

# Solvability and asymptotic analysis of a generalization of the Caginalp phase field system\*

GIACOMO CANEVARI AND PIERLUIGI COLLI

*Dipartimento di Matematica “F. Casorati”, Università di Pavia  
Via Ferrata, 1, 27100 Pavia, Italy*

*E-mail: giacomo.canevari@gmail.com pierluigi.colli@unipv.it*

## Abstract

We study a diffusion model of phase field type, which consists of a system of two partial differential equations involving as variables the thermal displacement, that is basically the time integration of temperature, and the order parameter. Our analysis covers the case of a non-smooth (maximal monotone) graph along with a smooth anti-monotone function in the phase equation. Thus, the system turns out a generalization of the well-known Caginalp phase field model for phase transitions when including a diffusive term for the thermal displacement in the balance equation. Systems of this kind have been extensively studied by Miranville and Quintanilla. We prove existence and uniqueness of a weak solution to the initial-boundary value problem, as well as various regularity results ensuring that the solution is strong and with bounded components. Then we investigate the asymptotic behaviour of the solutions as the coefficient of the diffusive term for the thermal displacement tends to 0 and prove convergence to the Caginalp phase field system as well as error estimates for the difference of the solutions.

**Key words:** phase field model, well-posedness, regularity, asymptotic behaviour, error estimates.

**AMS (MOS) Subject Classification:** 35K55, 35B30, 35B40, 80A22.

## 1 Introduction

This paper is concerned with the initial and boundary value problem:

$$w_{tt} - \alpha \Delta w_t - \beta \Delta w + u_t = f \quad \text{in } \Omega \times (0, T) \quad (1.1)$$

$$u_t - \Delta u + \gamma(u) + g(u) \ni w_t \quad \text{in } \Omega \times (0, T) \quad (1.2)$$

$$\partial_n w = \partial_n u = 0 \quad \text{on } \Gamma \times (0, T) \quad (1.3)$$

$$w(\cdot, 0) = w_0, \quad w_t(\cdot, 0) = v_0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega \quad (1.4)$$

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where  $\Omega \subset \mathbb{R}^3$  is a bounded domain with smooth boundary  $\Gamma$ ,  $T > 0$  represents some finite time, and  $\partial_n$  denotes the outward normal derivative on  $\Gamma$ . Moreover,  $\alpha$  and  $\beta$  are two positive parameters,  $\gamma : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  is a maximal monotone graph (one can see [2, in particular pp. 43–45] or [1]),  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz-continuous function,  $f$  is a given source term in equation (1.1) and  $w_0, v_0, u_0$  stand for initial data. The inclusion (in place of the equality) in (1.2) is due to the presence of the possibly multivalued graph  $\gamma$ .

Equations (1.1)–(1.2) yield a system of phase field type. Such systems have been introduced (cf. [3]) in order to include phase dissipation effects in the dynamics of moving interfaces arising in thermally induced phase transitions. In our case, we move from the following expression for the total free energy

$$\Psi(\theta, u) = \int_{\Omega} \left( -\frac{1}{2}\theta^2 - \theta u + \phi(u) + G(u) + \frac{1}{2}|\nabla u|^2 \right) \quad (1.5)$$

where the variables  $\theta$  and  $u$  denote the (relative) temperature and order parameter, respectively. Let us notice from the beginning that our  $w$  represents the thermal displacement variable, related to  $\theta$  by

$$w(\cdot, t) = w_0 + (1 * \theta)(\cdot, t) = w_0 + \int_0^t \theta(\cdot, s) ds, \quad t \in [0, T]. \quad (1.6)$$

In (1.5),  $\phi : [0, +\infty] \rightarrow \mathbb{R}$  is the convex and lower semicontinuous function such that  $\phi(0) = 0 = \min \phi$  and its subdifferential  $\partial\phi$  coincides with  $\gamma$ , while  $G$  stands for a smooth, in general concave, function such that  $G' = g$ . A typical example for  $\phi$  and  $G$  is the double obstacle case

$$\phi(u) = I_{[-1, +1]}(u) = \begin{cases} 0 & \text{if } |u| \leq 1 \\ +\infty & \text{if } |u| > 1 \end{cases}, \quad G(u) = 1 - u^2 \quad (1.7)$$

so that the two wells of the sum  $\phi(u) + G(u)$  are located in  $-1$  and  $+1$ , and one of the two is preferred as minimum of the potential in (1.5) according to whether the temperature  $\theta$  is negative or positive. Indeed, note the presence of the term  $-\theta u$  besides  $\phi(u) + G(u)$  in the expression of  $\Psi$ .

The example given in (1.7) is inspired by the systematic approach of Michel Frémond to non-smooth thermomechanics: we refer to the monography [7] which also deals with the phase change models. In the case of (1.7) the subdifferential of the indicator function of the interval  $[-1, +1]$  reads

$$\xi \in \partial I_{[-1, +1]}(u) \quad \text{if and only if} \quad \xi \begin{cases} \leq 0 & \text{if } u = -1 \\ = 0 & \text{if } |u| < 1 \\ \geq 0 & \text{if } u = +1 \end{cases} .$$

Let us point out that, with a different terminology motivated by earlier studies on the Stefan problem [6], some authors (cf. [7]) prefer to name “freezing index” the variable  $w$  defined by (1.6), having also in mind applications to frost propagation in porous media.

Another meaningful variable of the Stefan problem is the enthalpy  $e$ , which in our case is defined by

$$e = -d_{\theta}\Psi \quad (\text{— the variational derivative of } \Psi \text{ with respect to } \theta),$$

whence  $e = \theta + u = w_t + u$ . Then, the governing balance and phase equations are given by

$$e_t + \operatorname{div} \mathbf{q} = f \quad (1.8)$$

$$u_t + d_u \Psi = 0 \quad (1.9)$$

where  $\mathbf{q}$  denotes the thermal flux vector and  $d_u \Psi$  stands for the variational derivative of  $\Psi$  with respect to  $u$ . Hence, (1.9) reduces exactly to (1.2) along with the Neumann homogeneous boundary condition for  $u$ . If we assume the classical Fourier law  $\mathbf{q} = -\nabla \theta$  (for the moment let us take the heat conductivity coefficient just equal to 1), then (1.8) is nothing but the usual energy balance equation as in the Caginalp model [3]. This is also as in the weak formulation of the Stefan problem, in which the mere pointwise inclusion  $u \in (\partial I_{[-1,+1]})^{-1}(\theta)$ , or equivalently  $\theta \in \partial I_{[-1,+1]}(u)$ , replaces (1.2).

Another approach, which is by now well established, consists in adopting the so-called Cattaneo-Maxwell law (see, e.g., [4, 14] and references therein): such a law reads

$$\mathbf{q} + \varepsilon \mathbf{q}_t = -\nabla \theta, \quad \text{for } \varepsilon > 0 \text{ small,} \quad (1.10)$$

and leads to the following equation

$$\varepsilon \theta_{tt} + \theta - \Delta \theta + \varepsilon u_{tt} + u_t = f \quad \text{in } \Omega \times (0, T)$$

which has been investigated in [14]. On the other hand, if we solve (1.10) with respect to  $\mathbf{q}$  we find

$$\mathbf{q} = \mathbf{q}_0 + k * \nabla \theta, \quad \text{where } (k * \nabla \theta)(x, t) := \int_0^t k(t-s) \nabla \theta(x, s) ds,$$

$\mathbf{q}_0(x, t)$  is known and can be incorporated in the source term,  $k(t)$  is a given kernel (depending on  $\varepsilon$  of course): from (1.8) we obtain the balance equation for the standard phase field model with memory which has a hyperbolic character and has been extensively studied in [4, 5].

In [8, 9, 10, 11] Green and Naghdi presented an alternative approach based on a thermomechanical theory of deformable media. This theory takes advantage of an entropy balance rather than the usual entropy inequality. If we restrict our attention to the heat conduction, these authors proposed three different theories, labeled as type I, type II and type III, respectively. In particular, when type I is linearized, we recover the classical theory based on the Fourier law

$$\mathbf{q} = -\alpha \nabla w_t, \quad \alpha > 0 \quad (\text{type I}). \quad (1.11)$$

Furthermore, linearized versions of the two other theories yield

$$\mathbf{q} = -\beta \nabla w, \quad \beta > 0 \quad (\text{type II}) \quad (1.12)$$

and

$$\mathbf{q} = -\alpha \nabla w_t - \beta \nabla w \quad (\text{type III}). \quad (1.13)$$

Note that here we have used the thermal displacement (1.6) (instead of  $\theta$ ) to write such laws. We also point out that (1.12)–(1.13) have been recently discussed, applied and

compared by Miranville and Quintanilla in [15, 16, 17] (there the reader can find a rich list of references as well). In particular, (1.13) leads via (1.8) to our equation (1.1); further, a no flux boundary condition for  $\mathbf{q}$  corresponds to  $\partial_n w = 0$  in (1.3).

Thus, the system (1.1)–(1.4) results from (1.8)–(1.9) when (1.5) and (1.13) are postulated. We are interested in the study of existence, uniqueness, regularity of the solution to the initial-boundary value problem (1.1)–(1.4) when  $\gamma$  is an arbitrary maximal monotone graph, possibly multivalued, singular and with bounded domain. Of course, the case of  $\Psi$  shaped by a multiwell potential  $u \mapsto -w_t u + \phi(u) + G(u)$  is recovered as a sample. Then we study the asymptotic behaviour of the problem as  $\beta \searrow 0$ , obtaining convergence of solutions to the problem with  $\beta = 0$ , which corresponds to (1.11), the (type I) case of Green and Naghdi. We also prove two error estimates of the difference of solutions in suitable norms, showing a linear rate of convergence in both estimates. In a subsequent study we would like to address the investigation of the analogous limit  $\alpha \searrow 0$  to obtain the (type II) case in (1.12).

The paper is organized as follows. In Section 2 we state the main results related to the problem (1.1)–(1.4): existence and uniqueness of a weak solution, regularity results yielding a strong solution, further regularity results ensuring the boundedness of  $u$ ,  $w_t$  and of the appropriate selection of  $\gamma(u)$ . Section 3 contains the related statements. Then we investigate the asymptotic limit as  $\beta \searrow 0$ : precisely, the convergence result and the error estimates under different assumptions on the data. In Section 4 we introduce some notation and present the uniqueness proof. The approximation of the problem (1.1)–(1.4) via a Faedo-Galerkin scheme and the derivation of the uniform a priori estimates are carried out in Section 5. Regularity and boundedness properties for the solutions are proved in Sections 6–8. Finally, the details of the asymptotic analysis as  $\beta \searrow 0$  are developed in Section 9.

## 2 Well-posedness and regularity for $\alpha, \beta > 0$

We point out the assumptions on the data and state clearly the formulation of the problem and the main results we achieve. Let  $\Omega \subseteq \mathbb{R}^3$  be a bounded smooth domain with boundary  $\Gamma = \partial\Omega$  and let  $T > 0$ . Set  $Q := \Omega \times (0, T)$ . We assume that

$$\alpha, \beta \in (0, +\infty) \tag{2.1}$$

$$f \in L^2(0, T; H^1(\Omega)') + L^1(0, T; L^2(\Omega)) \tag{2.2}$$

$$\gamma \subseteq \mathbb{R} \times \mathbb{R} \text{ is a maximal monotone graph, with } \gamma(0) \ni 0 \tag{2.3}$$

$$\phi : \mathbb{R} \longrightarrow [0, +\infty] \text{ is convex and lower-semicontinuous} \tag{2.4}$$

$$\phi(0) = 0 \text{ and } \partial\phi = \gamma \tag{2.5}$$

$$g : \mathbb{R} \longrightarrow \mathbb{R} \text{ is Lipschitz-continuous} \tag{2.6}$$

$$w_0 \in H^1(\Omega), \quad v_0 \in L^2(\Omega), \quad u_0 \in L^2(\Omega), \quad \phi(u_0) \in L^1(\Omega). \tag{2.7}$$

The effective domain of  $\gamma$  will be denoted by  $D(\gamma)$ . We consider

**Problem** ( $\mathbf{P}_{\alpha,\beta}$ ). Find  $(w, u, \xi)$  satisfying

$$w \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \quad (2.8)$$

$$w_{tt} \in L^1(0, T; H^1(\Omega)') \quad (2.9)$$

$$u \in H^1(0, T; H^1(\Omega)') \cap C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad (2.10)$$

$$\xi \in L^2(Q), \quad u \in D(\gamma) \quad \text{and} \quad \xi \in \gamma(u) \quad \text{a.e. in } Q \quad (2.11)$$

$$\langle w_{tt}(t), v \rangle + \alpha (\nabla w_t(t), \nabla v)_{L^2(\Omega)} + \beta (\nabla w(t), \nabla v)_{L^2(\Omega)} + \langle u_t(t), v \rangle = \langle f(t), v \rangle \quad (2.12)$$

for all  $v \in H^1(\Omega)$  and a.a.  $t \in (0, T)$

$$\langle u_t(t), v \rangle + (\nabla u(t), \nabla v)_{L^2(\Omega)} + (\xi(t), v)_{L^2(\Omega)} + (g(u)(t), v)_{L^2(\Omega)} = (w_t(t), v)_{L^2(\Omega)} \quad (2.13)$$

for all  $v \in H^1(\Omega)$  and a.a.  $t \in (0, T)$

$$w(0) = w_0 \quad \text{in } H^1(\Omega), \quad w_t(0) = v_0 \quad \text{in } H^1(\Omega)', \quad u(0) = u_0 \quad \text{in } L^2(\Omega). \quad (2.14)$$

We can prove the well-posedness of this problem.

**Theorem 2.1** (Existence and uniqueness). *Let assumptions (2.1)–(2.7) hold. Then Problem ( $\mathbf{P}_{\alpha,\beta}$ ) has a unique solution.*

Next, in addition to (2.1)–(2.7), we suppose

$$f \in L^2(0, T; L^2(\Omega)) + L^1(0, T; H^1(\Omega)) \quad (2.15)$$

$$w_0 \in H^2(\Omega), \quad \partial_n w_0 = 0 \quad \text{on } \Gamma, \quad v_0 \in H^1(\Omega), \quad u_0 \in H^1(\Omega); \quad (2.16)$$

in this case, we are able to prove a regularity result, which allows us to solve a strong formulation of Problem ( $\mathbf{P}_{\alpha,\beta}$ ).

**Theorem 2.2** (Regularity and strong solution). *Assume (2.15)–(2.16) in addition to (2.1)–(2.7). Then the unique solution  $(w, u, \xi)$  of Problem ( $\mathbf{P}_{\alpha,\beta}$ ) fulfills*

$$w \in W^{1,\infty}(0, T; H^1(\Omega)) \cap H^1(0, T; H^2(\Omega)) \quad (2.17)$$

$$w_{tt} \in L^1(0, T; L^2(\Omega)) \quad (2.18)$$

$$u \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)). \quad (2.19)$$

*In particular,  $(w, u, \xi)$  solves Problem ( $\mathbf{P}_{\alpha,\beta}$ ) in a strong sense, that is,  $w$  and  $u$  satisfy*

$$\begin{aligned} w_{tt} - \alpha \Delta w_t - \beta \Delta w + u_t &= f \quad \text{a.e. in } Q \\ u_t - \Delta u + \xi + g(u) &= w_t, \quad \xi \in \gamma(u) \quad \text{a.e. in } Q \\ \partial_n w = \partial_n u &= 0 \quad \text{a.e. on } \Gamma \times (0, T). \end{aligned}$$

The aim of the subsequent results is to provide  $L^\infty$  estimates. We will need to strengthen again the hypotheses on the initial data. For  $s \in D(\gamma)$  let us denote by  $\gamma^0(s)$  the element of  $\gamma(s)$  having minimal modulus. Then, we require that

$$u_0 \in H^2(\Omega), \quad \partial_n u_0 = 0 \quad \text{on } \Gamma \quad (2.20)$$

$$u_0 \in D(\gamma) \quad \text{a.e. in } \Omega, \quad \gamma^0(u_0) \in L^2(\Omega). \quad (2.21)$$

**Theorem 2.3** (Further regularity). *If the conditions (2.1)–(2.7), (2.15)–(2.16) and (2.20)–(2.21) hold, then the solution  $(w, u, \xi)$  of Problem  $(\mathbf{P}_{\alpha,\beta})$  fulfills*

$$u \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)). \quad (2.22)$$

The above results still hold if the dimension  $N$  of the domain  $\Omega$  is arbitrary. On the other hand, since (2.22) implies in particular that  $u$  is continuous from  $[0, T]$  to the space  $H^s(\Omega)$  for all  $s < 2$ , then, if we let  $N \leq 3$  and  $s$  sufficiently large, it turns out that  $H^s(\Omega) \subset C^0(\overline{\Omega})$  and consequently

$$u \in C^0(\overline{Q}).$$

Finally, we assume for the data enough regularity to get  $L^\infty$  estimates for  $w_t$  and  $\xi$ . The hypothesis  $N \leq 3$  is essential in the proof of the following result.

**Theorem 2.4** ( $L^\infty$  estimate for  $w_t$  and  $\xi$ ). *In addition to assumptions (2.1)–(2.7), (2.15)–(2.16) and (2.20)–(2.21), we ask*

$$f \in L^\infty(0, T; L^2(\Omega)) + L^r(0, T; H^1(\Omega)) \quad \text{for some } r > 4/3 \quad (2.23)$$

$$\gamma^0(u_0) \in L^\infty(\Omega). \quad (2.24)$$

Then we have

$$w_t \in L^\infty(Q), \quad \xi \in L^\infty(Q).$$

*Remark 2.5.* All the statements contained in this paper still hold if  $\Omega \subseteq \mathbb{R}^3$  is, for instance, a convex polyhedron, for which standard results on Sobolev embeddings and regularity for elliptic problems apply.

### 3 Asymptotic behaviour as $\beta \searrow 0$

Let us fix the parameter  $\alpha$  once and for all. We shall concentrate on the asymptotic behaviour of the solution as  $\beta \searrow 0$ , so we let  $\beta$  vary in a bounded subset of  $(0, +\infty)$ . We allow the source term and the initial data in Problem  $(\mathbf{P}_{\alpha,\beta})$  to vary with  $\beta$ , by replacing  $f$ ,  $w_0$ ,  $v_0$  and  $u_0$  in (2.12) and (2.14) with  $f_\beta$ ,  $w_{0,\beta}$ ,  $v_{0,\beta}$  and  $u_{0,\beta}$  respectively. We will denote by  $(w_\beta, u_\beta, \xi_\beta)$  the solution to Problem  $(\mathbf{P}_{\alpha,\beta})$ .

If we set  $\beta = 0$  in the statement of Problem  $(\mathbf{P}_{\alpha,\beta})$ , we get a first-order system of differential equations, with respect to time, in the variable  $w_t$ , which is of physical relevance (recall that  $w_t = \theta$ ). Anyway, we avoid this change of variable, in order to preserve the formalism. We introduce the formulation of Problem  $(\mathbf{P}_\alpha)$ , in which  $\beta$  is set to be zero.

**Problem  $(\mathbf{P}_\alpha)$ .** Find  $(w, u, \xi)$  satisfying (2.8)–(2.11) as well as

$$\begin{aligned} \langle w_{tt}(t), v \rangle + \alpha (\nabla w_t(t), \nabla v)_{L^2(\Omega)} + \langle u_t(t), v \rangle &= \langle f(t), v \rangle \\ \text{for all } v \in H^1(\Omega) \quad \text{and a.a. } t \in (0, T) \end{aligned} \quad (3.1)$$

$$\langle u_t(t), v \rangle + (\nabla u(t), \nabla v)_{L^2(\Omega)} + ((\xi + g(u))(t), v)_{L^2(\Omega)} = (w_t(t), v)_{L^2(\Omega)} \quad (3.2)$$

for all  $v \in H^1(\Omega)$  and a.a.  $t \in (0, T)$

$$w(0) = w_0 \text{ in } H^1(\Omega), \quad w_t(0) = v_0 \text{ in } H^1(\Omega)', \quad u(0) = u_0 \text{ in } L^2(\Omega). \quad (3.3)$$

We state at first the well-posedness of Problem  $(\mathbf{P}_\alpha)$  and a convergence result.

**Theorem 3.1** (Well-posedness for  $(\mathbf{P}_\alpha)$ ). *If the hypotheses (2.2)–(2.7) hold, then Problem  $(\mathbf{P}_\alpha)$  admits exactly one solution.*

**Theorem 3.2** (Convergence as  $\beta \searrow 0$ ). *We assume (2.2)–(2.7) and*

$$f_\beta \rightharpoonup f \text{ in } L^2(0, T; H^1(\Omega)') + L^1(0, T; L^2(\Omega)) \quad (3.4)$$

$$w_{0,\beta} \rightharpoonup w_0 \text{ in } H^1(\Omega), \quad v_{0,\beta} \rightharpoonup v_0, \quad u_{0,\beta} \rightharpoonup u_0 \text{ in } L^2(\Omega). \quad (3.5)$$

*Then, the convergences*

$$w_\beta \rightharpoonup^* w \text{ in } W^{1,\infty}(0, T; L^2(\Omega)), \quad w_\beta \rightharpoonup w \text{ in } H^1(0, T; H^1(\Omega))$$

$$u_\beta \rightharpoonup u \text{ in } H^1(0, T; H^1(\Omega)') \cap L^2(0, T; H^1(\Omega))$$

$$\xi_\beta \rightharpoonup \xi \text{ in } L^2(Q).$$

*hold, where  $(w, u, \xi)$  denotes the solution to Problem  $(\mathbf{P}_\alpha)$ .*

With slightly strengthened hypotheses, we are able to prove the strong convergence for the solution and even to give an estimate for the convergence rate.

**Theorem 3.3** (First error estimate). *In addition to (2.3)–(2.6) and (3.4)–(3.5), we assume*

$$\|f_\beta - f\|_{L^2(0,T;H^1(\Omega)')+L^1(0,T;L^2(\Omega))} \leq c\beta \quad (3.6)$$

$$\|w_{0,\beta} - w_0\|_{H^1(\Omega)} + \|v_{0,\beta} - v_0\|_{H^1(\Omega)'} + \|u_{0,\beta} - u_0\|_{L^2(\Omega)} \leq c\beta \quad (3.7)$$

*for some constant  $c$  which is independent of  $\beta$ . Then one has the estimate*

$$\begin{aligned} & \|w_\beta - w\|_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))} \\ & + \|u_\beta - u\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))} \leq c\beta \end{aligned} \quad (3.8)$$

*where  $c$  does not depend on  $\beta$ .*

If  $\gamma$  is a (single-valued) smooth function, and if enough regularity on the data is assumed, it is possible to obtain much stronger estimates. The assumption  $N \leq 3$  on the spatial dimension is essential for the proof of the following result.

**Theorem 3.4** (Second error estimate). *Let (2.3)–(2.6), (3.4)–(3.5) hold and*

$$\gamma : D(\gamma) \longrightarrow \mathbb{R} \text{ be single-valued and locally Lipschitz-continuous.} \quad (3.9)$$

Moreover, assume that the data  $\{f_\beta, w_{0,\beta}, v_{0,\beta}, u_{0,\beta}\}$ , as well as  $\{f, w_0, v_0, u_0\}$ , satisfy (2.15)–(2.16), (2.20)–(2.21), (2.23)–(2.24) along with

$$\|f_\beta\|_{L^\infty(0,T;L^2(\Omega))+L^r(0,T;H^1(\Omega))} + \|u_{0,\beta}\|_{H^2(\Omega)} + \|\gamma(u_{0,\beta})\|_{L^\infty(\Omega)} \leq c \quad (3.10)$$

$$\|f_\beta - f\|_{L^2(0,T;L^2(\Omega))+L^1(0,T;H^1(\Omega))} \leq c\beta \quad (3.11)$$

$$\|w_{0,\beta} - w_0\|_{H^2(\Omega)} + \|v_{0,\beta} - v_0\|_{H^1(\Omega)} + \|u_{0,\beta} - u_0\|_{H^1(\Omega)} \leq c\beta \quad (3.12)$$

where  $r > 4/3$ . Then the estimate

$$\begin{aligned} & \|w_\beta - w\|_{W^{1,\infty}(0,T;H^1(\Omega)) \cap H^1(0,T;H^2(\Omega))} \\ & + \|u_\beta - u\|_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} \leq c\beta \end{aligned} \quad (3.13)$$

holds for a suitable constant  $c$ , which may depend on  $\alpha$  but not on  $\beta$ .

## 4 Notation and uniqueness proof

Before facing the proof of all the results, for the sake of convenience we fix some notation:

$$Q_t = \Omega \times (0, t) \quad \text{for } 0 \leq t \leq T, \quad Q = Q_T,$$

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad W = \{v \in H^2(\Omega) : \partial_n v = 0 \text{ a.e. on } \Gamma\}.$$

We embed  $H$  in  $V'$ , by means of the formula

$$\langle y, v \rangle = (y, v)_H \quad \text{for all } y \in H, v \in V.$$

Furthermore, the same symbol  $\|\cdot\|_H$  will denote both the norm in  $L^2(\Omega)$  and in  $L^2(\Omega)^N$ ; we behave similarly with  $\|\cdot\|_V$ . If  $a, b$  are functions of space and time variables, we introduce the convolution product with respect to time

$$(a * b)(t) = \int_0^t a(s)b(t-s)ds, \quad 0 \leq t \leq T.$$

We also point out that the symbols  $c, c_i$  – even in the same formula – stand for different constants, depending on  $\Omega, T$  and the data, but not on the parameters  $\alpha, \beta$ . However, as we will be interested in the study of convergence as  $\beta \searrow 0$ , if a constant  $c$  depends on  $\alpha, \beta$  in such a way that  $c$  is bounded whenever  $\alpha, \beta$  lie bounded, then we will accept the notation  $c$ . A constant depending on the data and on  $\alpha$ , but not on  $\beta$ , may be denoted by  $c_\alpha$  or  $c_{\alpha,i}$  or simply  $c$ , as it will happen in Section 9.

In our computations, we will often exploit the Hölder and Young inequalities to infer

$$\int_{Q_t} ab \leq \frac{1}{2\sigma} \int_0^t \|a(s)\|_H^2 ds + \frac{\sigma}{2} \int_0^t \|b(s)\|_H^2 ds$$



where  $a, b \in L^2(Q)$  and  $\sigma > 0$  is arbitrary. We point out another inequality which will turn out to be useful: if  $\varphi \in H^1(0, T; H)$ , then the fundamental theorem of calculus and the Hölder inequality entail

$$\|\varphi(t)\|_H^2 = \left\| \varphi(0) + \int_0^t \varphi_t(s) ds \right\|_H^2 \leq 2 \|\varphi(0)\|_H^2 + 2T \int_0^t \|\varphi_t(s)\|_H^2 ds \quad (4.1)$$

for all  $0 \leq t \leq T$ . Now, let us concentrate on the uniqueness proof.

Let  $(w_1, u_1, \xi_1)$  and  $(w_2, u_2, \xi_2)$  be solutions to the Problem  $(\mathbf{P}_{\alpha, \beta})$ ; we claim that they coincide. Setting  $w = w_1 - w_2$ ,  $u = u_1 - u_2$  and  $\xi = \xi_1 - \xi_2$ , we easily get

$$\langle w_t(t), v \rangle + \alpha (\nabla w_t(t), \nabla v)_H + \beta (\nabla w(t), \nabla v)_H + \langle u_t(t), v \rangle = 0 \quad (4.2)$$

$$\langle u_t(t), v \rangle + (\nabla u(t), \nabla v)_H + (\xi(t), v)_H + (g(u_1)(t) - g(u_2)(t), v)_H = (w_t(t), v)_H \quad (4.3)$$

for all  $v \in V$  and a.a.  $0 \leq t \leq T$ , along with the initial conditions

$$w(0) = w_t(0) = u(0) = 0. \quad (4.4)$$

We choose  $v = u(t)$  in equation (4.3) and integrate over  $(0, t)$ ; thus, we obtain

$$\frac{1}{2} \|u(t)\|_H^2 + \int_0^t \|\nabla u(s)\|_H^2 ds + \int_{Q_t} \xi u = - \int_{Q_t} (g(u_1) - g(u_2)) u + \int_{Q_t} w_t u.$$

Accounting for the Lipschitz-continuity of  $g$ , the Hölder inequality and the monotonicity of  $\gamma$ , from the above equality we easily derive

$$\frac{1}{2} \|u(t)\|_H^2 + \int_0^t \|\nabla u(s)\|_H^2 ds \leq c \int_0^t \|u(s)\|_H^2 ds + \int_{Q_t} w_t u. \quad (4.5)$$

Integrating in time the equation (4.2) (this is possible thanks to (2.8)) and taking the initial data (4.4) into account, we have

$$(w_t(t), v)_H + \alpha (\nabla w(t), \nabla v)_H + \beta (1 * \nabla w(t), \nabla v)_H + (u(t), v)_H = 0; \quad (4.6)$$

we choose  $v = w_t(t)$  in (4.6) and integrate over  $(0, t)$ . Noticing that the equality

$$(1 * \nabla w(t), \nabla w_t(t))_H = \frac{d}{dt} (1 * \nabla w(t), \nabla w(t))_H - \|\nabla w(t)\|_H^2 \quad (4.7)$$

holds, we get

$$\begin{aligned} \int_0^t \|w_t(s)\|_H^2 ds + \frac{\alpha}{2} \|\nabla w(t)\|_H^2 &= -\beta (1 * \nabla w(t), \nabla w(t))_H \\ &+ \beta \int_0^t \|\nabla w(s)\|_H^2 ds - \int_{Q_t} u w_t. \end{aligned} \quad (4.8)$$

The Hölder inequality and (4.1) allow us to deal with the right-hand side of this formula:

$$-\beta (1 * \nabla w(t), \nabla w(t))_H \leq \frac{c\beta^2}{\alpha} \int_0^t \|\nabla w(s)\|_H^2 ds + \frac{\alpha}{4} \|\nabla w(t)\|_H^2. \quad (4.9)$$

Collecting now (4.5), (4.8) and (4.9), it follows that

$$\begin{aligned} \frac{1}{2} \|u(t)\|_H^2 + \int_0^t \|\nabla u(s)\|_H^2 ds + \int_0^t \|w_t(s)\|_H^2 ds + \frac{\alpha}{4} \|\nabla w(t)\|_H^2 \\ \leq c \int_0^t \|u(s)\|_H^2 ds + c \left( \beta + \frac{\beta^2}{\alpha} \right) \int_0^t \|\nabla w(s)\|_H^2 ds; \end{aligned}$$

then, by applying the Gronwall lemma and recalling (4.4), we obtain  $u = w = 0$  almost everywhere in  $Q$ . A comparison in (2.13) and the density of  $H^1(Q)$  as a subspace of  $L^2(Q)$  entail  $\xi = 0$  almost everywhere in  $Q$ ; thus, the proof of uniqueness is complete.

## 5 Approximation and a priori estimates

We are going to prove the existence of a solution to Problem  $(\mathbf{P}_{\alpha,\beta})$  via a Faedo-Galerkin method. First, we approximate the graph  $\gamma$  with its Yosida regularization: for all  $\varepsilon \in (0, 1]$  say, we let

$$\gamma_\varepsilon := \frac{1}{\varepsilon} \{I - (I + \varepsilon\gamma)^{-1}\} \quad \text{and} \quad \phi_\varepsilon(s) := \min_{\tau \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon} |\tau - s|^2 + \phi(\tau) \right\} \quad \text{for } s \in \mathbb{R}$$

where  $I$  denotes the identity on  $\mathbb{R}$ . We recall that  $\phi_\varepsilon$  is a nonnegative, convex and differentiable function,  $\gamma_\varepsilon$  is Lipschitz-continuous, monotone and

$$\gamma_\varepsilon(0) = 0, \quad \phi'_\varepsilon = \gamma_\varepsilon, \quad 0 \leq \phi_\varepsilon(s) \leq \phi(s), \quad |\gamma_\varepsilon(s)| \leq |\gamma^0(s)| \quad \forall \varepsilon > 0, s \in \mathbb{R} \quad (5.1)$$

(see, e.g., [2, Prop. 2.6, p. 28 and Prop. 2.11, p.39] or [1, pp. 57–58]).

We look for a solution of the approximating problem in a finite-dimensional subspace  $V_n \subseteq V$ , choosing a sequence  $\{V_n\}$  filling up  $V$ ; then we get a priori estimates and use compactness arguments to take the limit as  $n \rightarrow +\infty$ . In a second step we let  $\varepsilon \searrow 0$ .

A special choice of the approximating subspaces will be useful. Let  $\{v_i\}_{i \in \mathbb{N}}$  be an orthonormal basis for  $V$  satisfying

$$-\Delta v_i = \lambda_i v_i \quad \text{in } \Omega, \quad \partial_n v_i = 0 \quad \text{on } \Gamma \quad (5.2)$$

where  $\{\lambda_i\}_{i \in \mathbb{N}}$  are the eigenvalues of the Laplace operator; also, let  $V_n$  be the subspace of  $V$  spanned by  $v_1, \dots, v_n$ , for all  $n \in \mathbb{N}$ . Thus, we have defined an increasing sequence of subspaces, whose union is dense in  $V$ , and hence in  $H$ ; furthermore, we notice that the regularity of  $\Omega$  implies  $V_n \subseteq W$ , for all  $n \in \mathbb{N}$ .

As approximations of the data  $w_0, v_0, u_0$  we choose the projections on  $V_n$ : let  $w_{0,n}$  be the projection of  $w_0$ , with respect to  $V$ , and let  $v_{0,n}, u_{0,n}$  be the projections of  $v_0, u_0$ , with respect to  $H$ . We notice that

$$w_{0,n} \rightarrow w_0 \quad \text{in } V, \quad v_{0,n} \rightarrow v_0 \quad \text{in } H, \quad u_{0,n} \rightarrow u_0 \quad \text{in } H. \quad (5.3)$$

We also need to regularize the source term  $f$ : so, we first write

$$f = f^{(1)} + f^{(2)}, \quad \text{where } f^{(1)} \in L^2(0, T; V') \quad \text{and} \quad f^{(2)} \in L^1(0, T; H), \quad (5.4)$$

then we assume  $f_n^{(1)}$ ,  $f_n^{(2)}$  to be functions in  $C^0([0, T]; V')$ ,  $C^0([0, T]; H)$  respectively, such that

$$f_n^{(1)} \longrightarrow f^{(1)} \text{ in } L^2(0, T; V'), \quad f_n^{(2)} \longrightarrow f^{(2)} \text{ in } L^1(0, T; H); \quad (5.5)$$

we also set  $f_n = f_n^{(1)} + f_n^{(2)}$ .

Now we are ready to state the approximated problem. For the sake of simplicity, we do not specify explicitly the dependency on  $\varepsilon$  in the solution.

**Problem  $(\mathbf{P}_{\alpha, \beta})_{n, \varepsilon}$ .** Find  $T_n \in (0, T]$  and  $(w_n, u_n)$  satisfying

$$\begin{aligned} w_n &\in C^2([0, T_n]; V_n), & u_n &\in C^1([0, T_n]; V_n) \\ (\partial_t^2 w_n(t), v)_H + \alpha (\nabla \partial_t w_n(t), \nabla v)_H + \beta (\nabla w_n(t), \nabla v)_H + (\partial_t u_n(t), v)_H \\ &= \langle f_n(t), v \rangle \quad \text{for all } v \in V_n \text{ and all } t \in [0, T_n] \end{aligned} \quad (5.6)$$

$$\begin{aligned} (\partial_t u_n(t), v)_H + (\nabla u_n(t), \nabla v)_H + (\gamma_\varepsilon(u_n)(t), v)_H + (g(u_n)(t), v)_H \\ = (\partial_t w_n(t), v)_H \quad \text{for all } v \in V_n \text{ and all } t \in [0, T_n] \end{aligned} \quad (5.7)$$

$$w_n(0) = w_{0,n}, \quad \partial_t w_n(0) = v_{0,n}, \quad u_n(0) = u_{0,n}. \quad (5.8)$$

Writing  $w_n$  and  $u_n$  as linear combinations of  $v_1, \dots, v_n$  with time-dependent coefficients, and testing equations (5.6) and (5.7) by  $v = v_1, \dots, v_n$ , we obtain a system of ordinary differential equations, for whose local existence and uniqueness standard results apply. Thus, Problem  $(\mathbf{P}_{\alpha, \beta})_{n, \varepsilon}$  admits a solution, defined on some interval  $[0, T_n]$ . The following estimates imply that these solutions can be extended over the whole interval  $[0, T]$ .

**First a priori estimate.** We choose  $v = u_n(t)$  in equation (5.7) and integrate over  $(0, t)$ :

$$\begin{aligned} \frac{1}{2} \|u_n(t)\|_H^2 + \int_0^t \|\nabla u_n(s)\|_H^2 ds + \int_{Q_t} \gamma_\varepsilon(u_n)u_n \\ = - \int_{Q_t} g(u_n)u_n + \int_{Q_t} u_n \partial_t w_n + \frac{1}{2} \|u_{0,n}\|_H^2. \end{aligned}$$

The last term in the left-hand side is non negative, because  $\gamma_\varepsilon$  is increasing and  $\gamma_\varepsilon(0) = 0$ ; it will be ignored in the following estimates. Meanwhile, the right-hand side can be easily estimated using the Lipschitz-continuity of  $g$  and (5.3); so we get

$$\frac{1}{2} \|u_n(t)\|_H^2 + \int_0^t \|\nabla u_n(s)\|_H^2 ds \leq c \int_0^t \|u_n(s)\|_H^2 ds + \int_{Q_t} u_n \partial_t w_n + c. \quad (5.9)$$

Following the same computation as in the uniqueness proof, we integrate equation (5.6) with respect to time:

$$\begin{aligned} (\partial_t w_n(t), v)_H + \alpha (\nabla w_n(t), \nabla v)_H + \beta (1 * \nabla w_n(t), \nabla v)_H + (u_n(t), v)_H \\ = \langle 1 * f_n^{(1)}(t), v \rangle + (1 * f_n^{(2)}(t), v)_H + (v_{0,n} + u_{0,n}, v)_H + \alpha (\nabla w_{0,n}, \nabla v)_H \end{aligned} \quad (5.10)$$

for all  $v \in V_n$  and  $0 \leq t \leq T_n$ . We take  $v = \partial_t w_n(t)$  in the previous equation and integrate over  $(0, t)$ . Recalling the identity (4.7), we have

$$\int_0^t \|\partial_t w_n(s)\|_H^2 ds + \frac{\alpha}{2} \|\nabla w_n(t)\|_H^2 = \sum_{i=1}^7 T_i(t) + \frac{\alpha}{2} \|\nabla w_{0,n}\|_H^2 \quad (5.11)$$

where we have set

$$\begin{aligned} T_1(t) &= \beta \int_0^t \|\nabla w_n(s)\|_H^2 ds, & T_2(t) &= -\beta (1 * \nabla w_n(t), \nabla w_n(t))_H \\ T_3(t) &= - \int_{Q_t} u_n \partial_t w_n, & T_4(t) &= \int_0^t \langle 1 * f_n^{(1)}(s), \partial_t w_n(s) \rangle ds, & T_5(t) &= \int_{Q_t} (1 * f_n^{(2)}) \partial_t w_n \\ T_6(t) &= \int_0^t (v_{0,n} + u_{0,n}, \partial_t w_n(s))_H ds, & T_7(t) &= \alpha \int_0^t (\nabla w_{0,n}, \nabla \partial_t w_n(s))_H ds. \end{aligned}$$

We do not need any estimate on terms  $T_1$  and  $T_3$ . With simple applications of the Hölder inequality, we estimate  $T_2$ ,  $T_5$  and  $T_6$ :

$$\begin{aligned} T_2(t) &\leq \frac{\alpha}{8} \|\nabla w_n(t)\|_H^2 + \frac{c\beta^2}{\alpha} \int_0^t \|\nabla w_n(s)\|_H^2 ds \\ T_5(t) &\leq \frac{1}{4} \int_0^t \|\partial_t w_n(s)\|_H^2 ds + \int_0^t \|1 * f_n^{(2)}(s)\|_H^2 ds \\ T_6(t) &\leq \frac{1}{4} \int_0^t \|\partial_t w_n(s)\|_H^2 ds + c \|v_{0,n}\|_H^2 + c \|u_{0,n}\|_H^2. \end{aligned}$$

We deal with  $T_7$  by direct integration and the use of the Hölder inequality:

$$T_7(t) = \alpha (\nabla w_n(t), \nabla w_{0,n})_H - \alpha \|\nabla w_{0,n}\|_H^2 \leq \frac{\alpha}{8} \|\nabla w_n(t)\|_H^2 + \alpha \|\nabla w_{0,n}\|_H^2.$$

Now we pay attention to  $T_4$  and integrate by parts in time:

$$\begin{aligned} T_4(t) &= \langle 1 * f_n^{(1)}(t), w_n(t) \rangle - \int_0^t \langle f_n^{(1)}(s), w_n(s) \rangle ds \leq \frac{1}{2\sigma} \|1 * f_n^{(1)}(t)\|_{V'}^2 \\ &\quad + \frac{\sigma}{2} \|w_n(t)\|_V^2 + \frac{1}{2} \int_0^t \|f_n^{(1)}(s)\|_{V'}^2 ds + \frac{1}{2} \int_0^t \|w_n(s)\|_V^2 ds, \end{aligned}$$

where  $\sigma > 0$  is arbitrary, to be set later. According to the definition of the norm in  $V$  and the inequality (4.1), we have

$$\begin{aligned} T_4(t) &\leq \frac{1}{2\sigma} \|1 * f_n^{(1)}(t)\|_{V'}^2 + \sigma T \int_0^t \|\partial_t w_n(s)\|_H^2 ds + \frac{\sigma}{2} \|\nabla w_n(t)\|_H^2 \\ &\quad + \frac{1}{2} \int_0^t \|f_n^{(1)}(s)\|_{V'}^2 ds + T \int_0^t \left( \int_0^s \|\partial_t w_n(\tau)\|_H^2 d\tau \right) ds \\ &\quad + \frac{1}{2} \int_0^t \|\nabla w_n(s)\|_H^2 ds + T(\sigma + 1) \|w_{0,n}\|_H^2. \end{aligned}$$

We collect all the terms containing  $\|\partial_t w_n\|_{L^2(0,t;H)}$  and  $\|\nabla w_n(t)\|_H$  in the left-hand side of (5.11); their coefficients turn out to be, respectively,

$$k_1 = \frac{1}{2} - T\sigma, \quad k_2 = \frac{1}{2} \left( \frac{\alpha}{2} - \sigma \right).$$

We choose  $\sigma \leq \min \{\alpha/4, 1/4T\}$ , so that  $k_1 \geq 1/4$ ,  $k_2 \geq \alpha/8$ . We also remark that the assumptions (5.5) and (5.3) enable us to get a bound for terms involving  $f_n^{(1)}$ ,  $f_n^{(2)}$  and

the initial data. Finally, adding (5.9) and (5.11) and taking into account all the previous inequalities, we obtain

$$\begin{aligned} & \frac{1}{2} \|u_n(t)\|_H^2 + \int_0^t \|\nabla u_n(s)\|_H^2 ds + \frac{1}{4} \int_0^t \|\partial_t w_n(s)\|_H^2 ds + \frac{\alpha}{8} \|\nabla w_n(t)\|_H^2 \\ & \leq c \int_0^t \|u_n(s)\|_H^2 ds + T \int_0^t \left( \int_0^s \|\partial_t w_n(\tau)\|_H^2 d\tau \right) ds + c_\alpha \int_0^t \|\nabla w_n(s)\|_H^2 ds + c_\alpha. \end{aligned}$$

The Gronwall lemma entails

$$\|u_n\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|w_n\|_{H^1(0,T;H)} + \sqrt{\alpha} \|w_n\|_{L^\infty(0,T;V)} \leq c_\alpha. \quad (5.12)$$

**Second a priori estimate.** Since  $\phi_\varepsilon$  is at most of quadratic growth, by definition, and  $\gamma_\varepsilon$  is Lipschitz-continuous, from the estimate (5.12) we directly derive

$$\|\phi_\varepsilon(u_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq c'_{\alpha,1} \quad (5.13)$$

$$\|\gamma_\varepsilon(u_n)\|_{L^2(Q)} \leq c'_{\alpha,2}; \quad (5.14)$$

where the symbols  $c'_{\alpha,i}$  denote positive constants, possibly depending on  $\varepsilon$  and  $\alpha$ , but not on  $n$  and  $\beta$ .

By (5.2), we can easily check that

$$(y, z)_H = (P_n y, z)_H \quad \text{for all } y \in V, \quad z \in V_n$$

where  $P_n y$  is the projection of  $y$  in  $V_n$ , with respect to  $V$ . Then, as we have a uniform estimate for  $u_n$  in  $L^2(0, T; V)$ , it is not difficult to extract from (5.7) the property

$$\|\partial_t u_n\|_{L^2(0,T;V')} \leq c'_{\alpha,3}. \quad (5.15)$$

**Third a priori estimate.** We take  $v = \partial_t w_n(t)$  as a test function in equation (5.6) and integrate over  $(0, t)$ ; thanks to the Hölder inequality, we get

$$\begin{aligned} & \frac{1}{2} \|\partial_t w_n(t)\|_H^2 + \alpha \int_0^t \|\nabla \partial_t w_n(s)\|_H^2 ds + \frac{\beta}{2} \|\nabla w_n(t)\|_H^2 \\ & \leq \int_0^t \langle f_n^{(1)} - \partial_t u_n(s), \partial_t w_n(s) \rangle ds + \int_0^t \|f_n^{(2)}(s)\|_H \|\partial_t w_n(s)\|_H ds \\ & \quad + \frac{1}{2} \|v_{0,n}\|_H^2 + \frac{\beta}{2} \|\nabla w_{0,n}\|_H^2. \end{aligned} \quad (5.16)$$

We consider the term involving  $f_n^{(1)} - \partial_t u_n$ :

$$\begin{aligned} \int_0^t \langle f_n^{(1)} - \partial_t u_n(s), \partial_t w_n(s) \rangle ds & \leq \frac{c}{\alpha} \int_0^t \|f_n^{(1)}(s)\|_{V'}^2 ds + \frac{c}{\alpha} \int_0^t \|\partial_t u_n(s)\|_{V'}^2 ds \\ & \quad + \frac{\alpha}{2} \int_0^t \|\partial_t w_n(s)\|_H^2 ds + \frac{\alpha}{2} \int_0^t \|\nabla \partial_t w_n(s)\|_H^2 ds. \end{aligned}$$

Because of the estimate (5.15) and the properties (5.5) and (5.3), from (5.16) we deduce

$$\begin{aligned} & \frac{1}{2} \|\partial_t w_n(t)\|_H^2 + \frac{\alpha}{2} \int_0^t \|\nabla \partial_t w_n(s)\|_H^2 ds + \frac{\beta}{2} \|\nabla w_n(t)\|_H^2 \\ & \leq c' + \frac{\alpha}{2} \int_0^t \|\partial_t w_n(s)\|_H^2 ds + \int_0^t \|f_n^{(2)}(s)\|_H \|\partial_t w_n(s)\|_H ds, \end{aligned}$$

where  $c'$  depends on  $\varepsilon, \alpha$ . Hence, by a generalized version of the Gronwall lemma (see, e.g., [2, pp. 156–157]), we infer that

$$\|w_n\|_{W^{1,\infty}(0,T;H)} + \sqrt{\alpha} \|w_n\|_{H^1(0,T;V)} \leq c'_{\alpha,4}. \quad (5.17)$$

**Passage to the limit as  $n \rightarrow +\infty$ .** From the estimates (5.12), (5.13)–(5.15), (5.17), with standard arguments of weak or weak\* compactness we can find functions  $(w_\varepsilon, u_\varepsilon)$  such that, possibly taking a subsequence as  $n \rightarrow +\infty$ ,

$$w_n \rightharpoonup^* w_\varepsilon \quad \text{in } W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V) \quad (5.18)$$

$$w_n \rightarrow w_\varepsilon \quad \text{in } H^1(0,T;V) \quad (5.19)$$

$$u_n \rightarrow u_\varepsilon \quad \text{in } H^1(0,T;V') \cap L^2(0,T;V) \quad (5.20)$$

$$u_n \rightharpoonup^* u_\varepsilon \quad \text{in } L^\infty(0,T;H). \quad (5.21)$$

Note that (5.19) implies the strong convergence

$$w_n \rightarrow w_\varepsilon \quad \text{in } C^0([0,T];H); \quad (5.22)$$

on the other hand, the generalised Ascoli theorem and the Aubin-Lions lemma (see, e.g., [13, pp. 57–58] and [18, Sect. 8, Cor. 4]) entail

$$u_n \rightarrow u_\varepsilon \quad \text{strongly in } C^0([0,T];V') \quad \text{and in } L^2(Q); \quad (5.23)$$

thus, since  $g$  and  $\gamma_\varepsilon$  are Lipschitz-continuous, we easily check that

$$g(u_n) \rightarrow g(u_\varepsilon) \quad \text{and} \quad \gamma_\varepsilon(u_n) \rightarrow \xi_\varepsilon \quad \text{strongly in } L^2(Q),$$

where  $\xi_\varepsilon = \gamma_\varepsilon(u_\varepsilon)$ . We then take the limit as  $n \rightarrow +\infty$  in (5.6)–(5.8) and see that  $(w_\varepsilon, u_\varepsilon, \xi_\varepsilon)$  fulfills equations (2.11)–(2.14), where  $\gamma$  is replaced by  $\gamma_\varepsilon$ . Indeed, by (5.22)–(5.23) and (5.3), it is obvious that  $w_\varepsilon(0) = w_0, u_\varepsilon(0) = u_0$ . To deal with the last initial condition properly, we fix a test function  $v \in V_m$ , where  $m \geq 1$  is arbitrary, and we integrate in time equation (5.6); we get equation (5.10), for  $0 \leq t \leq T$  and  $n \geq m$ . Arguing as in [13, pp. 12–13], we can take the limit in (5.10), (5.7) and check that  $(w_\varepsilon, u_\varepsilon, \xi_\varepsilon)$  fulfills

$$\begin{aligned} \langle \partial_t w_\varepsilon(t), v \rangle &= -\alpha (\nabla w_\varepsilon(t), \nabla v)_H - \beta (1 * \nabla w_\varepsilon(t), \nabla v)_H \\ &- \langle u_\varepsilon(t), v \rangle + \langle 1 * f(t), v \rangle + \alpha (\nabla w_0, \nabla v)_H + (v_0 + u_0, v)_H \end{aligned} \quad (5.24)$$

$$\langle \partial_t u_\varepsilon(t), v \rangle + (\nabla u_\varepsilon(t), \nabla v)_H + (\xi_\varepsilon(t), v)_H + (g(u_\varepsilon)(t), v)_H = (\partial_t w_\varepsilon(t), v)_H \quad (5.25)$$

for a.a.  $t \in (0, T)$ ,  $m \geq 1$  and  $v \in V_m$ ; by a density argument, the same equalities hold when  $v \in V$ . Since the right-hand side in (5.24) is a continuous function in  $[0, T]$ , taking  $t = 0$  we find that

$$\langle \partial_t w_\varepsilon(0), v \rangle = (v_0, v)_H \quad \text{for all } v \in V$$

whence the second of (2.14) follows.

**Fifth a priori estimate.** As a consequence of the weak lower semi-continuity of the norm in a Banach space,  $(w_\varepsilon, u_\varepsilon, \xi_\varepsilon)$  satisfy the estimate (5.12); we now need to improve estimates (5.13)–(5.15), (5.17).

We first notice that, because of the Lipschitz-continuity of  $\gamma_\varepsilon$ ,  $\xi_\varepsilon(t) \in V$  for all  $t$ ; thus, we can choose  $v = \xi_\varepsilon(t)$  in equation (5.25) and integrate over  $(0, t)$ , to get

$$\int_{Q_t} \partial_t u_\varepsilon \xi_\varepsilon + \int_{Q_t} \gamma'_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \int_0^t \|\xi_\varepsilon(s)\|_H^2 ds = \int_{Q_t} g(u_\varepsilon) \xi_\varepsilon + \int_{Q_t} \partial_t w_\varepsilon \xi_\varepsilon. \quad (5.26)$$

In view of (5.1), we have

$$\int_{Q_t} \partial_t u_\varepsilon \xi_\varepsilon = \int_{Q_t} \frac{\partial}{\partial t} (\phi_\varepsilon(u_\varepsilon)) = \|\phi_\varepsilon(u_\varepsilon(t))\|_{L^1(\Omega)} - \|\phi_\varepsilon(u_0)\|_{L^1(\Omega)};$$

on the other hand, because of the Lipschitz continuity of  $g$ ,

$$\int_{Q_t} g(u_\varepsilon) \xi_\varepsilon \leq c \int_{Q_t} (|u_\varepsilon| + 1) \xi_\varepsilon \leq c \int_0^t (\|u_\varepsilon(s)\|_H^2 + 1) ds + \frac{1}{2} \int_0^t \|\xi_\varepsilon(s)\|_H^2 ds.$$

From these estimates and (5.26), we derive

$$\begin{aligned} & \int_\Omega \phi_\varepsilon(u_\varepsilon)(t) + \int_{Q_t} \gamma'_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_0^t \|\xi_\varepsilon(s)\|_H^2 ds \\ & \leq c \int_0^t \|u_\varepsilon(s)\|_H^2 ds + c \int_0^t \|\partial_t w_\varepsilon(s)\|_H^2 ds + \int_\Omega \phi_\varepsilon(u_0) + c. \end{aligned}$$

We notice that the second term in the left-hand side is nonnegative, because of the monotonicity of  $\gamma_\varepsilon$ . Secondly, accounting for (5.12), (5.1) and (2.7), we infer that

$$\|\phi_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\gamma_\varepsilon(u_\varepsilon)\|_{L^2(Q)} \leq c_\alpha. \quad (5.27)$$

Now, by comparison in the equation (5.25), we have

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;V')} \leq c_\alpha; \quad (5.28)$$

and consequently we can also establish the estimate (5.17), now for a constant which is independent of  $\varepsilon$ .

**Passage to the limit as  $\varepsilon \searrow 0$ .** We are able to repeat the compactness argument as above and find  $(w, u, \xi)$ , a candidate for the solution to Problem  $(\mathbf{P}_{\alpha,\beta})$ , as a limit of a subsequence of  $(w_\varepsilon, u_\varepsilon, \xi_\varepsilon)$ . The proof will be easily completed by the passage to the limit as  $\varepsilon \searrow 0$ , provided that we deduce (2.11).

By construction, we can assume that

$$\xi_\varepsilon \rightharpoonup \xi \text{ in } L^2(Q), \quad u_\varepsilon \longrightarrow u \text{ in } L^2(Q),$$

from which the equality

$$\lim_{\varepsilon \searrow 0} \int_Q \xi_\varepsilon u_\varepsilon = \int_Q \xi u$$

follows; at this point, we apply [1, Prop. 1.1, p. 42] and deduce (2.11). Thus, the proof of the existence of a solution to Problem  $(\mathbf{P}_{\alpha,\beta})$  is complete.

## 6 Regularity and strong solutions

This section is devoted to the derivation of further a priori estimates on the approximating solutions  $(w_n, u_n, \xi_n)$ , which are independent of  $n$  and  $\varepsilon$ , under stronger assumptions. The same compactness – passage to the limit arguments then apply, and this will prove Theorem 2.2. We first notice that the hypothesis (2.16) and  $V_n \subseteq W$  make it possible to assume

$$w_{0,n} \longrightarrow w_0 \text{ in } W, \quad v_{0,n} \longrightarrow v_0 \text{ and } u_{0,n} \longrightarrow u_0 \text{ in } V; \quad (6.1)$$

on the other hand, owing to (2.15), we can require  $f_n^{(1)} \in L^2(Q)$ ,  $f_n^{(2)} \in L^1(0, T; V)$  for all  $n \in \mathbb{N}$  and

$$f_n^{(1)} \longrightarrow f^{(1)} \text{ in } L^2(Q), \quad f_n^{(2)} \longrightarrow f^{(2)} \text{ in } L^1(0, T; V). \quad (6.2)$$

**Sixth a priori estimate.** We choose  $v = \partial_t w_n(t)$  in the equation (5.6) and integrate over  $(0, t)$ ; an application of the Hölder inequality yields

$$\begin{aligned} \frac{1}{2} \|\partial_t w_n(t)\|_H^2 + \alpha \int_0^t \|\nabla \partial_t w_n(s)\|_H^2 ds + \frac{\beta}{2} \|\nabla w_n(t)\|_H^2 &\leq - \int_{Q_t} \partial_t u_n \partial_t w_n \\ &+ \int_0^t \|f_n(s)\|_H \|\partial_t w_n(s)\|_H ds + \frac{1}{2} \|v_{0,n}\|_H^2 + \frac{\beta}{2} \|\nabla w_{0,n}\|_H^2. \end{aligned} \quad (6.3)$$

Now, we take  $v = \partial_t u_n(t)$  in (5.7) and integrate over  $(0, t)$ ; recalling that  $\gamma_\varepsilon = \phi'_\varepsilon$ , using the Hölder inequality and the Lipschitz-continuity of  $g$ , we get

$$\begin{aligned} &\frac{1}{2} \int_0^t \|\partial_t u_n(s)\|_H^2 ds + \frac{1}{2} \|\nabla u_n(t)\|_H^2 + \|\phi_\varepsilon(u_n(t))\|_{L^1(\Omega)} \\ &\leq \int_{Q_t} \partial_t u_n \partial_t w_n + c \int_0^t (\|u_n(s)\|_H^2 + 1) ds + \|\phi_\varepsilon(u_{0,n})\|_{L^1(\Omega)}. \end{aligned} \quad (6.4)$$

Adding (6.3) and (6.4), thanks to the assumptions (2.7), (5.3), the inequality (4.1) and  $\phi_\varepsilon \leq \phi$ , we finally have

$$\begin{aligned} &\frac{1}{2} \|\partial_t w_n(t)\|_H^2 + \alpha \int_0^t \|\nabla \partial_t w_n(s)\|_H^2 ds + \frac{\beta}{2} \|\nabla w_n(t)\|_H^2 \\ &+ \frac{1}{2} \int_0^t \|\partial_t u_n(s)\|_H^2 ds + \frac{1}{2} \|\nabla u_n(t)\|_H^2 + \|\phi_\varepsilon(u_n(t))\|_{L^1(\Omega)} \\ &\leq c \int_0^t \left( \int_0^s \|\partial_t u_n(\tau)\|_H^2 d\tau \right) ds + \int_0^t \|f_n(s)\|_H \|\partial_t w_n(s)\|_H ds + c. \end{aligned}$$

The generalised Gronwall lemma (see, e.g., [2, pp. 156–157]) enables us to achieve

$$\|w_n\|_{W^{1,\infty}(0,T;H)} + \sqrt{\alpha} \|w_n\|_{H^1(0,T;V)} + \sqrt{\beta} \|w_n\|_{L^\infty(0,T;V)} \leq c_1 \quad (6.5)$$

$$\|u_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c_2. \quad (6.6)$$

*Remark 6.1.* Only the hypotheses (2.1)–(2.7) and  $f \in L^1(0, T; H)$  have been effectively exploited in the proof of this estimate.



*Remark 6.2.* By means of (6.5)–(6.6), the estimates (5.27)–(5.28) can be rewritten in terms of some constant which is independent of  $\alpha$ .

**Seventh a priori estimate.** We take  $v = -\Delta u_n(t)$  in equation (5.7); this is possible, because of the special choice of the approximating space  $V_n$ . We integrate over  $(0, t)$  and use the Hölder inequality and the Lipschitz continuity of  $g$ :

$$\begin{aligned} & \frac{1}{2} \|\nabla u_n(t)\|_H^2 + \int_0^t \|\Delta u_n(s)\|_H^2 ds + \int_{Q_t} \gamma'_\varepsilon(u_n) |\nabla u_n|^2 \\ &= - \int_{Q_t} g'(u_n) |\nabla u_n|^2 - \int_{Q_t} \partial_t w_n \Delta u_n + \frac{1}{2} \|\nabla u_{0,n}\|_H^2 \\ &\leq c \|\nabla u_n\|_{L^2(0,T;H)}^2 + \frac{1}{2} \|\partial_t w_n\|_{L^2(0,T;H)}^2 + \frac{1}{2} \int_0^t \|\Delta u_n(s)\|_H^2 ds + \frac{1}{2} \|\nabla u_{0,n}\|_H^2. \end{aligned}$$

The monotonicity of  $\gamma_\varepsilon$  yields that the last term in the left-hand side is non negative. Owing to conditions (6.1) on the data and estimates (6.5), (6.6), we have

$$\frac{1}{2} \int_0^t \|\Delta u_n(s)\|_H^2 ds \leq c \quad \text{for all } 0 \leq t \leq T;$$

hence, on account of this inequality, the estimate (6.6) and the boundary conditions for  $u_n$ , known regularity results for elliptic problems entail

$$\|u_n\|_{L^2(0,T;W)} \leq c_3, \quad (6.7)$$

where  $c_3$  does not depend on  $\alpha, \beta$ .

**Eigth a priori estimate.** Since  $w_n \in C^2([0, T]; V_n)$ , the special choice of  $V_n$  enables us to take  $v = -\Delta \partial_t w_n(t)$  as a test function in the equation (5.6). We integrate over  $(0, t)$  and use the Hölder inequality:

$$\begin{aligned} & \frac{1}{2} \|\nabla \partial_t w_n(t)\|_H^2 + \alpha \int_0^t \|\Delta \partial_t w_n(s)\|_H^2 ds + \frac{\beta}{2} \|\Delta w_n(t)\|_H^2 \\ &\leq \frac{\alpha}{2} \int_0^t \|\Delta \partial_t w_n(s)\|_H^2 ds + \frac{1}{\alpha} \int_0^t \|\partial_t u_n(s)\|_H^2 ds + \frac{1}{\alpha} \int_0^t \|f_n^{(1)}(s)\|_H^2 ds \\ &\quad - \int_{Q_t} f_n^{(2)} \Delta \partial_t w_n + \frac{1}{2} \|\nabla v_{0,n}\|_H^2 + \frac{\beta}{2} \|\Delta w_{0,n}\|_H^2. \end{aligned} \quad (6.8)$$

For the term involving  $f_n^{(2)}$ , we integrate by parts in space, recalling that  $\partial_n v = 0$  for all  $v \in V_n$ :

$$\left| \int_{Q_t} f_n^{(2)} \Delta \partial_t w_n \right| = \left| \int_{Q_t} \nabla f_n^{(2)} \cdot \nabla \partial_t w_n \right| \leq \int_0^t \|\nabla f_n^{(2)}(s)\|_H \|\nabla \partial_t w_n(s)\|_H ds. \quad (6.9)$$

Then, in view of (6.1), (6.2), (6.6) and owing to the generalized Gronwall lemma (see [2, pp. 156–157]), from (6.8)–(6.9) we obtain

$$\|w_n\|_{W^{1,\infty}(0,T;V)} + \sqrt{\alpha} \|w_n\|_{H^1(0,T;W)} + \sqrt{\beta} \|w_n\|_{L^\infty(0,T;W)} \leq c_{\alpha,4}. \quad (6.10)$$

Finally, if we choose  $v = \partial_t^2 w_n(t)$  in the equation (5.6), we get

$$\|\partial_t^2 w_n(t)\|_H^2 \leq \{\alpha \|\partial_t w_n(t)\|_W + \beta \|w_n(t)\|_W + \|\partial_t u_n(t)\|_H + \|f_n(t)\|_H\} \|\partial_t^2 w_n(t)\|_H ;$$

thanks to the estimates above, it is easy to derive

$$\|\partial_t^2 w_n\|_{L^1(0,T;H)} \leq c_{\alpha,5} . \quad (6.11)$$

Having established all the a priori estimates corresponding to (2.17)–(2.19) on the solutions of the approximating problem, we have completed the proof of Theorem 2.2.

## 7 Further regularity

Throughout this section we assume (2.20) and (2.21) in addition to all the hypotheses we had in Section 6. As we are interested in proving Theorem 2.3, we should get further estimates on the solution of the approximated problem. By the stronger assumptions on the initial data, we can require

$$u_{0,n} \longrightarrow u_0 \quad \text{in } W . \quad (7.1)$$

Consider the equation (5.7) and derive it, with respect to time, obtaining

$$\begin{aligned} (\partial_t^2 u_n(t), v)_H + (\nabla \partial_t u_n(t), \nabla v)_H + (\gamma'_\varepsilon(u_n(t)) \partial_t u_n(t), v)_H \\ + (g'(u_n(t)) \partial_t u_n(t), v)_H = (\partial_t^2 w_n(t), v)_H \end{aligned}$$

for all  $v \in V_n$  and a.a.  $t \in (0, T)$ . We choose  $v = \partial_t u_n(t)$  as an admissible test function, integrate over  $(0, t)$  and use the Lipschitz continuity of  $g$  to get

$$\begin{aligned} \frac{1}{2} \|\partial_t u_n(t)\|_H^2 + \int_0^t \|\nabla \partial_t u_n(s)\|_H^2 ds + \int_{Q_t} \gamma'_\varepsilon(u_n) |\partial_t u_n|^2 \leq c \int_0^t \|\partial_t u_n(s)\|_H^2 ds \\ + \int_0^t \|\partial_t^2 w_n(s)\|_H \|\partial_t u_n(s)\|_H ds + \frac{1}{2} \|\partial_t u_n(0)\|_H^2 . \end{aligned} \quad (7.2)$$

Since the last term in the left-hand side is non negative because of the monotonicity of  $\gamma_\varepsilon$ , if we had a bound for the last term in the right-hand side, we could use the generalized Gronwall lemma to conclude. In order to provide such an estimate, we set  $t = 0$ ,  $v = \partial_t u_n(0)$  in the equation (5.7); we obtain

$$\|\partial_t u_n(0)\|_H^2 \leq \{\|\Delta u_{0,n}\|_H + \|\gamma_\varepsilon(u_{0,n})\|_H + \|g(u_{0,n})\|_H + \|v_{0,n}\|_H\} \|\partial_t u_n(0)\|_H$$

and thus, taking into account the Lipschitz continuity of  $g$ , we infer

$$\begin{aligned} \|\partial_t u_n(0)\| \leq \|\Delta u_{0,n}\|_H + \|\gamma'_\varepsilon\|_{L^\infty(\mathbb{R})} \|u_{0,n} - u_0\|_H + \|\gamma_\varepsilon(u_0)\|_H \\ + c (\|u_{0,n}\|_H + 1) + \|v_{0,n}\|_H . \end{aligned}$$

Now, assumptions (7.1) and (5.3), as well as (2.21) and  $|\gamma_\varepsilon| \leq |\gamma^0|$ , enable us to achieve

$$\|\partial_t u_n(0)\| \leq c \quad (7.3)$$

for all  $\varepsilon > 0$  and  $n$  large enough, depending on  $\varepsilon$ ; these requests on parameters are not restrictive, as we first take the limit for  $n \rightarrow +\infty$ , then for  $\varepsilon \searrow 0$ . From (7.2) and (7.3) we deduce that

$$\|u_n\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V)} \leq c_{\alpha,6}. \quad (7.4)$$

Finally, we consider equation (5.7) and we rewrite it in the form

$$(\nabla u_n(t), \nabla v)_H + (\gamma_\varepsilon(u_n(t)), v)_H = (F_n(t), v),$$

for all  $v \in V_n$  and a.a.  $t \in (0, T)$ , where  $F_n = \partial_t w_n - \partial_t u_n - g(u_n)$ . Testing with  $v = -\Delta u_n(t)$  the previous equation and integrating by parts in space, we obtain

$$\|\Delta u_n(t)\|_H^2 + \int_\Omega \gamma'_\varepsilon(u_n(t)) |\nabla u_n(t)|^2 \leq \|F_n(t)\|_H \|\Delta u_n(t)\|_H \quad \text{for all } 0 \leq t \leq T.$$

Since the estimates (6.5) and (7.4) entail

$$\|F_n\|_{L^\infty(0,T;H)} \leq c$$

and we can apply the regularity results for elliptic problems, we deduce

$$\|u_n\|_{L^\infty(0,T;W)} \leq c_{\alpha,7}, \quad (7.5)$$

thus concluding the proof of Theorem 2.3.

## 8 $L^\infty$ estimates

The aim of this section is to obtain  $L^\infty$  estimates on  $w_t$  and on  $\xi$ , under the hypotheses (2.23) and (2.24).

We first deal with  $w_t$ . Setting  $\varphi = \alpha w_t + \beta w$ , Theorem 2.2 entail that the equalities

$$\frac{1}{\alpha} \varphi_t - \Delta \varphi = \frac{\beta}{\alpha} w_t - u_t + f \quad \text{in } Q, \quad \partial_n \varphi = 0 \quad \text{on } \Gamma \times (0, T)$$

hold almost everywhere. Furthermore, the assumption (2.23), the estimates (6.5) and (7.4) and the continuous embedding  $V \hookrightarrow L^6(\Omega)$  (valid if  $\Omega \subseteq \mathbb{R}^3$  is a bounded Lipschitz domain), yield

$$\frac{\beta}{\alpha} w_t - u_t + f \in L^\infty(0, T; H) + L^r(0, T; L^6(\Omega)), \quad \text{with } r > 4/3.$$

In these conditions, Theorem 7.1 in [12, p. 181] applies and ensures that  $\varphi \in L^\infty(Q)$ . Since we already know that  $w \in L^\infty(Q)$  (as it is implied, for example, by (6.10)), we have  $w_t \in L^\infty(Q)$  and

$$\|w_t\|_{L^\infty(Q)} \leq \frac{1}{\alpha} \|\varphi\|_{L^\infty(Q)} + \frac{c\beta}{\alpha} \|w\|_{L^\infty(0,T;W)} \leq c_{\alpha,8}. \quad (8.1)$$

We notice that, being  $\alpha$  fixed and letting  $\beta$  vary in a bounded set, we can find an upper bound for the constant  $c_{\alpha,8}$ .

In order to prove a  $L^\infty$  estimate for  $\xi$ , we consider the solution  $(w_\varepsilon, u_\varepsilon)$  to the approximating problem, in which the Yosida regularization appears; we then fix  $p \in (1, +\infty)$  and get a bound for  $\|\gamma_\varepsilon(u_\varepsilon)\|_{L^p(Q)}$ , which is independent of  $p, \varepsilon$ . From this, we will obtain a uniform bound for

$$\|\xi_\varepsilon\|_{L^\infty(Q)} = \lim_{p \rightarrow +\infty} \|\gamma_\varepsilon(u_\varepsilon)\|_{L^p(Q)},$$

and, via a weak\* compactness argument,  $\xi \in L^\infty(Q)$ . For the sake of simplicity, we do not plug in the subscript  $\varepsilon$  in the solution any more.

We know that the equalities

$$u_t - \Delta u + \gamma_\varepsilon(u) + g(u) = w_t \quad \text{in } Q, \quad (8.2)$$

$$\partial_n u = 0 \quad \text{on } \Gamma \times (0, T), \quad u(0) = u_0 \quad \text{in } \Omega$$

hold a.e.; we choose  $|\gamma_\varepsilon(u)|^{p-1} \gamma_\varepsilon(u)$  as a test function, by which we multiply both sides of the equation (8.2) – this is admissible since  $u \in L^\infty(Q)$ . Integrating over  $Q$ , we get

$$\begin{aligned} \int_Q \frac{\partial}{\partial t} \phi_{\varepsilon,p}(u) + \int_Q \nabla u \cdot \nabla (|\gamma_\varepsilon(u)|^{p-1} \gamma_\varepsilon(u)) + \int_Q |\gamma_\varepsilon(u)|^{p+1} \\ = \int_Q (w_t - g(u)) |\gamma_\varepsilon(u)|^{p-1} \gamma_\varepsilon(u), \end{aligned} \quad (8.3)$$

where we have set

$$\phi_{\varepsilon,p}(t) = \int_0^t |\gamma_\varepsilon(s)|^{p-1} \gamma_\varepsilon(s) ds \quad \text{for all } t \in \mathbb{R};$$

$\gamma_\varepsilon$  is increasing and  $\gamma_\varepsilon(0) = 0$ , so we have  $\phi_{\varepsilon,p} \geq 0$  for all  $\varepsilon, p$ . Since  $w_t, u \in L^\infty(Q)$  and  $g$  is continuous, for the right-hand side we have

$$\left| \int_Q (w_t - g(u)) |\gamma_\varepsilon(u)|^{p-1} \gamma_\varepsilon(u) \right| \leq c_\alpha \|\gamma_\varepsilon(u)\|_{L^p(\Omega)}^p;$$

on the other hand, a direct calculation and the monotonicity of  $\gamma_\varepsilon$  show that

$$\nabla u \cdot \nabla (|\gamma_\varepsilon(u)|^{p-1} \gamma_\varepsilon(u)) = p \gamma'_\varepsilon(u) |\gamma_\varepsilon(u)|^{p-1} |\nabla u|^2 \geq 0 \quad \text{a.e. in } Q.$$

Collecting all the information we have obtained so far, from (8.3) we derive

$$\int_\Omega \phi_{\varepsilon,p}(u(T)) + \|\gamma_\varepsilon(u)\|_{L^{p+1}(Q)}^{p+1} \leq c_\alpha \|\gamma_\varepsilon(u)\|_{L^p(Q)}^p + \int_\Omega \phi_{\varepsilon,p}(u_0) \quad (8.4)$$

and, since the first term can be ignored, we need only to find an estimate for the last term. We recall that, for the Yosida approximation of a maximal monotone graph, the inequality

$$|\gamma_\varepsilon(s)| \leq |\gamma^0(s)| \quad \text{for all } s \in D(\gamma), \quad \varepsilon > 0$$

holds (see, e.g., [2, Prop. 2.6, p. 28]); according to that, we have

$$\begin{aligned} \int_\Omega \phi_{\varepsilon,p}(u_0) &\leq \int_\Omega |\gamma^0(u_0)|^p |u_0| \leq \frac{p}{p+1} \int_\Omega |\gamma^0(u_0)|^{p+1} + \frac{1}{p+1} \int_\Omega |u_0|^{p+1} \\ &\leq \frac{p}{p+1} \int_\Omega |\gamma^0(u_0)|^{p+1} + \frac{1}{p+1} \|u_0\|_{L^{p+1}(\Omega)}^{p+1}, \end{aligned}$$

where the Hölder and Young inequalities have been used. We recall that  $u_0 \in L^\infty(\Omega)$  by the assumption (2.22) and also notice that the same inequalities imply

$$c_\alpha \|\gamma_\varepsilon(u)\|_{L^p(Q)}^p \leq \frac{p}{p+1} \|\gamma_\varepsilon(u)\|_{L^{p+1}(Q)}^{p+1} + \frac{c_\alpha}{p+1}.$$

Now, we come back to the equation (8.4); according to the previous estimates, we infer that

$$\frac{1}{p+1} \|\gamma_\varepsilon(u)\|_{L^{p+1}(Q)}^{p+1} \leq \frac{p}{p+1} \|\gamma^0(u_0)\|_{L^{p+1}(\Omega)}^{p+1} + \frac{1}{p+1} \|u_0\|_{L^{p+1}(\Omega)}^{p+1} + \frac{c_\alpha}{p+1}$$

and, hence,

$$\begin{aligned} \|\gamma_\varepsilon(u)\|_{L^{p+1}(Q)} &\leq \left\{ p \|\gamma^0(u_0)\|_{L^{p+1}(\Omega)}^{p+1} + \|u_0\|_{L^{p+1}(\Omega)}^{p+1} + c_\alpha \right\}^{1/(p+1)} \\ &\leq c_\alpha \left\{ \|\gamma^0(u_0)\|_{L^\infty(\Omega)} + \|u_0\|_{L^\infty(\Omega)} + 1 \right\}, \end{aligned}$$

which provides the desired estimate and concludes the proof.

## 9 Well-posedness of $(\mathbf{P}_\alpha)$ and convergence as $\beta \searrow 0$

Now we set the notation as in Section 3, since we are interested in the proof of Theorems 3.1–3.4. We assume that the hypotheses (2.1)–(2.7) are satisfied, and we start by studying the convergence as  $\beta \searrow 0$ , by a compactness argument.

**Convergence as  $\beta \searrow 0$ .** We recall the a priori estimates (5.17), (5.12), (5.27), (5.28) which are independent of  $\beta$  and thus holding also for  $(w_\beta, u_\beta, \xi_\beta)$ . Moreover, adopting the notation as in (5.4)–(5.5), by a comparison in (2.12) we find out that  $\{\partial_t^2 w_\beta - f_\beta^{(2)}\}$  is uniformly bounded in  $L^2(0, T; V')$ . Therefore, we can find a subsequence  $\beta_k \searrow 0$  and functions  $w, u, \xi$  such that

$$w_{\beta_k} \rightharpoonup^* w \text{ in } W^{1,\infty}(0, T; H), \quad w_{\beta_k} \rightharpoonup w \text{ in } H^1(0, T; V)$$

$$\partial_t^2 w_{\beta_k} - f_{\beta_k}^{(2)} \rightharpoonup w_{tt} - f^{(2)} \text{ in } L^2(0, T; V')$$

$$u_{\beta_k} \text{ tends to } u \text{ weakly in } H^1(0, T; V') \cap L^2(0, T; V), \text{ whence strongly in } L^2(Q),$$

$$\xi_{\beta_k} \rightharpoonup \xi \text{ in } L^2(Q)$$

as  $k \rightarrow +\infty$ , and here part of (3.4) has been used. Then, in view of (2.6), (3.4) and (3.5), we can pass to the limit in (2.12) and (2.13), as well as in the initial conditions (2.14) which can be recovered weakly in  $V'$  at least. On the other hand,  $u \in D(\gamma)$  and  $\xi \in \gamma(u)$  a.e. in  $Q$  follow as a consequence of the above convergences and [1, Lemma 1.3, p. 42].

**Uniqueness for  $(\mathbf{P}_\alpha)$ .** By applying the previous result with  $f_\beta = f$ ,  $w_{0,\beta} = w_0$ ,  $v_{0,\beta} = v_0$  and  $u_{0,\beta} = u_0$  given, we obtain the existence of a solution to Problem  $(\mathbf{P}_\alpha)$ ; we still have to prove the uniqueness. Let  $(w_1, u_1, \xi_1)$  and  $(w_2, u_2, \xi_2)$  be solutions of  $(\mathbf{P}_\alpha)$ ;

we write down the equations for the differences  $w = w_1 - w_2$ ,  $u = u_1 - u_2$ ,  $\xi = \xi_1 - \xi_2$  and integrate with respect to time the first one:

$$(w_t(t), v)_H + \alpha (\nabla w(t), \nabla v)_H + (u(t), v)_H = 0,$$

$$\langle u_t(t), v \rangle + (\nabla u(t), \nabla v)_H + (\xi(t), v)_H + (g(u_1)(t) - g(u_2)(t), v)_H = (w_t(t), v)_H,$$

to be complemented with null initial conditions as in (4.4). We set  $v = w_t(t)$  in the first equation and  $v = u(t)$  in the second one, integrate over  $(0, t)$  and add the two equations; it is straightforward to obtain

$$\int_0^t \|w_t(s)\|_H^2 ds + \frac{\alpha}{2} \|\nabla w_t(t)\|_H^2 + \frac{1}{2} \|u(t)\|_H^2 + \int_0^t \|\nabla u(s)\|_H^2 ds \leq c \int_0^t \|u(s)\|_H^2 ds.$$

According to the Gronwall lemma and owing to  $w(0) = 0$ , it turns out that  $w = u = 0$  a.e. in  $Q$  and, by comparison in the second equation,  $\xi = 0$  a.e. in  $Q$ .

**Error equations.** Because of the uniqueness, the whole family  $\{(w_\beta, u_\beta, \xi_\beta)\}_{\beta>0}$  converges, as  $\beta \searrow 0$ , to the solution  $(w, u, \xi)$  of Problem  $(\mathbf{P}_\alpha)$ . So, it makes sense to study the speed of this convergence. In order to perform that, we set  $\widehat{w}_\beta = w_\beta - w$ ,  $\widehat{u}_\beta = u_\beta - u$ ,  $\widehat{\xi}_\beta = \xi_\beta - \xi$  and consider the problem obtained for these variables, by subtracting side by side the equations of Problems  $(\mathbf{P}_{\alpha,\beta})$  and  $(\mathbf{P}_\alpha)$ . For all  $v \in V$  and a.a.  $t \in (0, T)$ , the equalities

$$\begin{aligned} \langle \partial_t^2 \widehat{w}_\beta(t), v \rangle + \alpha (\nabla \partial_t \widehat{w}_\beta(t), \nabla v)_H + \beta (\nabla w_\beta(t), \nabla v)_H + \langle \partial_t \widehat{u}_\beta(t), v \rangle \\ = \langle \widehat{f}_\beta(t), v \rangle \end{aligned} \quad (9.1)$$

$$\begin{aligned} \langle \partial_t \widehat{u}_\beta(t), v \rangle + (\nabla \widehat{u}_\beta(t), \nabla v)_H + \langle \widehat{\xi}_\beta(t), v \rangle + (g(u_\beta)(t) - g(u)(t), v)_H \\ = (\partial_t \widehat{w}_\beta(t), v)_H \end{aligned} \quad (9.2)$$

are satisfied, as well as the initial conditions

$$\widehat{w}_\beta(0) = \widehat{w}_{0,\beta}, \quad \partial_t \widehat{w}_\beta(0) = \widehat{v}_{0,\beta}, \quad \widehat{u}_\beta(0) = \widehat{u}_{0,\beta},$$

where  $\widehat{f}_\beta = f_\beta - f = \widehat{f}_\beta^{(1)} + \widehat{f}_\beta^{(2)}$ ,

$$\begin{aligned} \widehat{f}_\beta^{(1)} &= f_\beta^{(1)} - f^{(1)} \longrightarrow 0 \text{ in } L^2(0, T; V') \\ \widehat{f}_\beta^{(2)} &= f_\beta^{(2)} - f^{(2)} \longrightarrow 0 \text{ in } L^1(0, T; H) \end{aligned}$$

(cf. (3.6)),  $\widehat{w}_{0,\beta} := w_{0,\beta} - w_0$ ,  $\widehat{v}_{0,\beta} := v_{0,\beta} - v_0$ , and  $\widehat{u}_{0,\beta} := u_{0,\beta} - u_0$ .

**First estimate for the convergence error.** Now, we want to show Theorem 3.3, so we assume all the needed hypotheses. Choose  $v = \widehat{u}_\beta(t)$  in the equation (9.2) and integrate over  $(0, t)$ ; by the monotonicity of  $\gamma$  and the Lipschitz-continuity of  $g$ , we easily derive

$$\frac{1}{2} \|\widehat{u}_\beta(t)\|_H^2 + \int_0^t \|\nabla \widehat{u}_\beta(s)\|_H^2 ds \leq \frac{1}{2} \|\widehat{u}_{0,\beta}\|_H^2 + c \int_0^t \|\widehat{u}_\beta(s)\|_H^2 ds + \int_{Q_t} \widehat{u}_\beta \partial_t \widehat{w}_\beta. \quad (9.3)$$

We integrate with respect to time the equation (9.1):

$$\begin{aligned} (\partial_t \widehat{w}_\beta(t), v)_H + \alpha (\nabla \widehat{w}_\beta(t), \nabla v)_H + \beta (1 * \nabla w_\beta(t), \nabla v)_H + (\widehat{u}_\beta(t), v)_H \\ = \langle 1 * \widehat{f}_\beta(t), v \rangle + (\widehat{v}_{0,\beta} + \widehat{u}_{0,\beta}, v)_H + \alpha (\nabla \widehat{w}_{0,\beta}, \nabla v)_H. \end{aligned}$$

We set  $v = \partial_t \widehat{w}_\beta$  and integrate over  $(0, t)$ ; keeping only the first two terms in the left-hand side, we obtain

$$\begin{aligned} \int_0^t \|\partial_t \widehat{w}_\beta(s)\|_H^2 ds + \frac{\alpha}{2} \|\nabla \widehat{w}_\beta(t)\|_H^2 \leq \frac{\alpha}{2} \|\nabla \widehat{w}_{0,\beta}\|_H^2 \\ - \beta (1 * \nabla w_\beta(t), \nabla \widehat{w}_\beta(t))_H + \beta \int_0^t (\nabla w_\beta(s), \nabla \widehat{w}_\beta(s))_H ds - \int_{Q_t} \widehat{u}_\beta \partial_t \widehat{w}_\beta \\ + \int_0^t \langle 1 * \widehat{f}_\beta^{(1)}(s) + \widehat{v}_{0,\beta}, \partial_t \widehat{w}_\beta(s) \rangle ds + \int_{Q_t} (1 * \widehat{f}_\beta^{(2)} + \widehat{u}_{0,\beta}) \partial_t \widehat{w}_\beta + \alpha \int_{Q_t} \nabla \widehat{w}_{0,\beta} \nabla \partial_t \widehat{w}_\beta. \end{aligned} \quad (9.4)$$

Due to the Young and Hölder inequalities and the boundedness of  $\{w_\beta\}$  in  $L^2(0, T; V)$ , we have that

$$\begin{aligned} -\beta (1 * \nabla w_\beta(t), \nabla \widehat{w}_\beta(t))_H \leq \frac{c}{\alpha} \beta^2 \int_0^t \|\nabla w_\beta(s)\|_H^2 ds + \frac{\alpha}{12} \|\nabla \widehat{w}_\beta(t)\|_H^2 \\ \leq c\beta^2 + \frac{\alpha}{12} \|\nabla \widehat{w}_\beta(t)\|_H^2 \end{aligned} \quad (9.5)$$

and

$$\beta \int_0^t (\nabla w_\beta(s), \nabla \widehat{w}_\beta(s))_H ds \leq c\beta^2 + \alpha \int_0^t \|\nabla \widehat{w}_\beta(s)\|_H^2 ds, \quad (9.6)$$

$$\alpha \int_{Q_t} \nabla \widehat{w}_{0,\beta} \nabla \partial_t \widehat{w}_\beta \leq \frac{\alpha}{12} \|\nabla \widehat{w}_\beta(t)\|_H^2 + c\alpha \|\nabla \widehat{w}_{0,\beta}\|_H^2. \quad (9.7)$$

On the other hand, arguing as in the estimate of the term  $T_4(t)$  of (5.11) we deduce that

$$\begin{aligned} \int_0^t \langle 1 * \widehat{f}_\beta^{(1)}(s) + \widehat{v}_{0,\beta}, \partial_t \widehat{w}_\beta(s) \rangle ds \\ = \langle 1 * \widehat{f}_\beta^{(1)}(t) + \widehat{v}_{0,\beta}, \widehat{w}_\beta(t) \rangle - \int_0^t \langle \widehat{f}_\beta^{(1)}(s), \widehat{w}_\beta(s) \rangle ds \\ \leq c \left( \int_0^t \|\widehat{f}_\beta^{(1)}(s)\|_{V'}^2 ds + \|\widehat{v}_{0,\beta}\|_{V'}^2 + \|\widehat{w}_{0,\beta}\|_H^2 \right) + \frac{1}{4} \int_0^t \|\partial_t \widehat{w}_\beta(s)\|_H^2 ds \\ + \frac{\alpha}{12} \|\nabla \widehat{w}_\beta(t)\|_H^2 + c \int_0^t \left( \int_0^s \|\partial_t \widehat{w}_\beta(\tau)\|_H^2 d\tau \right) ds + c\alpha \int_0^t \|\nabla \widehat{w}_\beta(s)\|_H^2 ds \end{aligned} \quad (9.8)$$

Finally, we observe that

$$\int_{Q_t} (1 * \widehat{f}_\beta^{(2)} + \widehat{u}_{0,\beta}) \partial_t \widehat{w}_\beta \leq c \left( \|\widehat{f}_\beta^{(2)}\|_{L^1(0,T;H)}^2 + \|\widehat{u}_{0,\beta}\|_H^2 \right) + \frac{1}{4} \int_0^t \|\partial_t \widehat{w}_\beta(s)\|_H^2 ds \quad (9.9)$$

Now we add (9.3) and (9.4); collecting also all the estimates in (9.5)–(9.9), we find out that

$$\begin{aligned} & \frac{1}{2} \|\widehat{u}_\beta(t)\|_H^2 + \int_0^t \|\nabla \widehat{u}_\beta(s)\|_H^2 ds + \frac{1}{2} \int_0^t \|\partial_t \widehat{w}_\beta(s)\|_H^2 ds + \frac{\alpha}{4} \|\nabla \widehat{w}_\beta(t)\|_H^2 \\ & \leq c\beta^2 + c \left( \left\| \widehat{f}_\beta^{(1)} \right\|_{L^2(0,T;V')}^2 + \left\| \widehat{f}_\beta^{(2)} \right\|_{L^1(0,T;H)}^2 + \|\widehat{u}_{0,\beta}\|_H^2 + \|\widehat{v}_{0,\beta}\|_{V'}^2 + \|\widehat{w}_{0,\beta}\|_{V'}^2 \right) \\ & \quad + c \int_0^t \|\widehat{u}_\beta(s)\|_H^2 ds + c \int_0^t \left( \int_0^s \|\partial_t \widehat{w}_\beta(\tau)\|_H^2 d\tau \right) ds + c\alpha \int_0^t \|\nabla \widehat{w}_\beta(s)\|_H^2 ds. \end{aligned}$$

At this point, it suffices to recall (3.6)–(3.7) and apply the Gronwall lemma to obtain the thesis of Theorem 3.3.

**Second estimate for the convergence error.** Our aim is to prove Theorem 3.4, whose hypotheses are assumed to be satisfied. Thus, we can apply Theorems 2.3 and 2.4 to get a bound

$$\|u_\beta\|_{L^\infty(Q)} + \|u\|_{L^\infty(Q)} + \|\xi_\beta\|_{L^\infty(Q)} + \|\xi\|_{L^\infty(Q)} \leq c_\alpha \quad (9.10)$$

with  $c_\alpha$  which is independent of  $\beta$ . Now, if  $\gamma$  is a maximal monotone graph which reduces to a single-valued function in its domain, then  $D(\gamma)$  is an open interval  $(a, b)$  and, if  $b < +\infty$ , then  $\gamma(r) \nearrow +\infty$  as  $r \nearrow b$ ; similarly, if  $a > -\infty$  then  $\gamma(r) \searrow -\infty$  as  $r \searrow a$ . In any case, the condition (9.10) implies the existence of some compact interval  $K \subseteq D(\gamma)$  such that  $u_\beta(\overline{Q}) \subseteq K$  for all  $\beta > 0$ ,  $u(\overline{Q}) \subseteq K$ . Since  $\gamma$  is assumed to be locally Lipschitz-continuous (cf. (3.9)), thanks to (3.8) we immediately deduce that

$$\|\xi_\beta - \xi\|_{L^\infty(0,T;H)} \leq c \|u_\beta - u\|_{L^\infty(0,T;H)} \leq c\beta.$$

Moreover, by suitably modifying  $g$  we can set  $\widehat{\xi}_\beta \equiv 0$  in equation (9.2), without loss of generality.

We start by taking  $v = \partial_t \widehat{w}_\beta$  in (9.1),  $v = \partial_t \widehat{u}_\beta$  in (9.2), integrating both equations over  $(0, t)$  and adding side by side. Thanks to the Lipschitz-continuity of  $g$  and the Young and Hölder inequalities, it is straightforward to obtain

$$\begin{aligned} & \frac{1}{2} \|\partial_t \widehat{w}_\beta(t)\|_H^2 + \alpha \int_0^t \|\nabla \partial_t \widehat{w}_\beta(s)\|_H^2 ds + \int_0^t \|\partial_t \widehat{u}_\beta(s)\|_H^2 ds + \frac{1}{2} \|\nabla \widehat{u}_\beta(t)\|_H^2 \\ & \leq \frac{\beta^2}{\alpha} \|w_\beta\|_{L^2(0,T;V)}^2 + \frac{\alpha}{4} \int_0^t \|\nabla \partial_t \widehat{w}_\beta(s)\|_H^2 ds + c \int_0^t \|\widehat{u}_\beta(s)\|_H^2 ds + \frac{1}{2} \int_0^t \|\partial_t \widehat{u}_\beta(s)\|_H^2 ds \\ & \quad + \frac{1}{\alpha} \left\| \widehat{f}_\beta^{(1)} \right\|_{L^2(0,T;V')}^2 + \frac{\alpha}{4} \|\partial_t \widehat{w}_\beta\|_{L^2(0,T;H)}^2 + \frac{\alpha}{4} \int_0^t \|\nabla \partial_t \widehat{w}_\beta(s)\|_H^2 ds \\ & \quad + \int_0^t \left\| \widehat{f}_\beta^{(2)}(s) \right\|_H \|\partial_t \widehat{w}_\beta(s)\|_H ds + \frac{1}{2} \|\widehat{v}_{0,\beta}\|_H^2 + \frac{1}{2} \|\nabla \widehat{u}_{0,\beta}\|_H^2. \end{aligned}$$

Taking into account conditions (3.6), (3.12) and the previous estimate (3.8), we easily have

$$\begin{aligned} & \frac{1}{2} \|\partial_t \widehat{w}_\beta(t)\|_H^2 + \frac{\alpha}{2} \int_0^t \|\nabla \partial_t \widehat{w}_\beta(s)\|_H^2 ds + \frac{1}{2} \int_0^t \|\partial_t \widehat{u}_\beta(s)\|_H^2 ds + \frac{1}{2} \|\nabla \widehat{u}_\beta(t)\|_H^2 \\ & \leq c\beta^2 + \int_0^t \left\| \widehat{f}_\beta^{(2)}(s) \right\|_H \|\partial_t \widehat{w}_\beta(s)\|_H ds \end{aligned}$$



whence, by (3.6) and a generalised Gronwall lemma (cf., e.g., [2, Lemme A5, p. 157]), we infer that

$$\|\widehat{w}_\beta\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\widehat{u}_\beta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c\beta \quad (9.11)$$

where the constant  $c$  obviously depends on  $\alpha$ .

Next, observe that the assumptions on the data are strong enough to guarantee that (9.1) and (9.2) can be reformulated as

$$\partial_t^2 \widehat{w}_\beta - \alpha \Delta \partial_t \widehat{w}_\beta = \beta \Delta w_\beta - \partial_t \widehat{u}_\beta + \widehat{f}_\beta \quad \text{a.e. in } Q \quad (9.12)$$

$$\partial_t \widehat{u}_\beta - \Delta \widehat{u}_\beta + g(u_\beta) - g(u) = \partial_t \widehat{w}_\beta \quad \text{a.e. in } Q \quad (9.13)$$

along with the homogeneous Neumann boundary conditions for both  $\widehat{w}_\beta$  and  $\widehat{u}_\beta$ .

In view of (9.11), by a comparison of terms in (9.13) it is standard to deduce that  $\|\Delta \widehat{u}_\beta\|_{L^2(0,T;H)} \leq c\beta$  and consequently, owing to elliptic regularity estimates, we obtain

$$\|\widehat{u}_\beta\|_{L^2(0,T;W)} \leq c_\alpha \beta. \quad (9.14)$$

At this point, let us emphasize that for the proof of (9.11) and (9.14) we have just used the control (3.6) on the difference  $\widehat{f}_\beta$ .

We now pay attention to the equation (9.12) and multiply both sides by  $-\Delta \partial_t \widehat{w}_\beta$ , which belongs to  $L^2(Q)$  (cf. (2.17)), and integrate, also by parts, over  $Q_t$ . By means of the Hölder and Young inequalities, we infer that

$$\begin{aligned} \frac{1}{2} \|\nabla \partial_t \widehat{w}_\beta(t)\|_H^2 + \alpha \int_0^t \|\Delta \partial_t \widehat{w}_\beta(s)\|_H^2 ds &\leq \frac{1}{2} \|\nabla \widehat{v}_{0,\beta}\|_H^2 + \frac{\beta^2}{\alpha} \int_0^t \|\Delta w_\beta(s)\|_H^2 ds \\ &+ \frac{2}{\alpha} \int_0^t \|\partial_t \widehat{u}_\beta(s)\|_H^2 ds + \frac{2}{\alpha} \|\widehat{f}_\beta^{(1)}\|_{L^2(0,T;H)}^2 + \frac{\alpha}{2} \int_0^t \|\Delta \partial_t \widehat{w}_\beta(s)\|_H^2 ds \\ &+ \int_0^t \|\nabla \widehat{f}_\beta^{(2)}(s)\|_H \|\nabla \partial_t \widehat{w}_\beta(s)\|_H ds. \end{aligned}$$

Hence, recalling the uniform boundedness of  $\{w_\beta\}$  in  $L^2(0,T;W)$ , we use (3.12), (9.11), (3.11) and apply the generalised Gronwall lemma as before to obtain

$$\|\nabla \partial_t \widehat{w}_\beta(t)\|_H^2 + \int_0^t \|\Delta \partial_t \widehat{w}_\beta(s)\|_H^2 ds \leq c\beta^2.$$

Now, by virtue of (4.1) and (3.12) we also infer

$$\|\Delta \widehat{w}_\beta(t)\|_H \leq c\beta \quad \text{for all } t \in [0, T].$$

Then, standard elliptic regularity properties and the previous estimates (9.11) and (9.14) lead us to (3.13), thus completing the proof of Theorem 3.4.

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