QUADRATIC ESTIMATES FOR PERTURBED DIRAC TYPE OPERATORS ON DOUBLING MEASURE METRIC SPACES

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Abstract. We consider perturbations of Dirac type operators on complete, connected metric spaces equipped with a doubling measure. Under a suitable set of assumptions, we prove quadratic estimates for such operators and hence deduce that these operators have a bounded functional calculus. In particular, we deduce a Kato square root type estimate.

Contents

1. Introduction

Let $\mathcal{X}$ be a complete, connected metric space and μ a Borel-regular doubling measure. We consider densely defined, closed, nilpotent operators Γ on $\mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$ and perturbed Dirac type operators $\Pi_B = \Gamma + B_1 \Gamma^* B_2$, where B_i are strictly accretive L^∞ matrix valued functions. We prove quadratic estimates

$$\int_0^\infty \left\| t\Pi_B (1 + t^2 \Pi_B)^{-1} u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$ under a set of hypotheses (H1)-(H8) which are outlined in the sequel. These estimates are equivalent to Π_B having a bounded holomorphic functional calculus. This allows us to conclude that $\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1 \Gamma^* B_2)$ and that $\left\| \sqrt{\Pi_B^2} u \right\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|B_1 \Gamma^* B_2 u\|$. When $\mathcal{X} = \mathbb{R}^n$ and μ is the Lebesgue measure, it is shown by Axelsson, Keith and McIntosh in [5] that this implies $\mathcal{D}(\sqrt{\operatorname{div} A \nabla})) = \mathcal{D}(\nabla)$ and $\left\| \sqrt{\operatorname{div} A \nabla} u \right\| \simeq \|\nabla u\|$ for an appropriate class of perturbations A. Thus, we are justified in calling this a *Kato square root type estimate*.

We proceed to prove our theorem based on the ideas presented in [5]. These ideas date back to the resolution of the Kato conjecture by Auscher, Hofmann, Lacey, McIntosh and

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Tchamitchian in [2]. The exposition [10] by Hofmann is an excellent survey of the history and resolution of the Kato conjecture. Further historical references include the article by McIntosh [13] and by Auscher and Tchamitchian [3]. More recently, the proof in [5] was generalised by Morris in [14] for complete Riemannian manifolds with *exponential volume growth*. This work is beneficial to us since we rely upon the same abstract dyadic decomposition of Christ [7].

The main novelty of the work presented here is that we have have separated the assumptions on the operator Γ from the underlying differentiable structure of the space. In general, the spaces we consider may not admit a differentiable structure. However, we are motivated by the existence of measure metric spaces more general than Riemannian manifolds admitting such structures. See the work of Cheeger [6] and of Keith [12].

In our exposition, we follow the structure of the proof in [5]. We rephrase the proof purely in terms of Lipschitz functions. We use an upper gradient quantity, namely the *pointwise Lipschitz constant*, as a replacement for a gradient. This is the key feature that allows us to generalise the proof in [5].

The structure of this paper is as follows. In §2, we state the hypotheses (H1)-(H8) under which we obtain the quadratic estimates and state the main results. We devote §3 to illustrating some important consequences of the dyadic decomposition in [7]. In §4, we present some results about Carleson measures and maximal functions on doubling measure metric spaces. These tools are crucial since the proof of the main result proceeds by reducing the main estimate to a Carleson measure estimate. Lastly, we give a proof of the main theorem in §5, taking care to avoid unnecessary repetition of the work of [5] and [14], and highlight the key differences which we have introduced.

2. HYPOTHESES AND THE MAIN RESULTS

We list a set of hypotheses (H1)-(H8). These assumptions are similar those in [5], with the exception of (H6) and (H8) which require modification due to the lack of a differentiable structure. The assumptions (H1)-(H3) are purely operator theoretic and thus hold in sufficient generality. They are taken in verbatim from [5] but we list them here for completeness. We emphasise that here, $\mathscr{H}$ denotes an abstract Hilbert space.

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \to \mathscr{H}$ is closed, densely defined and *nilpotent* ($\Gamma^2 = 0$).

(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\begin{aligned} &\mathrm{Re}\, \langle B_1 u, u\rangle \geq \kappa_1 \|u\| &&\text{whenever } u \in \mathcal{R}(\Gamma^*) \\ &\mathrm{Re}\, \langle B_2 u, u\rangle \geq \kappa_2 \|u\| &&\text{whenever } u \in \mathcal{R}(\Gamma) \end{aligned}$$

where $\kappa_1, \kappa_2 > 0$ are constants.

(H3) The operators B_1, B_2 satisfy $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.

The full implications of these assumptions are listed in [5, §4]. However, for the sake of convenience, we include some relevant details from this reference. Define $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$. Furthermore, define the following associated bounded operators:

$$R_t^B = (1 + it\Pi_B)^{-1},\ P_t^B = (1 + t^2\Pi_B^2)^{-1},\ Q_t^B = t\Pi_B(1 + t^2\Pi_B^2)^{-1},\ \Theta_t^B = t\Gamma_B^*(1 + t^2\Pi_B^2)^{-1},$$

and write R_t, P_t, Q_t, Θ_t by setting $B_1 = B_2 = 1$. With this in mind, we bring the attention of the reader to the following important proposition.

Proposition 2.1 (Proposition 4.8 of [5]). *Suppose that* (Γ, B_1, B_2) *satisfy the hypotheses (H1)-(H3) and that there exists* $c > 0$ *such that*

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq c \left\|u\right\|^2$$

for all $u \in \mathcal{R}(\Gamma)$*, together with three similar estimates obtained by replacing* (Γ, B_1, B_2) *by* (Γ^*, B_2, B_1)*,* $(\Gamma^*, {B_2}^*, {B_1}^*)$ *and* $(\Gamma, {B_1}^*, {B_2}^*)$*. Then,* Π_B *satisfies*

$$\int_0^\infty \left\|Q_t^B u\right\|^2 \frac{dt}{t} \simeq \left\|u\right\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$*. Thus,* Π_B *has a bounded* H^∞ *functional calculus.*

For a fuller treatment of the theory of sectorial operators and holomorphic functional calculi, see [1] by Albrecht, Duong and McIntosh, and [11] by Kato. Furthermore, Morris deals with local quadratic estimates and their functional calculus implications in [15].

It is the conclusion of the above proposition that is our primary objective. We note as do the authors of [5] that we require additional assumptions on $\mathcal{X}$ and (Γ, B_1, B_2) in order to satisfy the hypothesis of the proposition. Thus, we start with the following definition.

Definition 2.2 (Doubling measure). *We say that* μ *is a doubling measure on* $\mathcal{X}$ *if there exists a constant* $C_D \geq 1$ *such that*

$$0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) < \infty.$$

We call C_D *the doubling constant and we let* $p = \log_2(C_D)$*.*

It is, in fact, easy to show that a measure is doubling if and only if $\mu(B(x, \kappa r)) \leq C_D \kappa^p \mu(B(x, r))$ for $\kappa > 1$.

We are now in a position to list (H4) and (H5).

- (H4) Let $\mathcal{X}$ be a complete, connected metric space and μ a Borel-regular measure on $\mathcal{X}$ that is doubling. Then set $\mathscr{H} = \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N; d\mu)$.
- (H5) $B_i \in L^\infty(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$ for $i = 1, 2$.

For convenience, we sometimes write $\mathscr{H} = \mathrm{L}^2(\mathcal{X})$ or $\mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

Note that the two hypotheses above are are the obvious adaptations of (H4) and (H5) in [5]. The matter of (H6) is a little more complicated since (H6) of [5] and [14] involves ∇ which in general does not exist for us. To circumvent this obstacle, we define the following quantity.

Definition 2.3 (Pointwise Lipschitz constant). *For* $\xi : \mathcal{X} \to \mathbb{C}^N$ *Lipschitz, define* $\operatorname{Lip} \xi : \mathcal{X} \to \mathbb{R}$ *by*

$$\operatorname{Lip} \xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x, y)}.$$

We take the convention that $\operatorname{Lip} \xi(x) = 0$ *when* x *is an isolated point.*

Letting $\mathbf{Lip}\,\xi$ denote the *Lipschitz constant* of ξ, we note that by construction, $\mathrm{Lip}\,\xi(x) \leq \mathbf{Lip}\,\xi$ for all $x \in \mathcal{X}$. Also, $\mathrm{Lip}\,\xi$ is a Borel function and therefore measurable. Many of the properties of $\mathrm{Lip}\,\xi$ are described in greater detail in [6]. We note that it is from this reference that we have borrowed this notation and the term pointwise Lipschitz constant.

(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by ξ preserves $\mathcal{D}(\Gamma)$ and $\mathrm{M}_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $|\mathrm{M}_\xi(x)| \leq m\,|\mathrm{Lip}\,\xi(x)|$ for almost all $x \in \mathcal{X}$.

We note that this implies the same hypothesis when Γ is replaced by Γ^* and Π. This observation is made in [14] and originated in [4].

When $\mathcal{X} = \mathbb{R}^n$ and μ is the Lebesgue measure (the setting in [5]), our (H6) is automatically satisfied since $|\nabla\xi(x)| = |\mathrm{Lip}\,\xi(x)|$ for almost all $x \in \mathbb{R}^n$.

The following is called the *cancellation hypothesis*. In the work of [14] and [4], this hypothesis is replaced by a weaker estimate which is applicable for local quadratic estimates [15]. The estimates we require are global and thus we assume the cancellation hypothesis in [5]. We denote the *support* of a function f by $\mathrm{spt}\, f$.

(H7) For each open ball B, we have

$$\int_B \Gamma u \; d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \; d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\, u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\, v \subset B$.

The last assumption we make is a *Poincaré hypothesis*. In [14] a Poincaré inequality on balls is assumed as a separate hypothesis. Their (H8) is a coercivity assumption following [5]. In our work, we find that a Poincaré type hypothesis with respect to the unperturbed operator Π is a sensible substitution.

(H8) There exists $C' > 0$ and $c > 0$ such that for all balls $B = B(y, r)$

$$\int_B |u(x) - u_B|^2 \; d\mu(x) \leq C' r^2 \int_{cB} |\Pi u(x)|^2 \; d\mu(x)$$

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$.

The authors of [5] reveal that (H1)-(H3) are adequate to set up the necessary operator theoretic framework. However, as we have noted before, the full set of assumptions (H1)-(H8) are necessary to obtain the desired estimates. It is under these assumptions that we present the main theorem of this paper.

Theorem 2.4. *Let $\mathcal{X}$, (Γ, B_1, B_2) satisfy (H1)-(H8). Then, Π_B satisfies the quadratic estimate*

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \; \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$ and hence has a bounded H^∞ functional calculus.

Let $E_B^\pm = \chi^\pm(\Pi_B)$, where $\chi^+(\zeta) = 1$ when $\mathrm{Re}\,(\zeta) > 0$ and 0 otherwise, and similarly, $\chi^-(\zeta) = 1$ when $\mathrm{Re}\,(\zeta) < 0$ and 0 otherwise. We have the following corollary resembling [5, Corollary 2.11].

Corollary 2.5 (Kato square root type estimate)**.**

(i) *There is a spectral decomposition*

$$\mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$$

(where the sum is in general non-orthogonal), and

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$ *with*

$$\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \left\| \sqrt{\Pi_B^2} u \right\|$$

for all $u \in \mathcal{D}(\Pi_B)$*.*

3. Abstract dyadic decomposition

We begin this section by quoting the following [7, Theorem 11].

Theorem 3.1. *There exists a countable collection of open subsets* $\left\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\right\}$ *with each* $z_\alpha^k \in Q_\alpha^k$*, where* I_k *are index sets (possibly finite), and constants* $\delta \in (0,1)$*,* $a_0 > 0$*,* $\eta > 0$ *and* $C_1, C_2 < \infty$ *satisfying:*

(i) *For all* $k \in \mathbb{Z}$*,* $\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$*,*
(ii) *If* $l \geq k$*, either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$*,*
(iii) *For each* (k,α) *and each* $l < k$ *there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$*,*
(iv) $\operatorname{diam} Q_\alpha^k \leq C_1 \delta^k$*,*
(v) $B(z_\alpha^k, a_0\delta^k) \subset Q_\alpha^k$*,*
(vi) *For all* k, α *and for all* $t > 0$*,* $\mu\left\{x \in Q_\alpha^k : d(x, \mathcal{X} \setminus Q_\alpha^k) \leq t\delta^k\right\} \leq C_2 t^\eta \mu(Q_\alpha^k)$*.*

Define $\mathscr{Q}^k = \left\{Q_\alpha^k : \alpha \in I_k\right\}$ to be the *level* k dyadic cubes and $\mathscr{Q} = \cup_k \mathscr{Q}^k$ to be the collection of dyadic cubes. For $Q_\alpha^k \in \mathscr{Q}^k$, define the *length* as $\ell(Q_\alpha^k) = \delta^k$ and the *centre* as z_α^k.

It is easy to see that each $\mathscr{Q}^k$ is a mutually disjoint collection. Furthermore, we have $\partial(\cup\mathscr{Q}^k) = \cup_{Q \in \mathscr{Q}^k} \partial Q$. These facts coupled with the assumption $\mu(B(x,r)) > 0$ implies that $\mathcal{X} = \overline{\cup\mathscr{Q}^k}$.

Fix a cube $Q \in \mathscr{Q}^j$ and denote the centre of this cube by z. We are interested in counting the number of cubes inside "shells" centred from this cube. We begin with the following definition.

Definition 3.2. *Whenever* $k \geq 1$*, define* $\mathscr{C}_k = \left\{Q_\alpha^j \in \mathscr{Q}^j : (k-1)C_1\delta^j \leq d(z, z_\alpha^j) \leq kC_1\delta^j\right\}$*. Also, let* $\tilde{\mathscr{C}}_k = \left\{Q_\alpha^j \in \mathscr{Q}^j : d(z, z_\alpha^j) \leq kC_1\delta^j\right\}$*.*

It is easy to see that $\mathscr{Q}^j = \cup_{k \geq 1} \mathscr{C}_k$. We compute a bound for $\operatorname{card} \mathscr{C}_k$ (where $\operatorname{card} S$ denotes the *cardinality* of a set S). First, we have the following proposition describing the distance of points in $\cup\mathscr{C}_k$ to z.

Proposition 3.3. *Let* $Q_\alpha^j \in \mathscr{C}_k$*. Then,*

(i) $0 \leq d(z,x) \leq (k+1)C_1\delta^j$ *for all* $x \in Q^j_\alpha$ *when* $k \leq 2$, *and*
(ii) $\frac{1}{3}kC_1\delta^j \leq d(z,x) \leq (k+1)C_1\delta^j$ *for all* $x \in Q^j_\alpha$ *when* $k \geq 3$.

Proof. Fix $Q^j_\alpha \in \mathscr{C}_k$ and fix $x \in Q^j_\alpha$. Then,

$$d(x,z) \leq d(x,z^j_\alpha) + d(z^j_\alpha, z) \leq \operatorname{diam} Q^j_\alpha + kC_1\delta^j \leq (k+1)C_1\delta^j.$$

Also,

$$(k-1)C_1\delta^j \leq d(z,z^j_\alpha) \leq d(x,z) + d(x,z^j_\alpha) \leq d(x,z) + C_1\delta^j.$$

Combining these two estimates we have

$$(k-2)C_1\delta^j \leq d(z,x) \leq (k+1)C_1\delta^j.$$

This gives us (i). To obtain (ii), note that whenever $k \geq 3$ we have $\frac{1}{3}k \leq k-2$. □

Next, we compare two balls which are separated by an arbitrary distance.

Proposition 3.4. *Fix balls* $B(x,r), B(y,r) \subset \mathcal{X}$. *Then, for all* $\varepsilon > 0$,

$$2^{-p}\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^{-p}\mu(B(y,r)) \leq \mu(B(x,r)) \leq 2^p\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^p\mu(B(y,r)).$$

Proof. Fix $\varepsilon > 0$ and note that $B(x,r), B(y,r) \subset B(x, d(x,y)+r+\varepsilon), B(y, d(x,y)+r+\varepsilon)$. Therefore,

$$\mu(B(y,r)) \leq \mu\left(B\left(x, \frac{d(x,y)+r+\varepsilon}{r}r\right)\right) \leq 2^p\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^p\mu(B(x,r)).$$

Similarly, we have

$$\mu(B(x,r)) \leq \mu\left(B\left(y, \frac{d(x,y)+r+\varepsilon}{r}r\right)\right) \leq 2^p\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^p\mu(B(y,r))$$

which establishes the claim. □

We make a parenthetical remark that our assumption $0 < \mu(B(x,r)) < \infty$ for all $x \in \mathcal{X}$ and $r > 0$ is not strong since the previous proposition, along with the doubling property, allows us to recover this assumption if we only required $0 < \mu(B(x_0,r_0)) < \infty$ to hold for some $x_0 \in \mathcal{X}$ and $r_0 > 0$.

We now return back to the problem of estimating $\operatorname{card}\mathscr{C}_k$. The reader will observe that we have been generous in our calculations.

Proposition 3.5. *We have* $\operatorname{card}\tilde{\mathscr{C}}_k \leq Ck^{2p}$ *where*

$$C = 4^p\left(\frac{C_1+2a_0}{a_0}\right)^p\left(\frac{2C_1}{a_0}\right)^p.$$

In particular, $\operatorname{card}\mathscr{C}_k \leq Ck^{2p}$.

Proof. Fix $k \geq 1$. Set $\varepsilon = r = a_0\delta^j$ and then

$$d(z,z^j_\alpha) + r + \varepsilon \leq kC_1\delta^j + 2a_0\delta^j \leq (C_1+2a_0)\delta^j k$$

when $Q^j_\alpha \in \tilde{\mathscr{C}}_k$. By Proposition 3.4,

$$2^{-p}\left(\frac{C_1+2a_0}{a_0}\right)^{-p}k^{-p}\mu(B(z,a_0\delta^j)) \leq \mu(B(z^k_\alpha, a_0\delta^j)).$$

Now, note that by Proposition 3.3, we have $\sup_{x\in Q_\alpha^j} d(x,z) \leq (k+1)C_1\delta^j$ and so $\cup\tilde{\mathscr{C}}_k \subset B(z,(k+1)C_1\delta^j)$. Then,

$$\mu(B(z,(k+1)C_1\delta^j)) \leq 2^p\left(\frac{(k+1)C_1}{a_0}\right)^p \mu(B(z,a_0\delta^j)) \leq 2^p\left(\frac{2C_1}{a_0}\right)^p k^p\mu(B(z,a_0\delta^j)).$$

Since $\mu(B(z,a_0\delta^j)) < \infty$ and by combining the two estimates, and the fact that $B(z_\alpha^k,a_0\delta^j) \subset Q_\alpha^j$ for each $Q_\alpha^j \in \tilde{\mathscr{C}}_k$, we compute

$$\operatorname{card}\mathscr{C}_k \leq 2^p\left(\frac{2C_1}{a_0}\right)^p k^p\, 2^p\left(\frac{C_1+2a_0}{a_0}\right)^p k^p = 4^p\left(\frac{C_1+2a_0}{a_0}\right)^p\left(\frac{2C_1}{a_0}\right)^p k^{2p}.$$

The observation that $\mathscr{C}_k \subset \tilde{\mathscr{C}}_k$ completes the proof. □

We have the following important consequences. They are useful in many of the calculations in §5. Following the notation of [5], we write $\langle x\rangle = 1+|x|$.

Corollary 3.6. *Fix $\delta^{j+1} < t \leq \delta^j$ and a cube $Q \in \mathscr{Q}^j$. Then,*

$$\sum_{R\in\mathscr{Q}^j}\left\langle\frac{\operatorname{dist}(R,Q)}{t}\right\rangle^{-M} \leq C\left(1+4^p+\left(\frac{3}{C_1}\right)^M\sum_{k=3}^\infty k^{2p-M}\right)$$

with C being the constant in the previous proposition.

Proof. First, we note that

$$1, \frac{\operatorname{dist}(R,Q)}{\delta^j} \leq 1+\frac{\operatorname{dist}(R,Q)}{t}.$$

Then,

$$\begin{aligned}\sum_{R\in\mathscr{Q}^j}\left\langle\frac{\operatorname{dist}(R,Q)}{t}\right\rangle^{-M} &\leq \operatorname{card}\mathscr{C}_1+\operatorname{card}\mathscr{C}_2+\sum_{k=3}^\infty\sum_{R\in\mathscr{C}_k}\left(\frac{\delta^j}{d(R,Q)}\right)^M \\ &\leq C+C2^{2p}+\sum_{k=3}^\infty\operatorname{card}\mathscr{C}_k\left(\frac{\delta^j}{\frac{1}{3}kC_1\delta^j}\right)^M \leq C\left(1+4^p+\left(\frac{3}{C_1}\right)^M\sum_{k=3}^\infty k^{2p-M}\right).\end{aligned}$$

□

Corollary 3.7. *For each $M > 2p+1$, there exists a constant $A_M > 0$ such that*

$$\sup_Q\sum_{R\in\mathscr{Q}^j}\left\langle\frac{\operatorname{dist}(R,Q)}{t}\right\rangle^{-M} \leq A_M.$$

4. MAXIMAL FUNCTIONS AND CARLESON MEASURES

A full treatment of the classical theory of maximal functions and Carleson measures can be found in the work of Stein [16, §4]. The objects of interest that we define in this section are taken from this book *mutatis mutandis*. Furthermore, we refer the reader to the books of Heinonen [9] and Coifman and Weiss [8] for two excellent expositions that touches on some of the issues and ideas presented here.

For a measurable subset S with $0 < \mu(S) < \infty$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X},\mathbb{C}^N)$, we define the *average* of f on S by $\fint_S f = \mu(S)^{-1}\int_S f$. Then, we make the following definition.

Definition 4.1 (Maximal function)**.** *Let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}, \mathbb{C}^N)$*. Define the uncentred maximal function of* f *by:*

$$\mathcal{M}f(x) = \sup_{B \ni x} \fint_B |f| \ d\mu$$

where the supremum is taken over all balls B *containing* x*.*

We want to deduce that this $\mathcal{M}$ exhibits a weak type $(1,1)$ estimate and is bounded in $\mathrm{L}^p(\mathcal{X}, \mathbb{C}^N)$ for $p > 1$. The proof of the following theorem is standard via a Vitali type covering theorem [8, Theorem 1.2].

Theorem 4.2 (Maximal theorem)**.** *There exists a constant* $C_1 > 0$ *such that whenever* $f \in \mathrm{L}^1(\mathcal{X}, \mathbb{C}^N)$*, we have*

$$\mu(\{x \in \mathcal{X} : \mathcal{M}f(x) > \alpha\}) \leq \frac{C_1}{\alpha} \int_{\mathcal{X}} |f| \ d\mu.$$

Whenever $f \in \mathrm{L}^q(\mathcal{X}, \mathbb{C}^N)$ *with* $q > 1$*,*

$$\|\mathcal{M}f\|_q \leq C_q \|f\|_q$$

where $C_q > 0$ *is a constant.*

In order to set up a theory of Carleson measures, we require an *upper half space.* We define this to be $\mathcal{X}_+ = \mathcal{X} \times \mathbb{R}^+$ where $\mathbb{R}^+ = (0, \infty)$. The *cone* over a point $x \in \mathcal{X}$ is then defined as $\Gamma(x) = \{(y,t) \in \mathcal{X}_+ : d(x,y) < t\}$ and this leads to the following.

Definition 4.3 (Nontangential maximal function)**.** *Let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}_+, \mathbb{C}^N)$*. Define*

$$\mathcal{M}^* f(x) = \sup_{(y,t) \in \Gamma(x)} |f(y,t)| .$$

Like its classical counterpart, this maximal function is measurable. This is the content of the following proposition.

Proposition 4.4. *The set* $\{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$ *is open and hence* $\mathcal{M}^* f$ *is measurable.*

Proof. Fix $x \in \mathcal{X}$ with $\mathcal{M}^* f(x) > \alpha$. Then, there exists a $(y,t) \in \Gamma(x)$ such that $|f(y,t)| > \alpha$. Consider the ball $B(y,t)$ and take any $z \in B(y,t)$. Note that since $d(z,y) < t$ we have $(y,t) \in \Gamma(z)$ and so $\mathcal{M}^* f(z) > \alpha$. Therefore, $x \in B(y,t) \subset \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$. ☐

Therefore, we define the following function space in an analogous way to the classical theory.

Definition 4.5 (Nontangential function space)**.** *Let* $\mathcal{N}$ *denote the space of Borel measurable functions* $f : \mathcal{X}_+ \to \mathbb{C}$ *such that* $\mathcal{M}^* f \in \mathrm{L}^1(\mathcal{X})$*. We equip this space with the norm* $\|f\|_{\mathcal{N}} = \|\mathcal{M}^* f\|_1$*.*

Now, let $B = B(x,r)$ and define the *tent over* B as $\mathrm{T}(B) = \{(y,t) \in \mathcal{X}_+ : d(x,y) \leq r - t\}$. For an arbitrary open set $O \subset \mathcal{X}$, we define the *tent over* O by $\mathrm{T}(O) = \mathcal{X}_+ \setminus \cup_{x \in \mathcal{X} \setminus O} \Gamma(x)$. The following is an equivalent characterisation of $\mathrm{T}(O)$.

Proposition 4.6. *Whenever* $(x,t) \in \mathrm{T}(O)$ *we have that* $(x,t) \in \mathrm{T}(B(x, d(x, \mathcal{X} \setminus O)))$ *and in particular,* $\mathrm{T}(O) = \cup_{x \in O} \mathrm{T}(B(x, d(x, \mathcal{X} \setminus O)))$*.*

Proof. First, note that de Morgen immediately gives us $\mathrm{T}(O) = \cap_{y\in\mathscr{X}\setminus O}\mathscr{X}_+ \setminus \Gamma(y)$. Fix $(x,t) \in \mathrm{T}(O)$. So, $(x,t) \in \mathscr{X}_+ \setminus \Gamma(y)$ for all $y \in \mathscr{X} \setminus O$. That is, for all $y \notin O$, we have $(x,t) \notin \Gamma(y)$ which implies $d(x,y) \geq t$. Therefore, $d(x,\mathscr{X}\setminus O) \geq t$. Then, by the definition of $\mathrm{T}(B(x,r))$ and setting $r = d(x,\mathscr{X}\setminus O)$, we conclude $(x,t) \in \mathrm{T}(B(x,d(x,\mathscr{X}\setminus O)))$. The converse inclusion is easy since $B(x,d(x,\mathscr{X}\setminus O)) \subset O$. □

Definition 4.7 (Carleson function)**.** *Let ν be any Borel measure on $\mathscr{X}_+$. Define*

$$\mathrm{C}(\nu)(x) = \sup_{B\ni x}\frac{\nu(\mathrm{T}(B))}{\mu(B)}.$$

Definition 4.8 (Space of Carleson measures)**.** *We define $\mathcal{C}$ to be the space of measures ν that are Borel on $\mathscr{X}_+$ and such that $\mathrm{C}(\nu)$ is bounded. Such a measure is called a* Carleson measure *and we define*

$$\|\nu\|_{\mathcal{C}} = \sup_{x\in\mathscr{X}}\mathrm{C}(\nu)(x)$$

to be the Carleson norm*.*

Since we have a dyadic structure, we can define the *Carleson box* over $Q \in \mathscr{Q}$ by $\mathrm{R}_Q = \overline{Q} \times (0,\ell(Q)]$. Unlike the classical definition, we are forced to take $\overline{Q}$ since $\mathscr{Q}$ is only guaranteed to cover $\mathscr{X}$ almost everywhere. The importance of this subtlety will become apparent in the proof of the following proposition that provides an alternative characterisation of a Carleson measure.

Proposition 4.9. *Let ν be a Borel measure on $\mathscr{X}_+$. Then the statement*

$$\sup_B\frac{\nu(\mathrm{T}(B))}{\mu(B)} < \infty \qquad \textit{for every ball } B$$

is equivalent to the statement

$$\sup_Q\frac{\nu(\mathrm{R}_Q)}{\mu(Q)} < \infty \qquad \textit{for every } Q \in \mathscr{Q}.$$

Proof. First, fix $Q \in \mathscr{Q}^j$ and let x_Q be its centre. Then, we have that $Q \subset B(x_Q, C_1\delta^j)$. Then, certainly, $\mathrm{R}_Q \subset \mathrm{T}(B(x_Q,(C_1+2)\delta^j))$. So,

$$\begin{aligned}\nu(\mathrm{R}_Q) &\leq \nu(\mathrm{T}(B(x_Q,(C_1+2)\delta^j))) \leq \|\nu\|_{\mathcal{C}}\,\mu(B(x_Q,(C_1+2)\delta^j)\\ &\leq 2^p\left(\frac{C_1+2}{a_0}\right)^p\|\nu\|_{\mathcal{C}}\,\mu(B(x_Q,a_0\delta^j)) \leq 2^p\left(\frac{C_1+2}{a_0}\right)^p\|\nu\|_{\mathcal{C}}\,\mu(Q).\end{aligned}$$

The converse is harder. Fix $B = B(x,r)$ and let $j \in \mathbb{Z}$ such that $\delta^{j+1} < r \leq \delta^j$. Let $N(B) = \{Q \in \mathscr{Q}^j : Q \cap B \neq \varnothing\}$. It is an easy fact that $N(B) \neq \varnothing$.

(i) First, we claim that $B \subset \cup_{Q\in N(B)}\overline{Q}$. Suppose $y \in B$ but $y \notin \cup N(B)$. That is, $y \notin Q$ for all $Q \in \mathscr{Q}^j$. Thus, there exists a $Q \in \mathscr{Q}^j$ such that $y \in \partial Q$. That is, for every $\varepsilon > 0$, $B(y,\varepsilon) \cap Q \neq \varnothing$. But there exists an $\varepsilon > 0$ such that $B(y,\varepsilon) \subset B$, and so $Q \cap B \neq \varnothing$. This means that $Q \in N(B)$ and establishes the claim.

(ii) Fix $Q \in N(B)$ as a reference cube and let $Q' \in N(B)$ be any other cube. Since $r < \delta^j$, we note that $d(x,x_Q), d(x,x_{Q'}) \leq \delta^j + C_1\delta^j$. Therefore, $d(x_Q,x_{Q'}) \leq 2(C_1+1)\delta^j$. That is, all the centres of cubes $Q' \in N(B)$ are inside the ball $B(x_Q, 2(C_1+1)\delta^j)$ and hence $\tilde{\mathscr{C}}_{2(C_1+1)}$. Thus, by Proposition 3.5,

$$\operatorname{card} N(B) \leq \operatorname{card}\tilde{\mathscr{C}}_{2(C_1+1)} \leq C2^p(C_1+1)^{2p}.$$

(iii) Now, suppose that $(y,t) \in \mathrm{T}(B)$. That is, $y \in B$ and We have $d(y,t) \leq r - t \leq \delta^j$. By (i), there exists a cube $Q \in N(B)$ such that $y \in \overline{Q}$. Therefore, $(y,t) \in \mathrm{R}_Q = \overline{Q}$ and shows that $\mathrm{T}(B) \subset \cup_{Q \in N(B)} \mathrm{R}_Q$.

(iv) Fix $Q \in N(B)$ and so $d(x, x_Q) \leq (C_1 + 1)\delta^j$. Set $\varepsilon = r = \delta^{j+1}$ in Proposition 3.4 so that

$$\begin{aligned} \mu(B(x_Q, \delta^{j+1})) &\leq 2^p \left(\frac{(C_1+1)\delta^j + 2\delta^{j+1}}{\delta^{j+1}} \right) \mu(B(x, \delta^{j+1})) \\ &\leq 2^p((C_1+1)\delta^{-1} + 2)^p \mu(B(x,r)). \end{aligned}$$

Now, by combining (i) - (iv),

$$\begin{aligned} \nu(\mathrm{T}(B)) \leq \sum_{Q \in N(B)} \nu(\mathrm{R}_Q) &\lesssim \sum_{Q \in N(B)} \mu(B(x_Q, C_1\delta^j)) \\ &\lesssim \sum_{Q \in N(B)} \mu(B(x_Q, \delta^{j+1})) \lesssim \operatorname{card} N(B) \mu(B(x,r)) \lesssim \mu(B(x,r)) \end{aligned}$$

which completes the proof. □

We quote the following covering theorem of Whitney [8, Theorem 1.3].

Theorem 4.10 (Whitney Covering Theorem)**.** *Let* $O \subsetneq \mathcal{X}$ *be open. Then, there exists a set of balls* $\mathscr{E} = \{B_j\}_{j \in \mathbb{N}}$ *and a constant* $c_1 < \infty$ *independent of* O *such that*

(i) The balls in $\mathscr{E}$ *are mutually disjoint,*
(ii) $O = \bigcup_{j \in \mathbb{N}} c_1 B_j$*,*
(iii) $4c_1 B_j \not\subset O$*.*

This allows us to prove the following theorem of Carleson.

Theorem 4.11 (Carleson's Theorem)**.** *Let* $f \in \mathcal{N}$ *and* $\nu \in \mathcal{C}$*. Then,*

$$\iint_{\mathcal{X}_+} |f(x,t)| \ d\nu(x,t) \lesssim \|f\|_{\mathcal{N}} \|\nu\|_{\mathcal{C}}$$

where the constant depends only on p *and the Whitney constant* c_1*.*

Proof. (i) We prove that $\{(x,t) \in \mathcal{X}_+ : |f(x,t)| > \alpha\} \subset \mathrm{T}(E_\alpha)$ where $E_\alpha = \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$. Fix $(x,t) \in \mathcal{X}_+$ such that $|f(x,t)| > \alpha$. Then, whenever $y \in B(x,t)$, we also have $x \in B(y,t)$ and

$$\mathcal{M}^* f(y) = \sup_{t>0} \sup_{z \in B(y,t)} |f(z,t)| > |f(x,t)| > \alpha.$$

Therefore, $B(x,t) \subset E_\alpha$ and $(x,t) \in \mathrm{T}(B(x,t)) \subset \mathrm{T}(E_\alpha)$.

(ii) Let $O \subsetneq \mathcal{X}$ be an open set, and let $\mathscr{E} = \{B_j\}_{j \in \mathbb{N}}$ be the Whitney covering guaranteed by Theorem 4.10. We prove that

$$\mathrm{T}(O) \subset \cup_j \mathrm{T}(9c_1 B_j).$$

Fix $x \in O$ and let $(x,t) \in T(B(x, d(x, \mathcal{X} \setminus O)))$. Then, there exists a ball $B_j = B_j(x_j, r_j) \in \mathscr{E}$ such that $x \in c_1 B_j$. Let $y \in B(x, d(x, \mathcal{X} \setminus O))$. Since $4c_1 B_j \cap \mathcal{X} \setminus O$, for any $z \in \mathcal{X} \setminus O$ $d(y, \mathcal{X} \setminus O) \leq d(x,z) \leq 8c_1 r_j$ Then,

$$d(y, x_j) \leq d(y,x) + d(x, x_k) \leq d(x, \mathcal{X} \setminus O) + d(x, x_k) < 8c_1 r_j + c_1 r_j = 9c_1 r_j.$$

This proves that $B(x, d(x, \mathcal{X} \setminus O)) \subset 9c_1 B_j$ and so $\mathrm{T}(B(x, d(x, \mathcal{X} \setminus O))) \subset \mathrm{T}(9c_1 B_j)$. We apply Proposition 4.6 to conclude that $\mathrm{T}(O) \subset \cup_j \mathrm{T}(9c_1 B_j)$.

(iii) Now, we prove that there exists a constant $C > 0$ such that for all open sets $O \subset \mathcal{X}$,

$$\nu(\mathrm{T}(O)) \leq C \|\nu\|_{\mathcal{C}} \, \mu(O).$$

First assume that $O = \mathcal{X}$. If $\mu(\mathcal{X}) = \infty$, then there is nothing to prove. So suppose otherwise. Now, for any $x \in \mathcal{X}$ and any ball $B_r = B(x, r)$,

$$\frac{1}{\mu(B_r)} \nu(\mathrm{T}(B_r)) \leq \mathrm{C}(\nu)(x) \leq \|\nu\|_{\mathcal{C}}$$

and therefore, $\nu(\mathrm{T}(B_r)) \leq \|\nu\|_{\mathcal{C}} \, \mu(\mathcal{X})$ for every ball B_r of radius r. Now, $\chi_{\mathrm{T}(B_n)} \leq 1$ for each $n \in \mathbb{N}$ and $\chi_{\mathrm{T}(B_n)} \to \chi_{\mathrm{T}(\mathcal{X})}$ and $n \to \infty$ pointwise. Then, by application of Dominated Convergence Theorem,

$$\nu(\mathrm{T}(\mathcal{X})) = \int_{\mathcal{X}_+} \lim_{n \to \infty} \chi_{\mathrm{T}(B_n)} \, d\nu = \lim_{n \to \infty} \int_{\mathcal{X}_+} \chi_{\mathrm{T}(B_n)} \, d\nu \leq \|\nu\|_{\mathcal{C}} \, \mu(\mathcal{X}).$$

Now, consider the case when $O \subsetneqq \mathcal{X}$. Then, by (ii) and the subadditivity of the measure,

$$\begin{aligned}\nu(\mathrm{T}(O)) &\leq \sum_j \nu(\mathrm{T}(9c_1 B_j)) \leq \|\nu\|_{\mathcal{C}} \sum_j \mu(9c_1 B_j) \\ &\leq 2^p (9c_1)^p \|\nu\|_{\mathcal{C}} \sum_j \mu(B_j) \leq (18c_1)^p \|\nu\|_{\mathcal{C}} \, \mu(O).\end{aligned}$$

(iv) By (i) and (iii),

$$\nu \{(x,t) \in \mathcal{X}_+ : |f(x,t)| > \alpha\} \lesssim \|\nu\|_{\mathcal{C}} \, \mu \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$$

and integrating both sides with respect to α completes the proof.

□

5. HARMONIC ANALYSIS OF Π_B

Let $\mathscr{Q}_t = \mathscr{Q}^j$ for $\delta^{j+1} < t \leq \delta^j$. Following the structure of the proof in [5], for $t \in \mathbb{R}^+$, we define the *dyadic averaging operator* $\mathcal{A}_t : \mathscr{H} \to \mathscr{H}$ as

$$\mathcal{A}_t(x) = \sum_{Q \in \mathscr{Q}_t} \chi_Q(x) \fint_Q u \, d\mu$$

when $x \in \cup \mathscr{Q}_t$ and 0 elsewhere. A straightforward calculation shows that $\mathcal{A}_t \in \mathcal{L}(\mathscr{H})$ and $\|\mathcal{A}_t\| \leq 1$ uniformly in t. Then, the *principal part* is defined as $\gamma_t(x) w = (\Theta_t^B \omega)(x)$ for $w \in \mathbb{C}^N$ and where $\omega(x) = w$ for all $x \in \mathcal{X}$.

Following [5], to prove Theorem 2.4 as a consequence of Proposition 2.1, we need to show that

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for $u \in \mathcal{R}(\Pi)$. Thus, we follow the paradigm in [5], [4] and [14] and decompose this problem in the following way:

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq \int_0^\infty \left\|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t u\right\|^2 \frac{dt}{t} \\ + \int_0^\infty \left\|\gamma_t \mathcal{A}_t (P_t - I)u\right\|^2 \frac{dt}{t} + \iint_{\mathcal{X}_+} |\mathcal{A}_t u(x)|^2 |\gamma_t(x)|^2 \frac{d\mu(x)dt}{t}.$$

The purpose of the first two terms is to reduce the estimate down to the third term which can be dealt with a Carleson measure estimate.

5.1. **Off Diagonal Estimates.**

The following lemma is a primary tool in our argument. Certainly, it was known to the authors of [5] since they use a similar result in the proof of their Proposition 5.2. The key difference is that we use $\mathbf{Lip}\,\xi$ instead of $\|\nabla\xi\|_\infty$ to control the "slope" of our cutoff. Furthermore, this lemma is used in the sequel to construct Lipschitz substitutions where [5], [4] and [14] use smooth cutoff functions. We include a detailed proof of this lemma since it is central to our work.

Lemma 5.1 (Lipschitz separation lemma)**.** *Let (X, d) be a metric space and suppose $E, F \subset X$ satisfy $d(E, F) > 0$. Then, there exists a Lipschitz function $\eta : X \to [0, 1]$, and a set $\tilde{E} \supset E$ with $d(\tilde{E}, F) > 0$ such that*

$$\eta|_E = 1, \quad \eta|_{X\setminus\tilde{E}} = 0 \quad \textit{and} \quad \mathbf{Lip}\,\eta \leq 4/d(E, F).$$

Proof. Define $\tilde{E} = \{x \in X : d(x, E) < 1/4d(E, F)\}$. By construction, $E \subset \tilde{E}$ and from the triangle inequality for d and taking infima,

$$d(\tilde{E}, F) + \sup_{x\in\tilde{E}} d(x, E) \geq d(E, F),$$

and since $\sup_{x\in\tilde{E}} d(x, E) \leq 1/4d(E, F)$, it follows that $d(\tilde{E}, F) \geq 3/4d(E, F) > 0$.

Now, define:

$$\eta(x) = \begin{cases} 1 - \frac{4d(x,E)}{d(E,F)} & x \in \tilde{E} \\ 0 & x \notin \tilde{E} \end{cases}.$$

We consider the three possible cases.

(i) First, suppose that $x, y \notin \tilde{E}$. Then,

$$|\eta(x) - \eta(y)| = 0 \leq \frac{4d(x, y)}{d(E, F)}.$$

(ii) Now, suppose that $x, y \in \tilde{E}$. By the triangle inequality, we have $d(x, z) \leq d(x, y) + d(y, z)$ and by taking an infima over $z \in E$ and invoking the symmetry of distance, $|d(x, E) - d(y, E)| \leq d(x, y)$. Therefore,

$$|\eta(x) - \eta(y)| = \left|\frac{1 - 4d(x, E)}{d(E, F)} - 1 + \frac{4d(y, E)}{d(E, F)}\right| \\ = \frac{4}{d(E, F)} |d(x, E) - d(y, E)| \leq \frac{4}{d(E, F)} d(x, y).$$

(iii) Lastly, suppose that $x \in \tilde{E}$ and $y \notin \tilde{E}$. Then $\eta(y) = 0$ and since $d(x, E) \leq \frac{1}{4}d(E, F)$,

$$|\eta(x) - \eta(y)| = |\eta(x)| = \eta(x) = 1 - \frac{4d(x, E)}{d(E, F)} = \frac{d(E, F) - 4d(x, E)}{d(E, F)}.$$

But we also have the triangle inequality $d(E,x)+d(x,y) \geq d(y,E)$ and by the choice of y we have that $d(y,E) \geq 1/4d(E,F)$. Therefore, $d(x,y) \geq d(y,E) - d(x,E) \geq \frac{1}{4}d(E,F) - d(x,E)$ which implies that

$$\frac{4d(x,y)}{d(E,F)} \geq \frac{d(E,F)-d(x,E)}{d(E,F)} = |\eta(x) - \eta(y)|\,.$$

□

A preliminary and immediate consequence are the following Off-diagonal estimates resembling those in [5, §5.1].

Proposition 5.2 (Off-diagonal estimates)**.** *Let U_t be either R_t^B for $t \in \mathbb{R}$ or P_t^B, Q_t^B, Θ_t^B for $t > 0$. Then, for each $M \in \mathbb{N}$, there exists a constant $C_M > 0$ (that depends only on M and (H1)-H6)) such that*

$$\|U_t u\|_{\mathrm{L}^2(E)} \leq C_M \left\langle \frac{\operatorname{dist}(E,F)}{t} \right\rangle^{-M} \|u\|_{\mathscr{H}}$$

whenever $E, F \subset \mathcal{X}$ are Borel sets and $u \in \mathscr{H}$ with $\operatorname{spt} u \subset F$.

We omit the proof since it is essentially the same as that of [5, Proposition 5.2] and relies on (H6). The following is an immediate consequence.

Corollary 5.3. *Let $Q \in \mathscr{Q}_t$ and $0 < s \leq t$ with U_s as specified in the proposition. Then*

$$\|U_s u\|_{\mathrm{L}^2(Q)} \leq C_M \sum_{R \in \mathscr{Q}_t} \left\langle \frac{\operatorname{dist}(R,Q)}{s} \right\rangle^{-M} \|u\|_{\mathrm{L}^2(R)}$$

whenever $u \in \mathscr{H}$.

In our setting, it is more convenient to deal with the following function space rather than $\mathrm{L}^2_{\mathrm{loc}}$ as used in [5].

Definition 5.4. *We define $\mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathbb{C}^N)$ to be the space of measurable functions $f : \mathcal{X} \to \mathbb{C}^N$ such that on each $Q \in \mathscr{Q}_t$,*

$$\int_Q |f|^2 \, d\mu < \infty.$$

We equip this space with the seminorms $\|\cdot\|_{\mathrm{L}^2(Q)}$ indexed by $\mathscr{Q}_t$.

We have the following observations analogous to those in [5, p478]. It follows from Propositions 3.3, 3.4, 3.5 coupled with the Off-Diagonal estimates and by choosing $M > \frac{5p}{2} + 1$.

Corollary 5.5. *There exists a $C' > 0$ such that for all $t > 0$, U_t extends to a continuous map $U_t : \mathrm{L}^\infty(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathbb{C}^N)$ with*

$$\|U_t u\|_{\mathrm{L}^2(Q)} \leq C' \mu(Q)^{\frac{1}{2}} \|u\|_{\mathrm{L}^\infty}\,.$$

Corollary 5.6. *We have $\gamma_t \in \mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$ and for all $Q \in \mathscr{Q}_t$ satisfy*

$$\fint_Q |\gamma_t(x)|^2_{\mathcal{L}(\mathbb{C}^N)} \, d\mu(x) \leq C'^2$$

In particular, $\|\gamma_t \mathcal{A}_t\|_{\mathcal{L}(\mathscr{H})} \leq C'$ uniformly for all $t > 0$. The constant C' is the same as that of the previous corollary.

5.2. **Weighted Poincaré inequality and bounding the first term.** Controlling the first term in [5] relies primarily on the weighted Poincaré inequality described in [5, Lemma 5.4]. We pursue a similar strategy and begin by noting the following simple consequence of (H8).

Lemma 5.7 (Dyadic Poincaré)**.** *Whenever* $Q \in \mathscr{Q}_t$ *and* $r \geq C_1 \delta^{-1}$ *we have*

$$\int_{B(x_Q,rt)} |u(x) - u_Q|^2 \ d\mu(x) \lesssim r^{p+2} \int_{B(x_Q,crt)} |t\Pi u(x)|^2 \ d\mu(x)$$

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$.

This yields the following proposition analogous to [5, Lemma 5.4].

Proposition 5.8 (Weighted Poincaré)**.** *Whenever* $Q \in \mathscr{Q}_t$ *and* $M > p+1$, *we have*

$$\int_{\mathcal{X}} |u(x) - u_Q|^2 \left\langle \frac{d(x,Q)}{t} \right\rangle^{-M} \ d\mu(x) \lesssim \int_{\mathcal{X}} |t\Pi u(x)|^2 \left\langle \frac{d(x,Q)}{t} \right\rangle^{p-M} \ d\mu(x)$$

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$, *where the constant depends on* M.

Proof. Observe that for $M > 1$, we have

$$\left\langle \frac{d(x,Q)}{t} \right\rangle^{-M} \leq \frac{2C_1}{\delta} \left\langle \frac{d(x,x_Q)}{t} \right\rangle^{-M}.$$

By evaluating the integral

$$\int_{\mathcal{X}} \int_{\theta(x)}^{\infty} |u(x) - u_Q| \ d\nu(r) \ d\mu(x),$$

where $d\nu(r) = Mr^{-M-1} \ dr$, and invoking Lemma 5.7 along with Fubini's Theorem establishes the claim. ☐

This leads to the following proposition which bounds the first term.

Proposition 5.9 (First term inequality)**.** *Whenever* $u \in \mathcal{R}(\Pi)$, *we have*

$$\int_0^{\infty} \left\| \Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u \right\|^2 \lesssim \|u\|^2.$$

We omit the proof since it is very similar to the proof of [5, Proposition 5.5]. It is a simple matter of verification using Corollary 3.7 and invoking the weighted Poincaré inequality.

5.3. **Bounding the second term.** The bounding of the second term relies on a suitable substitution for [5, Lemma 5.6]. The crux of the argument is to be able to perform a cutoff "close" to the boundary of the dyadic cube in question. First, we define the following sets.

Definition 5.10 ($\mathcal{E}_\tau, \tilde{\mathcal{E}}_\tau$)**.** *Let* $Q \in \mathscr{Q}_t$ *and* $\tau \leq t$ *Define*

$$\mathcal{E}_\tau = \left\{ x \in Q : d(x, \mathcal{X} \setminus Q) > \frac{a_0 \tau}{2} \right\} \quad \text{and} \quad \tilde{\mathcal{E}}_\tau = \left\{ x \in Q : d(x, \mathcal{X} \setminus Q) \leq \frac{a_0 \tau}{2} \right\}.$$

The following proposition renders a suitable Lipschitz substitution to the smooth cutoff used in [5, Lemma 5.6] and [14, Lemma 4.4.9].

Proposition 5.11. *There exists a Lipschitz function* $\xi : Q \to [0,1]$ *such that* $\xi = 1$ *on* $\mathcal{E}_\tau$, $\operatorname{spt}\ (\operatorname{Lip}\xi) \subset \tilde{\mathcal{E}}_\tau$, *and*

$$\mathbf{Lip}\,\xi \leq \frac{16}{a_0}\frac{1}{\tau}.$$

Proof. Set

$$F = \left\{x \in Q : d(x, \mathcal{X} \setminus Q) \leq \frac{a_0\tau}{4}\right\}$$

and note that $F \subset \tilde{\mathcal{E}}_\tau$. Then,

$$\frac{a_0\tau}{2} \leq \operatorname{dist}(\mathcal{X} \setminus Q, \mathcal{E}_\tau) \leq \operatorname{dist}(\mathcal{E}_\tau, F) + \operatorname{dist}(\mathcal{X} \setminus Q, F) \leq \operatorname{dist}(\mathcal{E}_\tau, F) + \frac{a_0\tau}{4}$$

and so $\operatorname{dist}(\mathcal{E}_\tau, F) > \frac{a_0\tau}{4}$. By application of Lemma 5.1, we find $\xi = 1$ on $\mathcal{E}_\tau$, $\xi = 0$ on $Q \setminus F$ and

$$\mathbf{Lip}\,\xi \leq \frac{4}{\frac{a_0\tau}{4}} = \frac{16}{a_0}\frac{1}{\tau}.$$

Now, fix $x \in \mathcal{E}_\tau$. It is a simple matter to verify that $\mathcal{E}_\tau$ is open and nonempty. So there exists an $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subset \mathcal{E}_\tau$. Therefore,

$$\operatorname{Lip}\xi(x) = \limsup_{y\to x} \frac{|\xi(x) - \xi(y)|}{d(x,y)} = \limsup_{\varepsilon\to 0} \left\{\frac{|\xi(x) - \xi(y)|}{d(x,y)} : y \in \mathcal{E}_\tau \cap B(x,\varepsilon) \setminus \{a\}\right\} = 0.$$

Thus, $\operatorname{spt}\,\xi \subset \tilde{\mathcal{E}}_\tau$. □

This enables us to prove the following lemma. It is of key importance in bounding the second term, as well as in the Carleson measure estimate which allows us to bound the last term.

Lemma 5.12. *Let* Υ *be* Γ, Γ^* *or* Π. *Then, whenever* $Q \in \mathscr{Q}_t$,

$$\left|\fint_Q \Upsilon u\ d\mu\right|^2 \lesssim \frac{1}{t^\eta}\left(\fint_Q |u|^2\ d\mu\right)^{\frac{\eta}{2}}\left(\fint_Q |\Upsilon u|^2\ d\mu\right)^{1-\frac{\eta}{2}}$$

where the constant depends only on C_1, C_2, a_0, η *and* p.

Proof. Let $\tau = \left(\fint_Q |u|^2\ d\mu\right)^{\frac{1}{2}}\left(\fint_Q |\Upsilon u|^2\ d\mu\right)^{-\frac{1}{2}}$. The case of $t \leq \tau$ is easy. So, suppose that $\tau \leq t \leq \delta^j$ and let ξ be the Lipschitz function guaranteed in Proposition 5.11 extended to 0 outside of Q. and so write

$$\left|\int_Q \Upsilon u\ d\mu\right| \leq \left|\int_Q (1-\xi)\Upsilon u\ d\mu\right| + \left|\int_Q [\xi, \Upsilon]u\ d\mu\right| + \left|\int_Q \Upsilon(\xi u)\ d\mu\right|.$$

The last term is 0 by (H7) and so we are left with estimating the two remaining terms. First, noting that $\operatorname{spt}\ (1-\xi) \subset \tilde{\mathcal{E}}_\tau$ we compute

$$\begin{aligned}\left|\int_Q (1-\xi)\Upsilon u\ d\mu\right| &\leq \left|\int_{\tilde{\mathcal{E}}_\tau} (1-\xi)\Upsilon u\ d\mu\right| \leq \left(\int_{\tilde{\mathcal{E}}_\tau} |\Upsilon u|^2\ d\mu\right)^{\frac{1}{2}} \mu(\tilde{\mathcal{E}}_\tau) \\ &\leq C_2^{\frac{1}{2}}\left(\frac{a_0\tau}{2\delta^j}\right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}}\left(\int_Q |\Upsilon u|^2\ d\mu\right)^{\frac{1}{2}} \leq C_2^{\frac{1}{2}}\left(\frac{a_0\tau}{2t}\right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}}\left(\int_Q |\Upsilon u|^2\ d\mu\right)^{\frac{1}{2}}.\end{aligned}$$

Now, for the second term. We note that $\operatorname{spt} M_\xi \subset \operatorname{spt}\ \operatorname{Lip}\xi \subset \tilde{\mathcal{E}}_\tau$ and compute

$$\begin{aligned}\left|\int_Q [\xi,\Upsilon]u\right| &= \left|\int_{\tilde{\mathcal{E}}_\tau} M_\xi(x)u(x)\ d\mu(x)\right| \leq \left(\int_{\tilde{\mathcal{E}}_\tau} |M_\xi|^2\ d\mu\right)^{\frac{1}{2}} \left(\int_{\tilde{\mathcal{E}}_\tau} |u|^2\ d\mu\right)^{\frac{1}{2}} \\ &\leq \mathbf{Lip}\,\xi\ \mu(\tilde{\mathcal{E}}_\tau)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \leq \frac{16}{a_0} C_2^{\frac{1}{2}} \left(\frac{a_0\tau}{2t}\right)^{\frac{\eta}{2}} \frac{1}{\tau}\mu(Q)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \\ &\leq \frac{16}{a_0} C_2^{\frac{1}{2}} \left(\frac{a_0\tau}{2t}\right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2\right)^{\frac{1}{2}}\end{aligned}$$

by making the substitution for $\frac{1}{\tau}$. Combining these estimates, we have

$$\left|\int_Q \Upsilon u\ d\mu\right| \leq D\frac{1}{t^{\frac{\eta}{2}}}\tau^{\frac{\eta}{2}}\mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2\ d\mu\right)^{\frac{1}{2}}$$

where

$$D = C_2^{\frac{1}{2}}\left(\frac{a_0}{2}\right)^{\frac{\eta}{2}} + \frac{16}{a_0}C_2^{\frac{1}{2}}\left(\frac{a_0}{2}\right)^{\frac{\eta}{2}} \quad \text{and} \quad \tilde{D} = C(2^p C_1^p a_0^{-p})^{\frac{1}{2}}.$$

By Cauchy-Schwartz and multiplying both sides by $\mu(Q)^{-2}$, we find

$$\left|\fint_Q \Upsilon u\ d\mu\right|^2 \leq 2D^2\frac{1}{t^\eta}\tau^\eta \fint_Q |\Upsilon u|^2\ d\mu.$$

The proof is complete by making a substitution for τ^η. ☐

Proposition 5.13 (Second term estimate)**.** *For all $u \in \mathscr{H}$, we have*

$$\int_0^\infty \|\gamma_t \mathcal{A}_t(P_t - I)u\|\ \frac{dt}{t} \lesssim \|u\|^2.$$

Again, the proof of this proposition is omitted since it resembles the proof of [5, Proposition 5.7] with minor differences.

5.4. **Carleson measure estimate.** We begin this section with the following proposition which illustrates that the final term can be dealt with a Carleson measure estimate.

Proposition 5.14. *For all $u \in \mathscr{H}$, we have*

$$\iint_{\mathcal{X}_+} |\mathcal{A}_t u(x)|^2\ d\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}}\|u\|^2$$

for every $\nu \in \mathcal{C}$.

Proof. First, we show that for for almost every $x \in \mathcal{X}$,

$$\mathcal{M}^*|\mathcal{A}.u|^2(x) \lesssim \mathcal{M}u(x)^2$$

where the constant depends only on p, C_1, δ and a_0. Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}_+, \mathbb{C}^N)$. Then, we note that

$$\mathcal{M}^*f(x) = \sup_{t>0}\sup_{y\in B(x,t)} |f(y,t)|.$$

Fix t such that $\delta^{j+1} < t \leq \delta^j$ and fix $x \in \cup\mathscr{Q}_t$. Since $\mathcal{A}_t u(z) = 0$ when $z \not\in \cup\mathscr{Q}_t$, take $y \in \cup\mathscr{Q}_t$ such that $d(x,y) < t$. Let $Q \in \mathscr{Q}_t$ be the unique cube with $y \in Q$ and let $y_Q \in Q$ such that $B(y_Q, a_0\delta^j) \subset Q \subset B(y_Q, C_1\delta^j)$. Then, $d(y_Q,x) \leq d(y_Q,y) + d(y,x) \leq Ct$, where $C = (C_1\delta^{-1} + 1)$.

Also $\mu(B(y_Q, Ct)) \leq \mu(B(y_Q, C\delta^j)) \leq 2^p C^p a_0^{-p} \mu(B(y_Q, a_0\delta^j)) \leq 2^p C^p a_0^{-p}\mu(Q)$ and therefore,

$$|\mathcal{A}_t u(y)| \leq \fint_Q |u| \ d\mu \leq 2^p C^p a_0^{-p} \fint_{B(y_Q, Ct)} |u| \ d\mu.$$

Moreover,

$$|\mathcal{A}_t u(y)|^2 \leq C' \left(\fint_{B(y_Q, Ct)} |u| \ d\mu \right)^2$$

where $C' = 2^2pC^2pa_0^{-2p}$.

Now, since we have established that $x \in B(y_Q, Ct)$,

$$\sup_{y \in B(x,t)} |\mathcal{A}_t u(y)|^2 \leq C' \sup_{y \in B(x,t)} \left(\fint_{B(y_Q(y), Ct)} |u| \ d\mu \right)^2 \leq C'(\mathcal{M}u(x))^2.$$

Let $\tilde{\mathscr{X}} = \cap_j \cup \mathscr{Q}^j$ and so $\mu(\mathscr{X} \setminus \tilde{\mathscr{X}}) = \mu(\cup_j \mathscr{X} \setminus \cup \mathscr{Q}^j) \leq \sum_j \mu(\mathscr{X} \setminus \cup \mathscr{Q}^j) = 0$. Therefore, $x \in \tilde{\mathscr{X}}$, then $x \in \cup \mathscr{Q}_t$ for all $t > 0$. So, fix $x \in \tilde{\mathscr{X}}$. Then,

$$\mathcal{M}^* \left|\mathcal{A}_. u\right|^2 (x) = \sup_{t>0} \sup_{y \in B(x,t)} |\mathcal{A}_t u(y)|^2 \leq C' \mathcal{M}u(x)^2$$

which completes the proof.

Next, let $f(x,t) = |\mathcal{A}_t u(x)|^2$. Then, $\|f\|_{\mathcal{N}} = \|\mathcal{M}^* f\|_1 \lesssim \|\mathcal{M}u\|^2 < \infty$ by the Maximal Theorem 4.2. Invoking Carleson's Theorem 4.11 completes the proof. □

Thus, to bound the final term, it suffices to prove

$$A \mapsto \iint_A |\gamma_t(x)|^2 \ d\mu(x) \frac{dt}{t}$$

is a Carleson measure. We follow [5] and fix $\delta > 0$ to be chosen later. Let

$$K_\nu = \left\{ \nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left| \frac{\nu'}{|\nu'|} - \nu \right| \leq \sigma \right\}$$

and let $\mathcal{F}$ be a finite set of $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$ such that $\cup_{\nu \in \mathcal{F}} K_\nu = \mathcal{L}(\mathbb{C}^N) \setminus \{0\}$. We note as do the authors of [5] that it is enough to show

$$\iint_{(x,t) \in \mathrm{R}_Q, \gamma_t \in K_\nu} |\gamma_t(x)|^2 \ d\mu(x) \frac{dt}{t} \lesssim \mu(Q)$$

for each $\nu \in \mathcal{F}$. A stopping time argument allows us to reduce this to the following.

Proposition 5.15. *There exists a* $0 < \beta < 1$ *such that for every dyadic cube* $Q \in \mathscr{Q}$ *and* $\nu \in \mathcal{L}(\mathbb{C}^N)$ *with* $|\nu| = 1$, *there exists a collection* $\{Q_k\} \subset \mathscr{Q}$ *of disjoint subcubes of* Q *satisfying* $\mu(E_{Q,\nu}) > \beta\mu(Q)$ *and such that*

$$\iint_{(x,t) \in E^*_{Q,\nu}, \ \gamma_t(x) \in K_\nu} |\gamma_t(x)|^2 \ d\mu(x) \frac{dt}{t} \lesssim \mu(Q)$$

where $E_{Q,\nu} = Q \setminus \cup_k Q_k$ *and* $E^*_{Q,\nu} = \mathrm{R}_Q \setminus \cup_k \mathrm{R}Q_k$.

We prove this via defining a test function similar to that of [5, p484]. Here, the authors use a smooth cutoff function in their construction. Again, we rephrase this in terms of a Lipschitz cutoff function whose existence is guaranteed by the following lemma.

Lemma 5.16. *Let $Q \in \mathscr{Q}$. Then, there exists a Lipschitz function $\eta : \mathcal{X} \to [0,1]$ such that $\eta = 1$ on $B(x_Q, \tau C_1\, \ell(Q))$ and $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, 2\tau C_1\, \ell(Q))$ with*

$$\mathbf{Lip}\,\eta \leq \frac{4}{\tau C_1} \frac{1}{\ell(Q)}$$

whenever $\tau > 1$.

Proof. Fix $Q \in \mathscr{Q}^j$, and we have $Q \subset B(x_Q, \tau C_1 \delta^j) \subset B(x_Q, 2\tau C_1 \delta^j)$. Also,

$$d(B(x_Q, \tau C_1 \delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j)) \geq (2\tau C_1 - \tau C_1)\delta^j = \tau C_1 \delta^j.$$

Now, we invoke the Lipschitz separation Lemma 5.1 with $E = B(x_Q, \tau C_1 \delta^j)$ and $F = \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j)$ to find a Lipschitz $\eta : \mathcal{X} \to [0,1]$ with $\eta = 1$ on $B(x_Q, \tau C_1 \delta^j)$, $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, \tau C_1 \delta^j)$ and

$$\mathbf{Lip}\,\eta \leq \frac{4}{d(B(x_Q, \tau C_1 \delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j))} \leq \frac{4}{\tau C_1} \frac{1}{\delta^j} = \frac{4}{\tau C_1} \frac{1}{\ell(Q)}$$

which completes the proof. □

The test function is now defined as follows. Let $Q \in \mathscr{Q}$ and fix $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$. Let η_Q be the Lipschitz map guaranteed by Lemma 5.16 and let $w, \hat{w} \in \mathbb{C}^N$ such that $\nu^*(\hat{w}) = w$ with $|w| = |\hat{w}| = 1$. Furthermore, let $w_Q = \eta_Q w$ and define

$$f^w_{Q,\varepsilon} = w_Q - \varepsilon\, \ell(Q) \imath \Gamma (I + \varepsilon\, \ell(Q) \imath \Pi_B)^{-1} w_Q = (1 + \varepsilon\, \ell(Q) \imath \Gamma^*_B)(1 + \varepsilon\, \ell(Q) \imath \Pi_B)^{-1} w_Q.$$

It is then an easy fact that $\|w_Q\|^2 \leq (4\tau C_1 a_0^{-1})^p \mu(Q)$ and we obtain the following lemma analogous to [5, Lemma 5.10].

Lemma 5.17. *There exists $c > 0$ such that for all $\varepsilon > 0$,*

$$\|f^w_{Q,\varepsilon}\| \leq c\mu(Q)^{\frac{1}{2}}, \quad \iint_{\mathrm{R}_Q} \left|\Theta^B_t f^w_{Q,\varepsilon}\right|^2 \, d\mu(x) \frac{dt}{t} \leq c \frac{1}{\varepsilon^2} \mu(Q), \quad \text{and} \quad \left| \fint_Q f^w_{Q,\varepsilon} - w \right| \leq c\varepsilon^{\frac{\eta}{2}}.$$

Proof. The proof of the first two estimates are essentially the same as that of [5, Lemma 5.10]. To prove the last estimate, note that since $\eta_Q = 1$ on Q, we have on Q that

$$\begin{aligned} f^w_{Q,\varepsilon} - w &= w_Q - \varepsilon\, \ell(Q) \imath (1 + \varepsilon\, \ell(Q) \imath \Pi_B)^{-1} w_Q - w \\ &= (\eta_Q - 1) w - \varepsilon\, \ell(Q) \imath (1 + \varepsilon\, \ell(Q) \imath \Pi_B)^{-1} w_Q = -\varepsilon\, \ell(Q) \imath (1 + \varepsilon\, \ell(Q) \imath \Pi_B)^{-1} w_Q. \end{aligned}$$

Setting $u = (1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q$ and $\Upsilon = \Gamma$, we apply Lemma 5.12

$$\begin{aligned}
\left|\fint_Q f^w_{Q,\varepsilon} - w\right| &= \left|\fint_Q \varepsilon\,\ell(Q)\imath(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right| \\
&= \varepsilon\,\ell(Q)\left|\fint Q\imath(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right| \\
&\lesssim \frac{\varepsilon\,\ell(Q)}{t^{\frac{\eta}{2}}}\left(\fint_Q \left|(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right|\, d\mu\right)^{\frac{\eta}{4}} \\
&\qquad\qquad \left(\fint_Q \left|\Gamma(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right|^2\, d\mu\right)^{\frac{1}{2}-\frac{\eta}{4}} \\
&= \left(\frac{\varepsilon\,\ell(Q)}{t}\right)^{\frac{\eta}{2}}\left(\fint_Q \left|(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right|\, d\mu\right)^{\frac{\eta}{4}} \\
&\qquad\qquad \left(\fint_Q \left|\varepsilon\,\ell(Q)\imath\Gamma(1+\varepsilon\,\ell(Q)\imath\Pi_B)^{-1}w_Q\right|^2\, d\mu\right)^{\frac{1}{2}-\frac{\eta}{4}}.
\end{aligned}$$

The proof is completed by noting $t \simeq \ell(Q)$ and invoking Proposition 2.5 and Lemma 4.2 of [5]. ☐

The proof of Proposition 5.15 then follows a procedure similar to that which is used to prove [5, Lemma 5.12].

We note that our hypotheses (H1)-(H8) remain unchanged upon replacing (Γ, B_1, B_2) by (Γ^*, B_2, B_1), $(\Gamma^*, {B_2}^*, {B_1}^*)$ and $(\Gamma, {B_1}^*, {B_2}^*)$. Thus, the hypothesis of Proposition 2.1 is satisfied and Theorem 2.4 is proved.

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