UNIVERSALITY OF THE LATTICE OF TRANSFORMATION MONOIDS

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Abstract. The set of all transformation monoids on a fixed set of infinite cardinality λ, equipped with the order of inclusion, forms an algebraic lattice $\mathrm{Mon}(\lambda)$ with 2^λ compact elements. We show that this lattice is universal, i.e., every algebraic lattice with at most 2^λ compact elements is isomorphic to a complete sublattice of $\mathrm{Mon}(\lambda)$.

1. Definitions and the result

Fix an infinite set – for the sake of simpler notation, we identify the set with its cardinality λ. By a *transformation monoid* on λ we mean a subset of λ^λ which is closed under composition and which contains the identity function. The set of transformation monoids acting on λ, ordered by inclusion, forms a complete lattice $\mathrm{Mon}(\lambda)$, in which the meet of a set of monoids is simply their intersection. This lattice is *algebraic*, i.e., every element is a join of compact elements – an element a in a complete lattice L is called *compact* iff whenever $A \subseteq L$ and $a \leq \bigvee A$, then there is a finite $A' \subseteq A$ such that $a \leq \bigvee A'$. In the case of $\mathrm{Mon}(\lambda)$, the compact elements are precisely the finitely generated monoids, i.e., those monoids which contain a finite set of functions such that every function of the monoid can be composed from functions of this finite set. Consequently, the number of compact elements of $\mathrm{Mon}(\lambda)$ equals 2^λ.

It is well-known and not hard to see that the algebraic lattices with 2^λ compact elements are, up to isomorphism, precisely the subalgebra lattices of algebras whose domain have 2^λ elements (we refer to the textbook [CD73] as a general reference of lattice theory). For example, $\mathrm{Mon}(\lambda)$ is the subalgebra lattice of the algebra which has domain λ^λ, a binary operation which is the function composition on λ^λ, as well as a constant operation whose value is the identity function on λ.

Let K be a complete algebraic lattice and L be a complete sublattice of K, i.e., arbitrary joins and meets in L exist and equal the corresponding

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joins and meets in K. Then it is a folklore fact that L is algebraic as well. Moreover, the number of compact elements of L cannot be larger than the corresponding number for K. Therefore, any complete sublattice of $\mathrm{Mon}(\lambda)$ is algebraic and has at most 2^λ compact elements. In this paper, we prove the converse of this fact. This had been stated as an open problem in [GP08, Problem C] (we remark that it is clear from the context in [GP08] that the word "subinterval" in the formulation of Problem C is an error; it is Problem B which is asks about subintervals).

Theorem 1. $\mathrm{Mon}(\lambda)$ *is* universal *for algebraic lattices with at most* 2^λ *compact elements with respect to complete embeddings, i.e., the complete sublattices of* $\mathrm{Mon}(\lambda)$ *are, up to isomorphism, precisely the algebraic lattices with at most* 2^λ *compact elements.*

2. Related work and possible extensions

A *clone* on λ is a set of finitary operations on λ which is closed under composition and which contains all finitary projections; in other words, it is a set of finitary operations closed under building of terms (without constants). The set of all clones on λ, ordered by inclusion, also forms a complete algebraic lattice $\mathrm{Cl}(\lambda)$ with 2^λ compact elements, into which $\mathrm{Mon}(\lambda)$ embeds naturally, since a transformation monoid can be viewed as a clone all of whose operations depend on at most one variable. Universality of $\mathrm{Cl}(\lambda)$ for algebraic lattices with at most 2^λ compact elements and complete embeddings has been shown in [Pin07] – our result is a strengthening of this result.

Observe that similarly to transformation monoids and clones, the set of *permutation groups* on λ forms a complete algebraic lattice $\mathrm{Gr}(\lambda)$ with respect to inclusion. By virtue of the identity embedding, $\mathrm{Gr}(\lambda)$ is a complete sublattice of $\mathrm{Mon}(\lambda)$. We do not know the following.

Problem 2. Is every algebraic lattice with at most 2^λ compact elements a complete sublattice of $\mathrm{Gr}(\lambda)$?

A related problem is which lattices appear as *intervals* of $\mathrm{Gr}(\lambda)$, $\mathrm{Mon}(\lambda)$, and $\mathrm{Cl}(\lambda)$. This remains open – for the latter two lattices this question has been posed as an open problem in [GP08] (Problems B and A, respectively). By a deep theorem due to Tůma [Tům89], every algebraic lattice with λ compact elements is isomorphic to an interval of the subgroup lattice of a group of size λ; from this it only follows that $\mathrm{Gr}(\lambda)$ contains all algebraic lattices with at most λ compact elements as intervals. Proving that $\mathrm{Gr}(\lambda)$ contains all algebraic lattices with at most 2^λ compact elements as intervals would be a common strengthening of Tůma's result and a positive answer to Problem 2.

3. Proof of the theorem

3.1. **Independent composition engines.** For a cardinal κ and a natural number $n \geq 1$, we write $\Lambda_\kappa^n := \{(\eta,\phi) : \eta \in \kappa^n \wedge \phi \in 2^n\}$. We set $\Lambda_\kappa := \bigcup_{n\geq 1} \Lambda_\kappa^n$. For sequences p, q, we write $p \triangleleft q$ if p is a non-empty initial segment of q (we consider q to be an initial segment of itself). For (η,ϕ) and (η',ϕ') in Λ_κ, we also write $(\eta,\phi) \triangleleft (\eta',\phi')$ if $\eta \triangleleft \eta'$ and $\phi \triangleleft \phi'$. If p is a sequence and r a set, then $p * r$ denotes the extension of p by the element r. We write $<r>$ for the one-element sequence containing only r.

A sequence P of elements of Λ_κ is *reduced* iff it does not contain both $(\eta * \alpha,\ \phi * 0)$ and $(\eta * \alpha,\ \phi * 1)$ for any $(\eta,\phi) \in \Lambda_\kappa$ and $\alpha \in \kappa$. We call two sequences P, Q *equivalent* iff P can be transformed into Q by permuting its elements.

For a set W and a cardinal κ, a *κ-branching independent composition engine (κ-ICE) on* W is an indexed set $\{f_{(\eta,\phi)} : (\eta,\phi) \in \Lambda_\kappa\}$ of permutations on W satisfying all of the following:

(i) (Composition) For all $(\eta,\phi) \in \Lambda_\kappa$ and for all $\alpha \in \kappa$ we have $f_{(\eta,\phi)} = f_{(\eta*\alpha,\ \phi*0)} \circ f_{(\eta*\alpha,\ \phi*1)}$;
(ii) (Commutativity) For all $a, b \in \Lambda_\kappa$ we have that $f_a \circ f_b = f_b \circ f_a$.
(iii) (Independence) Whenever $P = (p_1,\dots,p_n), Q = (q_1,\dots,q_m) \subseteq \Lambda_\kappa$ are inequivalent reduced sequences, then $t_P := f_{p_1} \circ \cdots \circ f_{p_n}$ and $t_Q := f_{q_1} \circ \cdots \circ f_{q_m}$ are not equal.

Note that by the commutativity of the system, the order of the elements of the sequences P and Q in condition (iii) is not of importance.

Lemma 3. *There exists a 2^λ-ICE on λ.*

Proof. We show that there exists a 2^λ-ICE on $W := \lambda \times \mathbb{Z}$. Let

$$\{A_{(\eta,\phi)} : (\eta,\phi) \in \Lambda_{2^\lambda} \text{ and the last entry of } \phi \text{ equals } 0\}$$

be an independent family of subsets of λ, i.e., any non-trivial finite Boolean combination of these sets is non-empty (see, for example, [Jec78] for a proof of the existence of such a family). For all $(\eta,\phi) \in \Lambda_{2^\lambda}$, set $\#A_{(\eta,\phi)}$ to equal $A_{(\eta,\phi)}$, if the last entry of ϕ equals 0, and $\lambda \setminus A_{(\eta,\phi')}$ otherwise, where ϕ' is obtained from ϕ by changing the last entry to 0. Now define $B_{(\eta,\phi)} := \bigcap_{s \triangleleft (\eta,\phi)} \#A_s$, for all $(\eta,\phi) \in \Lambda_{2^\lambda}$.

We will define the 2^λ-ICE by means of the family $\{B_{(\eta,\phi)} : (\eta,\phi) \in \Lambda_{2^\lambda}\}$ as follows. For all $(\eta,\phi) \in \Lambda_{2^\lambda}$ and all $(\alpha,i) \in W$, we set

$$f_{(\eta,\phi)}(\alpha,i) = \begin{cases} (\alpha, i+1) & , \text{ if } \alpha \in B_{(\eta,\phi)}, \\ (\alpha,i) & , \text{ otherwise.} \end{cases}$$

We claim that this defines a 2^λ-ICE on W. Clearly, (ii) of the definition is satisfied. Property (i) is a direct consequence of the fact that for all $(\eta,\phi) \in \Lambda_{2^\lambda}$ and all $\alpha < \lambda$, $B_{(\eta,\phi)}$ is the disjoint union of $B_{(\eta*\alpha,\phi*0)}$ and $B_{(\eta*\alpha,\phi*1)}$. To see (iii), let P and Q be inequivalent; without loss of generality, we may assume that there exists an element (η, ϕ) of Λ_{2^λ} which appears i times in P, and $j < i$ times in Q. By deleting j occurrences of (η, ϕ) from both P and Q, which is the same as composing t_P and t_Q with $f^{-j}_{(\eta,\phi)}$, we may even assume that (η, ϕ) does not occur in Q at all. Let A be the union of all B_q for which q appears in Q. Then it follows from the independence of the family $\{A_{(\eta,\phi)} : (\eta, \phi) \in \Lambda_{2^\lambda}\}$ and from the fact that Q is reduced that $B_{(\eta,\phi)} \setminus A$ is non-empty. Let α be an element of the latter set. Then $t_P(\alpha, 0) = (\alpha, k)$ for some $k \geq i > 0$, whereas $t_Q(\alpha, 0) = (\alpha, 0)$. Hence, $t_P \neq t_Q$. □

3.2. **From lattices to monoids.** For a κ-ICE $\{f_{(\eta,\ \phi)} : (\eta, \phi) \in \Lambda_\kappa\}$ on W and a subset S of Λ_κ, we set $F(S)$ to be the monoid *generated* by the functions with index in S, i.e., the smallest monoid of functions from W to W which contains all the functions with index in S. Another way to put it is that $F(S)$ contains precisely the composites of functions with index in S as well as the identity function on W.

In the following, fix a 2^λ-ICE $\{f_{(\eta,\ \phi)} : (\eta, \phi) \in \Lambda_{2^\lambda}\}$ on λ. Let $L = (X, \vee, \wedge)$ be any algebraic lattice with 2^λ compact elements. Enumerate the set $C \subseteq X$ of these elements, possibly with repetitions, by $\{c_{(\eta,\phi)} : (\eta, \phi) \in \Lambda_{2^\lambda}\}$, and such that the following hold:

(1) Every compact element is equal to $c_{(<\alpha>,<0>)}$ for some $\alpha < 2^\lambda$;
(2) For all $(\eta, \phi) \in \Lambda_{2^\lambda}$ and all $\alpha \in 2^\lambda$ we have $c_{(\eta,\phi)} \leq c_{(\eta*\alpha,\phi*0)} \vee c_{(\eta*\alpha,\phi*1)}$;
(3) For all $(\eta, \phi) \in \Lambda_{2^\lambda}$ and all $d, d' \in C$ with $c_{(\eta,\phi)} \leq d \vee d'$, there exists $\alpha < 2^\lambda$ such that $d = c_{(\eta*\alpha,\phi*0)}$ and $d' = c_{(\eta*\alpha,\phi*1)}$.

It is a well-known fact that L is isomorphic to the lattice of ideals (i.e., join- and downward closed subsets) of the semilattice $(C, \vee)$ of its compact elements. The meet $\bigwedge_{u\in U} I_u$ of a set of ideals $\{I_u : u \in U\}$ in this lattice is just their intersection; their join $\bigvee_{u\in U} I_u$ the smallest ideal containing all I_u, and contains all elements c of C for which there exist $c_1, \ldots, c_n \in \bigcup_{u\in U} I_u$ such that $c \leq c_1 \vee \cdots \vee c_n$.

To every ideal $I \subseteq C$, assign the sets $S(I) := \{(\eta, \phi) \in \Lambda_{2^\lambda} : c_{(\eta,\phi)} \in I\}$, and $F(I) := F(S(I))$.

Lemma 4. *If $\{I_u : u \in U\}$ is a set of ideals of $(C, \vee)$, then $\bigvee_{u\in U} F(I_u) = F(\bigvee_{u\in U} I_u)$.*

Proof. The inclusion $\subseteq$ is trivial. For the other direction, it is enough to show that if $c_{(\eta,\phi)}$ is an element of $\bigvee_{u\in U} I_u$, then $f_{(\eta,\phi)}$ is an element of $\bigvee_{u\in U} F(I_u)$. We have that there exist $c_{(\eta_1,\phi_1)}, \ldots, c_{(\eta_n,\phi_n)} \in \bigcup_{u\in U} I_u$ such that $c_{(\eta,\phi)} \leq c_{(\eta_1,\phi_1)} \vee \cdots \vee c_{(\eta_n,\phi_n)}$. We use induction over n. If $n = 1$, then $c_{(\eta,\phi)} \leq c_{(\eta_1,\phi_1)} \in I_u$ for some $u \in U$, so $c_{(\eta,\phi)} \in I_u$. Hence, $f_{(\eta,\phi)} \in F(I_u)$, and we are done. In the induction step, suppose the claim holds for all $1 \leq k < n$. Set $d := c_{(\eta_1,\phi_1)} \vee \cdots \vee c_{(\eta_{n-1},\phi_{n-1})}$ and $d' := c_{(\eta_n,\phi_n)}$. Since $c_{(\eta,\phi)} \leq d \vee d'$, there exist $\alpha < 2^\lambda$ such that $(d, d') = (c_{(\eta*\alpha,\phi*0)}, c_{(\eta*\alpha,\phi*1)})$, by Property (3) of our enumeration. By the induction hypothesis, we have

$f_d, f_{d'} \in \bigvee_{u\in U} F(I_u)$. Since $f_{(\eta,\phi)} = f_{(\eta*\alpha,\phi*0)} \circ f_{(\eta*\alpha,\phi*1)}$, we get that $f_{(\eta,\phi)} \in \bigvee_{u\in U} F(I_u)$ as well, proving the lemma. □

Lemma 5. *If $\{I_u : u \in U\}$ is a set of ideals of $(C, \vee)$, then $\bigcap_{u\in U} F(I_u) = F(\bigcap_{u\in U} I_u)$.*

Proof. This time, the inclusion $\supseteq$ is trivial. For the other direction, let $t \in \bigcap_{u\in U} F(I_u)$. Then there is a unique reduced set P such that $t = t_P$, by Property (iii) of an independent composition engine. Now let $u \in U$ be arbitrary. Then there exists a sequence Q in $S(I_u)$ such that $t = t_Q$. By subsequently replacing two entries $(\eta * \alpha, \phi * 0), (\eta * \alpha, \phi * 1)$ in Q by (η, ϕ), we obtain a reduced sequence Q' which still satisfies $t = t_{Q'}$. Since I_u is closed under joins, and by Property (2) of our enumeration of the compact elements, all entries of Q' are still elements of $S(I_u)$. By the independence property, P and Q' are equivalent, and hence P is a sequence in $S(I_u)$. But u was arbitrary, so P is a sequence in $\bigcap_{u\in U} S(I_u)$. This proves $t \in F(\bigcap_{u\in U} I_u)$. □

It follows from the above that the mapping $I \mapsto F(I)$ is a mapping from the lattice of ideals of $(C, \vee)$ to $\mathrm{Mon}(\lambda)$ which preserves arbitrary joins and meets, proving our theorem.

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