e) *if R/I is level and $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$, then*
  (i) *$h_{d-1} = h_d = h_{d+1} = d + 1$, or*
  (ii) *$h_{d-1} < h_d < h_{d+1}$ and $\Delta h_d = \Delta h_{d+1}$."*

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Department of Mathematics Education, Kongju National University, 182, Shinkwan-dong, Kongju, Chungnam 314-701, Republic of Korea
E-mail address: `jeamanahn@kongju.ac.kr`

Department of Mathematics, Sungshin Women's University, Seoul, Korea, 136-742
E-mail address: `ysshin@sungshin.ac.kr`