An Ergodic Dilation of Completely Positive Maps

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Abstract

We shall prove the following Stinespring-type theorem: there exists a triple $(\pi, \mathcal{H}, \mathbf{V})$ associated with an unital completely positive map $\Phi : \mathfrak{A} \to \mathfrak{A}$ on C*-algebra $\mathfrak{A}$ with unit, where $\mathcal{H}$ is a Hilbert space, $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ is a faithful representation and $\mathbf{V}$ is a linear isometry on $\mathcal{H}$ such that $\pi(\Phi(a) = \mathbf{V}^*\pi(a)\mathbf{V}$ for all a belong to $\mathfrak{A}$. The Nagy dilation theorem, applied to isometry $\mathbf{V}$, allows to construct a dilation of ucp-map, Φ, in the sense of Arveson, that satisfies ergodic properties of a Φ-invariante state φ on $\mathfrak{A}$, if Φ admit a φ-adjoint.

1 Introduction

A discrete quantum process is a pair $(\mathfrak{M}, \Phi)$ consisting of a von Neumann algebra $\mathfrak{M}$ and a normal unital completely positive map Φ on $\mathfrak{M}$. In this work we shall prove that any quantum process is possible dilate to quantum process where the dynamic Φ is a *-endomorphism of a larger von Neumann algebra.
In dynamical systems, the process of dilation has taken different meanings. Here we adopt the following definition (See Ref. Muhly-Solel [6]):
Suppose $\mathfrak{M}$ acts on Hilbert space $\mathcal{H}$, a dilation of a quantum process $(\mathfrak{M}, \Phi)$ is a quadruple $(\mathfrak{R}, \Theta, \mathcal{K}, z)$ where $(\mathfrak{R}, \Theta)$ is a quantum process with $\mathfrak{R}$ acts on Hilbert space $\mathcal{K}$ and Θ is a homomorphism (i.e. *-endomorphism on von Neumann algebra $\mathfrak{R}$) with $z : \mathcal{H} \to \mathcal{K}$ isometric embedding such that:

- $z\mathfrak{M}z^* \subset \mathfrak{R}$ and $z^*\mathfrak{R}z \subset \mathfrak{M}$;
- $\Phi^n(a) = z^*\Theta^n(zaz^*)z$ for all $a \in \mathfrak{M}$ and $n \in \mathbb{N}$;
- $z^*\Theta^n(X)z = \Phi^n(z^*Xz)$ for all $X \in \mathfrak{R}$ and $n \in \mathbb{N}$.

Many authors in the past have been applied to problems very similar to the one we described above. We remember the work of Arveson [2] on the Eo-semigroups, of Baht-Parthasarathy on the dilations of nonconservative dynamical semigroups [3] and finally, the most recent work of Mhulay-Solel [6].
We shall prove the existence of dilation using the Nagy theorem for linear contraction (See Fojas-Nagy Ref.[7]) and of a particular covariat representation obtained through the Stinespring's theorem for completely positive maps (See Stinespring Ref.[10]).
We recall that a covariant representation of discrete quantum process $(\mathfrak{M}, \Phi)$ is a triple $(\pi, \mathcal{H}, \mathbf{V})$ where $\pi : \mathfrak{M} \to \mathfrak{B}(\mathcal{H})$ is a normal faithful representation on the Hilbert space $\mathcal{H}$ and $\mathbf{V}$ is an isometry on $\mathcal{H}$ such that for $a \in \mathfrak{M}$ and $a \in \mathbb{N}$,

$$\pi(\Phi^n(a)) = \mathbf{V}^{n*}\pi(a)\mathbf{V}^n.$$

Since the covariant representation is faithful and normal, we identify the von neuman algebra $\mathfrak{M}$ with $\pi(\mathfrak{M})$ and in sec. 3 we construct a dilation of the quantum process $(\pi(\mathfrak{M}), \Psi)$ where Ψ is the following completely positive map $\Psi(\pi(x)) = \pi(\Phi(x))$ for all $n \in \mathfrak{M}$.
In fact, if the triple $(\widehat{\mathbf{V}}, \widehat{\mathcal{H}}, z)$ is the minimal unitary dilation of isometry $\mathbf{V}$, we can construct a von Neumann algebras $\widehat{\mathfrak{M}} \subset \mathfrak{B}(\widehat{\mathcal{H}})$ with following properties: $\widehat{\mathbf{V}}^*\widehat{\mathfrak{M}}\widehat{\mathbf{V}} \subset \widehat{\mathfrak{M}}$ and $z^*\widehat{\mathfrak{M}}z = \mathfrak{M}$.
Of fundamental importance to quantum process theory, is the φ-adjointness properties. The dynamic Φ admit a φ-adjoint (See Kummerer Ref.[4]) relative to the normal Φ-invariant state φ on $\mathfrak{M}$, if there is a normal unital completely positive map $\Phi_\natural : \mathfrak{M} \to \mathfrak{M}$ such that for $a, b \in \mathfrak{M}$,

$$\varphi(\Phi(a)b) = \varphi(a\Phi_\natural(b)).$$

The relationship between reversible process, modular operator and φ-adjointness has been studied by Accardi-Cecchini in [1] and Majewski in [5].
In sec. 4 we shall prove that our dilation satisfies ergodic properties of a Φ-invariante state φ on $\mathfrak{M}$ if the dynamic Φ admit a φ-adjoint.
More precisely, let $(\mathfrak{R},\Theta)$ be our dilation of quantum process $(\mathfrak{M},\Phi)$, we shall prove that if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\varphi(a\Phi^k(b))-\varphi(a)\varphi(b)|=0,$$

for all $a,b\in\mathfrak{M}$, we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\varphi(z^*X\Theta^k(Y)z)-\varphi(z^*Xz)\varphi(z^*Yz)|=0,$$

for all $X,Y\in\mathfrak{R}$.
For generality, we will work with concrete unital C*-algebras $\mathfrak{A}$ and unital completely positive map Φ (briefly ucp-map). The results obtained are easily extended to the quantum process $(\mathfrak{M},\Phi)$.
Before introducing the proof about existence of dilation of discrete quantum process, it is necessary to recall the fundamental Nagy dilation theorem, subject of the next section.

2 Nagy dilation theorem

If $\mathbf{V}$ is a linear isometry on Hilbert space $\mathcal{H}$, there is a triple $(\widehat{\mathbf{V}},\widehat{\mathcal{H}},\mathbf{Z})$ where $\widehat{\mathcal{H}}$ is a Hilbert space, $\mathbf{Z}:\mathcal{H}\to\widehat{\mathcal{H}}$ is a lineary isometry, while $\widehat{\mathbf{V}}$ is an unitary operator on $\widehat{\mathcal{H}}$ such that for $n\in\mathbb{N}$,

$$\widehat{\mathbf{V}}^n\mathbf{Z}=\mathbf{Z}\mathbf{V}^n, \tag{1}$$

with the following minimal properties:

$$\widehat{\mathcal{H}}=\bigvee_{k\in\mathbb{Z}}\widehat{\mathbf{V}}^k\mathbf{Z}\mathcal{H}. \tag{2}$$

For our purposes it is useful to recall here the structure of the unitary minimal dilation of a contraction (See Fojas-Nagy Ref.[7]).
Let $\mathcal{K}$ be a Hilbert space, by $l^2(\mathcal{K})$ we denote the Hilbert space $\{\xi:\mathbb{N}\to\mathcal{K}:\sum_{n\geq 0}\|\xi(n)\|^2<\infty\}$.
We now get the orthogonal projection $\mathbf{F}=\mathbf{I}-\mathbf{V}\mathbf{V}^*$ and the following Hilbert space $\widehat{\mathcal{H}}=\mathcal{H}\oplus l^2(\mathbf{F}\mathcal{H})$ and define the following unitary operator on the Hilbert space $\widehat{\mathcal{H}}$:

$$\widehat{\mathbf{V}}=\left|\begin{array}{cc}\mathbf{V} & \mathbf{F}\Pi_0\\ \mathbf{0} & \mathbf{W}\end{array}\right|,$$

where for each $j\in\mathbb{N}$ we have set with $\Pi_j:l^2(\mathbf{F}\mathcal{H})\to\mathcal{H}$ the canonical projections:

$$\Pi_j(\xi_0,\xi_1...\xi_n...)=\xi_j,$$

while $\mathbf{W}:l^2(\mathbf{F}\mathcal{H})\to l^2(\mathbf{F}\mathcal{H})$ is the linear operator

$$\mathbf{W}(\xi_0,\xi_1...\xi_n...)=(\xi_1,\xi_2...),$$

for all $(\xi_0,\xi_1...\xi_n...)\in l^2(\mathbf{F}\mathcal{H})$.
If $Z:\mathcal{H}\to\widehat{\mathcal{H}}$ is the isometry defined by $\mathbf{Z}\Psi=\Psi\oplus 0$ for all $\Psi\in\mathcal{H}$, it's simple to prove that the relationships 1 and 2 are given.
We observe that for each $n\in\mathbb{N}$ we have

$$\widehat{\mathbf{V}}^n=\left|\begin{array}{cc}\mathbf{V}^n & C(n)\\ \mathbf{0} & \mathbf{W}^n\end{array}\right|, \tag{3}$$

where $C(n): l^2(\mathbf{F}\mathcal{H}) \to \mathcal{H}$ are the following operators:

$$C(n) = \sum_{j=1}^{n} \mathbf{V}^{n-j}\mathbf{F}\Pi_{j-1}, \qquad n \geq 1. \tag{4}$$

Furthermore, for each $n, m \in \mathbb{N}$ we obtain:

$$\Pi_n \mathbf{W}^m = \Pi_{n+m} \quad \text{and} \quad \Pi_n \mathbf{W}^{m^*} = \begin{cases} \Pi_{n-m} & n \geq m \\ 0 & n < m \end{cases}, \tag{5}$$

since

$$\mathbf{W}^{m*}(\xi_0, \xi_1 ... \xi_n ..) = (0, 0....0, \overbrace{\xi_0}^{m+1}, \xi_1 ...),$$

while for each k and p natural number, we obtain:

$$\Pi_p \mathbf{C}(k)^* = \begin{cases} \mathbf{F}\mathbf{V}^{(k-p-1)^*} & k > p \\ \mathbf{0} & \text{elsewhere} \end{cases} \tag{6}$$

since

$$C(k)^*\Psi = (\overbrace{\mathbf{F}\mathbf{V}^{(k-1)^*}\Psi ......\mathbf{F}\mathbf{V}^{*}\Psi, \mathbf{F}\Psi}^{k-time}, 0, .0..).$$

for all $\Psi \in \mathcal{H}$.

3 Invariant algebra

Let be $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$ a C*-algebras with unit and $\mathbf{V}$ an isometry on Hilbert space $\mathcal{H}$ such that

$$\mathbf{V}^*\mathfrak{A}\mathbf{V} \subset \mathfrak{A}.$$

If $(\widehat{\mathbf{V}}, \widehat{\mathcal{H}}, \mathbf{Z})$ denotes the minimal unitary dilation of the isometry $\mathbf{V}$ we shall prove the following proposition:

Proposition 1. *There exists a C^*-algebra with unit $\widehat{\mathfrak{A}} \subset \mathfrak{B}(\widehat{\mathcal{H}})$ such that:*
1 - $\mathbf{Z}\mathfrak{A}\mathbf{Z}^* \subset \widehat{\mathfrak{A}}$ *and* $\mathbf{Z}^*\widehat{\mathfrak{A}}\mathbf{Z} \subset \mathfrak{A}$,
2 - $\widehat{\mathbf{V}}^*\widehat{\mathfrak{A}}\widehat{\mathbf{V}} \subset \widehat{\mathfrak{A}}$,
3 - $\mathbf{Z}^*\widehat{\mathbf{V}}^* X \widehat{\mathbf{V}}\mathbf{Z} = \mathbf{V}^*\mathbf{Z}^* X \mathbf{Z}\mathbf{V}$, *for all* $X \in \widehat{\mathfrak{A}}$,
4 - $\mathbf{Z}^*\widehat{\mathbf{V}}^*(\mathbf{Z}A\mathbf{Z}^*)\widehat{\mathbf{V}} = \mathbf{V}^*A\mathbf{V}$, *for all* $A \in \mathfrak{A}$.

first of all we want to consider some special operators on Hilbert space $\mathcal{H}$.

3.1 The gamma operators associated to pair $(\mathfrak{A}, V)$

The sequences of elements of type $\alpha = (n_1, n_2 .... n_r, A_1, A_2 ... A_r)$, with $n_j \in \mathbb{N}$ and $A_j \in \mathfrak{A}$ for all $j = 1, 2 ... r$, are called strings of $\mathfrak{A}$ of length r and weight $\sum_{i=1}^{n} n_i$.
For each α string of $\mathfrak{A}$, we associate the following operators of $\mathfrak{B}(\mathcal{H})$:

$$|\alpha) = A_1\mathbf{V}^{n_1} \cdots A_r\mathbf{V} \quad \text{and} \quad (\alpha| = \mathbf{V}^{n_r^*}A_r \cdots \mathbf{V}^{n_1^*}A_1,$$

furthermore $\dot{\alpha} = \sum_{i=1}^{n} n_i$ and $l(\alpha) = r$, while $|n)$ denote the set operators $|\alpha)$ with $\dot{\alpha} = n$ and usually

$$|n)\mathfrak{A} = \Big\{ |\alpha)A : A \in \mathfrak{A} \text{ and } \alpha\text{-string of } \mathfrak{A} \text{ with } \dot{\alpha} = n \Big\}.$$

The symbols $(n|$ and $\mathfrak{A}\,(n|$ have the same obvious meaning of above.

Proposition 2. *Let α and β are strings of $\mathfrak{A}$ for each $R \in \mathfrak{A}$ we have:*

$$(\alpha|\, R\, |\beta) \in \begin{cases} \mathfrak{A}\left(\dot{\alpha} - \dot{\beta}\right| & if \;\; \dot{\alpha} \geq \dot{\beta} \\ \left|\dot{\beta} - \dot{\alpha}\right)\mathfrak{A} & if \;\; \dot{\alpha} < \dot{\beta} \end{cases}, \tag{7}$$

and with a simple calculation

$$|\alpha)\, R\, |\beta) \in \left|\dot{\alpha} + \dot{\beta}\right). \tag{8}$$

Proof. For each $m, n \in \mathbb{N}$ and $R \in \mathfrak{A}$ we have:

$$\mathbf{V}^{m^*} R \mathbf{V}^{n} \in \begin{cases} \mathbf{V}^{(m-n)^*}\mathfrak{A} & m \geq n \\ \mathfrak{A}\mathbf{V}^{(n-m)} & m < n \end{cases} \tag{9}$$

Let $\alpha = (m_1, m_2....m_r, A_1, A_2...A_r)$ and $\beta = (n_1, n_2....n_s, B_1, B_2...B_S)$ strings of $\mathfrak{A}$, we obtain:

$$(\alpha|\, R\, |\beta) = \mathbf{V}^{m_r^*} A_r \cdots \mathbf{V}^{m_1^*} A_1 R B_1 \mathbf{V}^{n_1} \cdots B_s \mathbf{V}^{n_s} = (\widetilde{\alpha}|\, \mathbf{I} \left|\widetilde{\beta}\right)$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are strings of $\mathfrak{A}$ with $l(\widetilde{\alpha}) + l\left(\widetilde{\beta}\right) = l(\alpha) + l(\beta) - 1$. Moreover if $\dot{\alpha} \geq \dot{\beta}$ we have $\dot{\widetilde{\alpha}} \geq \dot{\widetilde{\beta}}$ while if $\dot{\alpha} < \dot{\beta}$ it follows that $\dot{\widetilde{\alpha}} < \dot{\widetilde{\beta}}$.
In fact if $m_1 \geq n_1$ we obtain:

$$(\alpha|\, R\, |\beta) = \mathbf{V}^{m_r^*} A_r \cdots A_2 \mathbf{V}^{(m_1-n_1)^*} R_1 B_2 \mathbf{V}^{n_2} \cdots B_s \mathbf{V}^{n_s} = (\widetilde{\alpha}|\, \mathbf{I} \left|\widetilde{\beta}\right),$$

where $R_1 = \mathbf{V}^{n_1^*} A_1 R B_1 \mathbf{V}^{n_1}$, $\widetilde{\alpha} = (m_1 - n_1, m_2....m_r, R_1, A_2...A_r)$ and $\widetilde{\beta} = (n_2....n_s, B_2...B_S)$.
If $m_1 < n_1$ we can write:

$$(\alpha|\, R\, |\beta) = \mathbf{V}^{m_r^*} A_r \cdots \mathbf{V}^{m_2^*} A_2 R_1 \mathbf{V}^{(n_1-m_1)} B_2 \cdots B_s \mathbf{V}^{n_s} = (\widetilde{\alpha}|\, \mathbf{I} \left|\widetilde{\beta}\right),$$

where $R_1 = \mathbf{V}^{m_1^*} A_1 R B_1 \mathbf{V}^{m_1}$, $\widetilde{\alpha} = (m_2....m_r, A_2...A_r)$ and $\widetilde{\beta} = (n_1 - m_1, n_2....n_s, R_1, B_2...B_S)$.
Then by induction on number $\nu = l(\alpha) + l(\beta)$ we have the relationship 7. □

For each α string of $\mathfrak{A}$ with $\dot{\alpha} \geq 1$, we define the linear operators:

$$\Gamma(\alpha) = (\alpha|\mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-1},$$

that will be the gamma associated operators to the pair $(\mathfrak{A}, \mathbf{V})$.

Proposition 3. *For each α and β strings of $\mathfrak{A}$ with $\dot{\alpha}, \dot{\beta} \geq 1$, the gamma operators associated to $(\mathfrak{A}, \mathbf{V})$ satisfy the following relationship:*

$$\Gamma(\alpha) \cdot \Gamma(\beta)^* \in \mathfrak{A}.$$

Proof. We obtain:

$$\Gamma(\alpha) \cdot \Gamma(\beta)^* = (\alpha|\mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-1}\mathbf{\Pi}^*_{\dot{\beta}-1}\mathbf{F}\, |\beta) = \begin{cases} (\alpha|\, \mathbf{F}\, |\beta) & \dot{\alpha} = \dot{\beta} \\ 0 & \dot{\alpha} \neq \dot{\beta} \end{cases},$$

in fact

$$(\alpha|\, \mathbf{F}\, |\beta) = (\alpha|\, (\mathbf{I} - \mathbf{V}\mathbf{V}^*)\, |\alpha) = (\alpha|\, \mathbf{I}\, |\alpha) - (\alpha|\, \mathbf{V}\mathbf{V}^*\, |\alpha) \in \mathfrak{A},$$

since we have $(\alpha|\, \mathbf{V} \in \left(\dot{\alpha} - 1\right|$ while $\mathbf{V}^*\, |\alpha) \in \left|\dot{\alpha} - 1\right)$ and by relationship 7 follows that:

$$\left(\dot{\alpha} - 1\right| \mathbf{I} \left|\dot{\alpha} - 1\right) \subset \mathfrak{A}.$$

□

We have an operator system Σ of $\mathfrak{B}(l^2(\mathbf{F}\mathcal{H}))$ this is:

$$\Sigma = \left\{\mathbf{T} \in \mathfrak{B}(l^2(\mathbf{F}\mathcal{H})) : \Gamma_1 \mathbf{T} \Gamma_2^* \in \mathfrak{A} \ \ \textit{for all gamma operators } \Gamma_i \textit{ associated to } (\mathfrak{A}, \mathbf{V}\right\}. \tag{10}$$

We observe that $\mathbf{I} \in \Sigma$ and $\Gamma_1^* \mathfrak{A} \Gamma_2 \in \Sigma$ for all gamma operators Γ_i. Moreover Σ is a norm closed, while it is a weakly closed if $\mathfrak{A}$ is a W*-algebra.

3.2 The napla operators

For each α, β strings of $\mathfrak{A}$, $A \in \mathfrak{A}$ and $k \in \mathbb{N}$ we define the napla operators of $\mathfrak{B}(l^2(\mathbf{F}\mathcal{H}))$:

$$\Delta_k(A, \alpha, \beta) = \Pi^*_{\dot{\alpha}+k} \mathbf{F}|\alpha) A (\beta| \mathbf{F} \Pi_{\dot{\beta}+k}.$$

For each $h, k \geq 0$ we obtain the following results:

$$\Delta_k(A, \alpha, \beta)^* = \Delta_k(A^*, \beta, \alpha),$$

and

$$\Delta_k(A, \alpha, \beta) \cdot \Delta_h(B, \gamma, \delta) = \begin{cases} 0 & k + \dot{\beta} \neq h + \dot{\gamma}, \\ \Delta_k (R, \alpha, \vartheta) & k + \dot{\beta} = h + \dot{\gamma}, \ h - k \geq 0, \ \textit{with } \dot{\vartheta} = \dot{\delta} + h - k \textit{ and } R \in \mathfrak{A} \\ \Delta_h (R, \vartheta, \delta) & k + \dot{\beta} = h + \dot{\gamma}, \ k - h > 0, \ \textit{with } \dot{\vartheta} = \dot{\delta} + k - h \textit{ and } R \in \mathfrak{A} \end{cases} \tag{11}$$

In fact we have:

$$\Delta_k (A, \alpha, \beta) \cdot \Delta_h (B, \gamma, \delta) = \Pi^*_{\dot{\alpha}+k} \mathbf{F} |\alpha) A (\beta| \mathbf{F} \Pi_{\dot{\beta}+k} \Pi^*_{\dot{\gamma}+h} \mathbf{F} |\gamma) B (\delta| \mathbf{F} \Pi_{\dot{\delta}+h}$$

and if $k + \dot{\beta} \neq h + \dot{\gamma}$ follows that $\Pi_{\dot{\beta}+k} \Pi^*_{\dot{\gamma}+h} = 0$, while if $k + \dot{\beta} = h + \dot{\gamma}$, without losing generality we can get $h \geq k$, and we obtain $\dot{\beta} = \dot{\gamma} + h - k \geq \dot{\gamma}$. Moreover by relationship 7

$$(\beta| \mathbf{F} |\gamma) \in \mathfrak{A} \left(\dot{\beta} - \dot{\gamma}\right|$$

then

$$A (\beta| \mathbf{F} |\gamma) B (\delta| \in \mathfrak{A} \left(\dot{\delta} + \dot{\beta} - \dot{\gamma}\right|,$$

there exists ϑ string of $\mathfrak{A}$ with $\dot{\vartheta} = \dot{\delta} + \dot{\beta} - \dot{\gamma}$ and a $R \in \mathfrak{A}$ such that:

$$A (\beta| \mathbf{F} |\gamma) B (\delta| = R (\vartheta| .$$

Since $\dot{\vartheta} = \dot{\delta} + h - k$ we have:

$$\Delta_k (A, \alpha, \beta) \cdot \Delta_h (B, \gamma, \delta) = \Pi^*_{\dot{\alpha}+k} \mathbf{F} |\alpha) R (\vartheta| \mathbf{F} \Pi_{\dot{\delta}+h} = \Pi^*_{\dot{\alpha}+k} \mathbf{F} |\alpha) R (\vartheta| \mathbf{F} \Pi_{\dot{\vartheta}+k} = \Delta_k (R, \alpha, \vartheta) .$$

Proposition 4. *The linear space $\mathfrak{X}_o$ generated by napla operators, is a *-subalgebra of $\mathfrak{B}\left(l^2\left(\mathbf{F}\mathcal{H}\right)\right)$ included in the operator systems Σ defined in 10.*

Proof. From relationship 11 the linear space $\mathfrak{X}_o$ is a *-algebra. Moreover for each gamma operators $\Gamma(\alpha)$ and $\Gamma(\beta)$ we obtain:

$$\Gamma(\alpha) \Delta_k (A, \gamma, \delta) \Gamma(\beta)^* = (\alpha| \mathbf{F} \mathbf{\Pi}_{\dot{\alpha}-1} \Pi^*_{\dot{\gamma}+k} \mathbf{F} |\gamma) A (\delta| \mathbf{F} \Pi_{\dot{\delta}+k} \mathbf{\Pi}_{\dot{\beta}-1} \mathbf{F} |\beta) \in \mathfrak{A},$$

since by the relationships 7 and 8 we have

$$(\alpha| \mathbf{F} \mathbf{\Pi}_{\dot{\alpha}-1} \Pi^*_{\dot{\gamma}+k} \mathbf{F} |\gamma) A (\delta| \mathbf{F} \Pi_{\dot{\delta}+k} \mathbf{\Pi}_{\dot{\beta}-1} \mathbf{F} |\beta) \in \begin{cases} (k+1| \mathfrak{A} |k+1) & \dot{\alpha} - 1 = \dot{\gamma} + k, \ \dot{\beta} - 1 = \dot{\delta} + k \\ \mathbf{0} & \text{elsewhere} \end{cases}$$

In fact if $\dot{\alpha} = \dot{\gamma} + k + 1$ we can write:

$$(\alpha|\, \mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-1}\Pi^*_{\dot{\gamma}+k}\mathbf{F}\,|\gamma) = (\alpha|\, \mathbf{F}\,|\gamma) = (\alpha|\, \mathbf{I}\,|\gamma) - (\alpha|\, \mathbf{V}\mathbf{V}^*\,|\gamma) \in \mathfrak{A}\,(k+1|$$

since

$$(\alpha|\, \mathbf{I}\,|\gamma) \in \mathfrak{A}\,(k+1| \text{ and } (\alpha|\, \mathbf{V}\mathbf{V}^*\,|\gamma) \in \mathfrak{A}\,(k+1|$$

while if $\dot{\beta} = \dot{\delta} + k + 1$ we obtain

$$(\delta|\, \mathbf{F}\Pi_{\dot{\delta}+k}\mathbf{\Pi}_{\dot{\beta}-1}\mathbf{F}\,|\beta) \in (k+1|\,\mathfrak{A}.$$

□

Corollary 1. *The *-algebra $\mathfrak{X}_o$ and the operator systems Σ are $\mathbf{W}$-invariant:*

$$\mathbf{W}^*\mathfrak{X}_o\mathbf{W} \subset \mathfrak{X}_o \quad \text{and} \quad \mathbf{W}^*\Sigma\mathbf{W} \subset \Sigma.$$

Proof. Let be $\mathbf{T}$ belong to Σ, for each gamma operators $\Gamma(\alpha)$ and $\Gamma(\beta)$ we have:

$$\begin{aligned}\Gamma(\alpha)\,(\mathbf{W}^*\mathbf{T}\mathbf{W})\,\Gamma(\beta)^* &= (\alpha|\, \mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-1}\mathbf{W}^*\mathbf{T}\mathbf{W}\mathbf{\Pi}_{\dot{\beta}-1}\mathbf{F}\,|\beta) = \\ &= (\alpha|\, \mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-\mathbf{2}}\mathbf{T}\mathbf{\Pi}_{\dot{\beta}-\mathbf{2}}\mathbf{F}\,|\beta) \in \mathfrak{A}\mathbf{V}^*\Gamma_1(\alpha_o)\,\mathbf{T}\Gamma_2(\beta_o)\,\mathbf{V}\mathfrak{A} \subset \mathbf{V}^*\mathfrak{A}\mathbf{V} \subset \mathfrak{A}.\end{aligned}$$

where α_o and β_o are strings of $\mathfrak{A}$ with $\dot{\alpha_o} = \dot{\alpha} - 1$ and $\dot{\beta_o} = \dot{\beta} - 1$.
In fact let $\alpha = (m_1, m_2....m_r, A_1, A_2...A_r)$ by definition of gamma operator, there is $i \leq r$ with $m_i \geq 1$ such that

$$(\alpha|\, \mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-\mathbf{2}} = A_1 \cdots A_i\mathbf{V}^*(\alpha_\mathbf{o}|\, \mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-\mathbf{2}} = A_1 \cdots A_i\mathbf{V}^*\Gamma(\alpha_o)\,,$$

where $\alpha_o = (0, ..0, m_i - 1, m_{i+1}..m_r, A_1, A_2...A_r)$ with $\dot{\alpha_o} = \dot{\alpha} - 1$. □

Let $\mathfrak{X}$ be the closure in norm of the *-algebra $\mathfrak{X}_o$. Since Σ is a norm closed set, we have $\mathfrak{X} \subset \Sigma$ while if $\mathfrak{A}$ is a von Neumann algebra of $\mathfrak{B}(\mathcal{H})$ then Σ is weakly closed and $\mathfrak{X}_o'' \subset \Sigma$.

Proposition 5. *The set*

$$\mathcal{S} = \left\{ \begin{vmatrix} A & \Gamma_1 \\ \Gamma_2^* & \mathbf{T} \end{vmatrix} : A \in \mathfrak{A},\ \mathbf{T} \in \mathfrak{X} \text{ and } \Gamma_i \text{ are gamma op.of } (\mathfrak{A}, \mathbf{V}) \right\}, \tag{12}$$

is an operator system of $\mathfrak{B}\left(\widehat{\mathcal{H}}\right)$ such that:

$$\widehat{\mathbf{V}}^*\mathcal{S}\widehat{\mathbf{V}} \subset \mathcal{S}.$$

Furthermore

$$\widehat{\mathbf{V}}^*C^*(\mathcal{S})\,\widehat{\mathbf{V}} \subset C^*(\mathcal{S})\,,$$

where $C^*(\mathcal{S})$ is the C^*-algebra generated by the set $\mathcal{S}$.

Proof. We obtain:

$$\widehat{\mathbf{V}}^*\mathcal{S}\widehat{\mathbf{V}} = \begin{vmatrix} \mathbf{V}^*A\mathbf{V} & \mathbf{V}^*A\mathbf{C}(1) + \mathbf{V}^*\Gamma_1\mathbf{W} \\ \mathbf{C}(1)^*A\mathbf{V} + \mathbf{W}^*\Gamma_2^*\mathbf{V} & \mathbf{C}(1)^*A\mathbf{C}(1) + \mathbf{W}^*\Gamma_2^*\mathbf{C}(1) + \mathbf{C}(1)^*\Gamma_1\mathbf{W} + \mathbf{W}^*\mathbf{T}\mathbf{W} \end{vmatrix},$$

where the operators $\mathbf{V}^*\Gamma(\alpha)\,\mathbf{W}$ and $\mathbf{V}^*A\mathbf{C}(1)$ are gamma operators associated to pair $(\mathfrak{A}, \mathbf{V})$, while $\mathbf{C}(1)^*A\mathbf{C}(1)$, $\mathbf{C}(1)^*\Gamma(\alpha)\,\mathbf{W}$, and $\mathbf{W}^*T\mathbf{W}$ are operators belonging to $\mathfrak{X}$.
In fact we have the following relationships:

$$\mathbf{V}^*A\mathbf{C}(1) = \mathbf{V}^*A\mathbf{F}\mathbf{\Pi}_0 = \Gamma(\vartheta) \quad \text{with } \vartheta = (1, A)\,.$$

while if $\alpha = (m_1, m_2..m_r, A_1, A_2...A_r)$ we obtain:

$$\mathbf{V}^{*}\Gamma(\alpha)\mathbf{W} = \mathbf{V}^{*}(\alpha|\mathbf{F}\mathbf{\Pi}_{\dot{\alpha}-1}\mathbf{W} = \Gamma(\vartheta),$$

with $\vartheta = (m_1 + 1, m_2..m_r, A_1, A_2...A_r)$ since $\mathbf{\Pi}_{\dot{\alpha}-1}\mathbf{W} = \mathbf{\Pi}_{\dot{\alpha}}$.
Furthermore

$$\mathbf{C}(1)^{*} A\mathbf{C}(1) = \mathbf{\Pi}_0^{*}\mathbf{F}A\mathbf{F}\mathbf{\Pi}_0 = \Delta_0(A, \alpha, \beta) \quad \text{with } \alpha = \beta = (0, \mathbf{I})$$

while

$$\mathbf{C}(1)^{*}\Gamma(\alpha)\mathbf{W} = \mathbf{\Pi}_0^{*}\mathbf{F}(\alpha|\mathbf{F}\Pi_{\dot{\alpha}-1}\mathbf{W} = \mathbf{\Pi}_0^{*}\mathbf{F}|\gamma)(\alpha|\mathbf{F}\Pi_{\dot{\alpha}+0} = \Delta_0(\mathbf{I}, \gamma, \alpha) \text{ with } \gamma = (0, \mathbf{I}).$$

□

We observe that the *-algebra $\mathcal{A}^{*}(\mathcal{S})$ generated by the operator system $\mathcal{S}$ is given by

$$\mathcal{A}^{*}(\mathcal{S}) = \begin{vmatrix} \mathfrak{A} & \mathfrak{A}\Gamma\mathfrak{X} \\ \mathfrak{X}\Gamma^{*}\mathfrak{A} & \mathfrak{X} \end{vmatrix}. \tag{13}$$

Now we can easily prove proposition 1.

Proof. We get $C^{*}(\mathcal{S})$, the C*-algebra generated by $\mathcal{S}$ defined in 12, by the definition $\mathbf{Z}\mathfrak{A}\mathbf{Z}^{*} \subset \mathcal{S}$ then

$$\mathbf{Z}^{*}C^{*}(\mathcal{S})\mathbf{Z} \subset \mathfrak{A}.$$

Moreover for $X \in C^{*}(\mathcal{S})$ we have:

$$\mathbf{Z}^{*}\widehat{\mathbf{V}}^{*}X\widehat{\mathbf{V}}\mathbf{Z} = \mathbf{V}\mathbf{Z}^{*}X\mathbf{Z}\mathbf{V},$$

since $\widehat{\mathbf{V}}\mathbf{Z} = \mathbf{Z}\mathbf{V}$.
Let be $\mathfrak{F}$ the family of C*-subalgebras $\widehat{\mathfrak{B}}$ with unit of $C^{*}(\mathcal{S})$ such that $\mathbf{Z}\mathfrak{A}\mathbf{Z}^{*} \subset \widehat{\mathfrak{B}}$ and $\widehat{\mathbf{V}}^{*}\widehat{\mathfrak{B}}\widehat{\mathbf{V}} \subset \widehat{\mathfrak{B}}$. The family $\mathfrak{F}$ with inclusion is partially ordered set, then for Zorn lemma's exists a minimal element that we shall denote with $\widehat{\mathfrak{A}}$. □

4 Stinespring's theorem and dilations

We examine a concrete C*-algebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ with unit and an ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$. By the Stinespring theorem for the ucp-map Φ, we can deduce a triple $(\mathbf{V}_{\Phi}, \sigma_{\Phi}, \mathcal{L}_{\Phi})$ constituted by a Hilbert space $\mathcal{L}_{\Phi}$, a representation $\sigma_{\Phi} : \mathfrak{A} \to \mathcal{B}(\mathcal{L}_{\Phi})$ and a linear contraction $\mathbf{V}_{\Phi} : \mathcal{H} \to \mathcal{L}_{\Phi}$ such thata for $\in \mathfrak{A}$,

$$\Phi(a) = \mathbf{V}_{\Phi}^{*}\sigma_{\Phi}(a)\mathbf{V}_{\Phi}. \tag{14}$$

We recall that on the algebraic tensor $\mathfrak{A} \otimes \mathcal{H}$ we can define a semi-inner product by

$$\langle a_1 \otimes \Psi_1, a_2 \otimes \Psi_2 \rangle_{\Phi} = \langle \Psi_1, \Phi(a_1^{*}a_2)\Psi_2 \rangle_{\mathcal{H}},$$

for all $a_1, a_2 \in \mathfrak{A}$ and $\Psi_1, \Psi_2 \in \mathcal{H}$ furthermore the Hilbert space $\mathcal{L}_{\Phi}$ is the completion of the quotient space $\mathfrak{A}\overline{\otimes}_{\Phi}\mathcal{H}$ of $\mathfrak{A} \otimes \mathcal{H}$ by the linear subspace

$$\{X \in \mathfrak{A} \otimes \mathcal{H} : \langle X, X \rangle_{\Phi} = 0\}$$

with inner product induced by $\langle \cdot, \cdot \rangle_{\Phi}$. We shall denote the image at $a \otimes \Psi \in \mathfrak{A} \otimes \mathcal{H}$ in $\mathfrak{A}\overline{\otimes}_{\Phi}\mathcal{H}$ by $a\overline{\otimes}_{\Phi}\Psi$, so that we have

$$\langle a_1\overline{\otimes}_{\Phi}\Psi_2, a_2\overline{\otimes}_{\Phi}\Psi_2 \rangle_{\mathcal{L}_{\Phi}} = \langle \Psi_1, \Phi(a_1^{*}a_2)\Psi_2 \rangle_{\mathcal{H}},$$

for all $a_1, a_2 \in \mathfrak{A}$ and $\Psi_1, \Psi_2 \in \mathcal{H}$.
Moreover $\sigma_{\Phi}(a)(x\overline{\otimes}_{\Phi}\Psi) = ax \otimes_{\Phi} \Psi$, for each $x\overline{\otimes}_{\Phi}\Psi \in \mathcal{L}_{\Phi}$ and $\mathbf{V}_{\Phi}\Psi = \mathbf{1}\overline{\otimes}_{\Phi}\Psi$ for each $\Psi \in \mathcal{H}$.
Since Φ is unital map, the linear operator $\mathbf{V}_{\Phi}$ is an isometry with adjoint $\mathbf{V}_{\Phi}^{*}$ defined by

$$\mathbf{V}_{\Phi}^{*}a\overline{\otimes}_{\Phi}\Psi = \Phi(a)\Psi,$$

for all $a \in \mathfrak{A}$ and $\Psi \in \mathcal{H}$.
We recall that the multiplicative domain of the ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$ is the C*-subalgebra of $\mathfrak{A}$ such defined:

$$\mathcal{D}_\Phi = \{a \in \mathfrak{A} : \Phi(a^*)\Phi(a) = \Phi(a^*a) \text{ and } \Phi(a)\Phi(a^*) = \Phi(aa^*)\},$$

we have the following implications (See Paulsen Ref.[9]):
$a \in \mathcal{D}_\Phi$ if and only if $\Phi(a)\Phi(x) = \Phi(ax)$ and $\Phi(x)\Phi(a) = \Phi(xa)$ for all $x \in \mathfrak{A}$.

Proposition 6. *The ucp-map Φ is a multiplicative if and only if $\mathbf{V}_\Phi$ is an unitary. Moreover if $x \in \mathcal{D}(\Phi)$ we have:*

$$\sigma_\Phi(x)\,\mathbf{V}_\Phi\mathbf{V}_\Phi^* = \mathbf{V}_\Phi\mathbf{V}_\Phi^*\sigma_\Phi(x)\,.$$

Proof. For each $\Psi \in \mathcal{H}$ we obtain the following implications:

$$a\overline{\otimes}_\Phi\Psi = \mathbf{1}\overline{\otimes}_\Phi\Phi(a)\,\Psi \quad \Longleftrightarrow \quad \Phi(a^*a) = \Phi(a^*)\,\Phi(a)\,,$$

since

$$\|a\overline{\otimes}_\Phi\Psi - 1\overline{\otimes}_\Phi\Phi(a)\,\Psi\| = \langle\Psi, \Phi(a^*a)\,\Psi\rangle - \langle\Psi, \Phi(a^*)\,\Phi(a)\,\Psi\rangle\,.$$

Furthermore, for each $a \in \mathfrak{A}$ and $\Psi \in \mathcal{H}$ we have $\mathbf{V}_\Phi\mathbf{V}_\Phi^* a\overline{\otimes}_\Phi\Psi = \mathbf{1}\overline{\otimes}_\Phi\Phi(a)\,\Psi$. □

Now we prove the following Stinespring-type theorem (See Zsido Ref.[11]):

Proposition 7. *Let $\mathfrak{A}$ be a concrete C*-subalgebra with unit of $\mathcal{B}(\mathcal{H})$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ an ucp-map, then there exists a faithful representation $(\pi_\infty, \mathcal{H}_\infty)$ of $\mathfrak{A}$ and an isometry $\mathbf{V}_\infty$ on Hilbert Space $\mathcal{H}_\infty$ such that for $a \in \mathfrak{A}$,*

$$\mathbf{V}_\infty^*\pi_\infty(a)\,\mathbf{V}_\infty = \pi_\infty(\Phi(a))\,, \tag{15}$$

where

$$\sigma_0 = id, \qquad \Phi_n = \sigma_n \circ \Phi$$

and $(\mathbf{V}_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ is the Stinespring dilation of Φ_n for every $n \geq 0$,

$$\mathcal{H}_\infty = \bigoplus_{j=0}^{\infty}\mathcal{H}_j, \qquad \mathcal{H}_j = \mathfrak{A}\overline{\otimes}_{\Phi_{j-1}}\mathcal{H}_{j-1}, \quad \textit{for } j \geq 1 \textit{ and } \mathcal{H}_0 = \mathcal{H}; \tag{16}$$

and

$$\mathbf{V}_\infty(\Psi_0, \Psi_1, \Psi_2, ...) = (0, \mathbf{V}_0\Psi_0,\ \mathbf{V}_1\Psi_1, ...)$$

for each $(\Psi_0, \Psi_1, \Psi_2, ...) \in \mathcal{H}_\infty$.
Furthermore the map Φ is a homomorphism if and only if $\mathbf{V}_\infty\mathbf{V}_\infty^* \in \pi_\infty(\mathfrak{A})'$.

Proof. By the Stinespring theorem there is triple $(\mathbf{V}_0, \sigma_1, \mathcal{H}_1)$ such that for each $a \in \mathfrak{A}$ we have $\Phi(a) = \mathbf{V}_0^*\sigma_1(a)\mathbf{V}_0$. The application $a \in \mathfrak{A} \to \sigma_1(\Phi(a)) \in \mathcal{B}(\mathcal{H}_1)$ is a composition of cp-maps therefore it is also a cp map. Set $\Phi_1(a) = \sigma_1(\Phi(a))$. By appling the Stinespring's theorem to Φ_1, we have a new triple $(\mathbf{V}_1, \sigma_2, \mathcal{H}_2)$ such that $\Phi_1(a) = \mathbf{V}_1^*\sigma_2(a)\mathbf{V}_1$. By induction for $n \geq 1$ we define $\Phi_n(a) = \sigma_n(\Phi(a))$ and we have a triple $(\mathbf{V}_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ such that $\mathbf{V}_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ and $\Phi_n(a) = \mathbf{V}_n^*\sigma_{n+1}(a)\mathbf{V}_n$.
We get the Hilbert space $\mathcal{H}_\infty$ defined in 16 and the injective representation of the C*-algebra $\mathfrak{A}$ on $\mathcal{H}_\infty$:

$$\pi_\infty(a) = \bigoplus_{n\geq 0}\sigma_n(a) \tag{17}$$

with $\sigma_0(a) = a$, for each $a \in \mathfrak{A}$.
Let $\mathbf{V}_\infty : \mathcal{H}_\infty \to \mathcal{H}_\infty$ be the isometry defined by

$$\mathbf{V}_\infty(\Psi_0, \Psi_1....\Psi_n...) = (0, \mathbf{V}_0\Psi_0, \mathbf{V}_1\Psi_1....\mathbf{V}_n\Psi_n...), \tag{18}$$

for all $\Psi_i \in \mathcal{H}_i$ with $i \in \mathbb{N}$.
The adjoint operator of $\mathbf{V}_\infty$ is

$$\mathbf{V}_\infty^*(\Psi_0, \Psi_1, ...\Psi_n...) = (\mathbf{V}_0^*\Psi_1, \mathbf{V}_1^*\Psi_2....\mathbf{V}_{n-1}^*\Psi_n...) \tag{19}$$

for all $\Psi_i \in \mathcal{H}_i$ with $i \in \mathbb{N}$, therefore

$$\mathbf{V}_\infty^*\pi_\infty(a)\,\mathbf{V}_\infty \bigoplus_{n\geq 0}\Psi_n = \bigoplus_{n\geq 0}\mathbf{V}_n^*\sigma_{n+1}(a)\,\mathbf{V}_n\Psi_n = \bigoplus_{n\geq 0}\Phi_n(a)\,\Psi_n =$$
$$= \bigoplus_{n\geq 0}\sigma_n(\Phi(a))\,\Psi_n = \pi_\infty(\Phi(a)) \bigoplus_{n\geq 0}\Psi_n.$$

We notice that $\mathbf{E}_n = \mathbf{V}_n\mathbf{V}_n^*$ be the orthogonal projection of $\mathcal{B}(\mathcal{H}_{n-1})$, we have:

$$\mathbf{E}(\Psi_0, \Psi_1...\Psi_n..) = (0, \mathbf{E}_0\Psi_1, \mathbf{E}_1\Psi_2, ...\mathbf{E}_n\Psi_{n+1}...)\,.$$

Finally for the proof of the last statement we only need to note that x belong to multiplicative domains $\mathcal{D}(\Phi)$ if and only if we have:

$$\pi_\infty(x)\,\mathbf{V}_\infty\mathbf{V}_\infty^* = \mathbf{V}_\infty\mathbf{V}_\infty^*\pi_\infty(x)\,.$$

. □

Remark 1. *Let $(\mathfrak{M}, \Phi)$ be a quantum process, the representation $\pi_\infty(a) : \mathfrak{M} \to \mathfrak{B}(\mathcal{H}_\infty)$ defined in proposition 7 is normal, since the Stinespring representation $\sigma_\Phi : \mathfrak{A} \to \mathcal{B}(\mathcal{L}_\Phi)$ is a normal map. Then $(\pi_\infty, \mathcal{H}_\infty, \mathbf{V}_\infty)$ is a covariant representation of quantum process.*

4.1 Dilations of ucp-Maps

If $(\mathcal{H}_\infty, \pi_\infty, \mathbf{V}_\infty)$ is the Stinespring representation of proposition 7, we have that $\mathbf{V}_\infty^*\pi_\infty(\mathfrak{A})\,\mathbf{V}_\infty \subset \pi_\infty(\mathfrak{A})$ and by proposition 1 there exists a C*-algebra with unit of $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$ such that:
1 - $\mathbf{Z}\pi_\infty(\mathfrak{A})\,\mathbf{Z}^* \subset \widehat{\mathfrak{A}}$,
2 - $\mathbf{Z}^*\widehat{\mathfrak{A}}\mathbf{Z} = \pi_\infty(\mathfrak{A})$,
3 - $\mathbf{Z}^*\widehat{\mathbf{V}}^*X\widehat{\mathbf{V}}\mathbf{Z} = \mathbf{V}\pi_\infty(\mathbf{Z}^*\mathbf{X}\mathbf{Z})\,\mathbf{V}$, for all $\mathbf{X} \in \widehat{\mathfrak{A}}$.
Furthermore, we have a homomorphism $\widehat{\Phi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ thus defined

$$\widehat{\Phi}(X) = \widehat{\mathbf{V}}^*X\widehat{\mathbf{V}}$$

for all $X \in \widehat{\mathfrak{A}}$, such that for $A \in \mathfrak{A}$, $X \in \widehat{\mathfrak{A}}$ and $n \in \mathbb{N}$ we have:

$$\Phi^n(A) = \mathbf{Z}^*\widehat{\Phi}^n(\mathbf{Z}A\mathbf{Z}^*)\mathbf{Z}$$

and

$$\mathbf{Z}^*\widehat{\Phi}^n(X)\mathbf{Z} = \Phi^n(\mathbf{Z}^*X\mathbf{Z}).$$

The quadruple $(\widehat{\Phi}, \widehat{\mathfrak{A}}, \mathcal{H}, \mathbf{Z})$ with the above properties, is said to be a multiplicative dilation of ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$.

Remark 2. *It is clear that these results are easily extended to the von Neumann algebras $\mathfrak{M}$ with Φ normal ucp-map. In this way we obtain a dilation of discrete quantum process $(\mathfrak{M}, \Phi)$.*

5 Ergodic properties

Let $\mathfrak{A}$ be a concrete C*-algebra of $\mathcal{B}(\mathcal{H})$ with unit, $\Phi : \mathfrak{A} \to \mathfrak{A}$ an ucp-map and φ a state on $\mathfrak{A}$ such that $\varphi \circ \Phi = \varphi$. We recall (See N.S.Z. Ref.[8]) that the state φ is a ergodic state, relative to the ucp-map Φ, if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}[\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)] = 0,$$

for all $a, b \in \mathfrak{A}$, while is weakly mixing if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\varphi(a\Phi^k(b))-\varphi(a)\varphi(b)|=0,$$

for all $a, b \in \mathfrak{A}$.
We observe that by the Stinepring-type theorem 7 we can assume, without losing generality, that $\mathfrak{A}$ is a concrete C*-algebra of $\mathfrak{B}(\mathcal{H})$, and that there is a linear isometry $\mathbf{V}$ on $\mathcal{H}$ such that:

$$\Phi(A)=\mathbf{V}^*A\mathbf{V} \text{ for all } A\in\mathfrak{A}.$$

Then $\left(\widehat{\mathbf{V}}, \widehat{\mathcal{H}}, \mathbf{Z}\right)$ is the minimal unitary dilation of $(\mathbf{V}, \mathcal{H})$ and the C*-algebra $\widehat{\mathfrak{A}}$ defined in proposition 1 is included in $\mathfrak{B}(\widehat{\mathcal{H}})$.
We want to prove the following ergodic theorem, for dilation ucp-map $(\widehat{\Phi}, \widehat{\mathfrak{A}}, \mathcal{H}, \mathbf{Z})$ previously defined:

Proposition 8. *If the ucp-map Φ admits a φ-adjoint and φ is a ergodic state, we obtain:*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}[\varphi(\mathbf{Z}^*X\widehat{\mathbf{V}}^{k^*}Y\widehat{\mathbf{V}}^k\mathbf{Z})-\varphi(\mathbf{Z}^*X\mathbf{Z}\varphi(\mathbf{Z}^*Y\mathbf{Z})]=0,$$

while if φ is weakly mixing:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}|\varphi(\mathbf{Z}^*X\widehat{\mathbf{V}}^{k^*}Y\widehat{\mathbf{V}}^k\mathbf{Z})-\varphi(\mathbf{Z}^*X\mathbf{Z})\varphi(\mathbf{Z}^*Y\mathbf{Z})|=0,$$

for all $X, Y \in \widehat{\mathfrak{A}}$.

If we write every element X of $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$ in matrix form $X=\begin{vmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2}\end{vmatrix}$ with $\widehat{\mathcal{H}}=\mathcal{H}\oplus l^2(\mathbf{F}\mathcal{H})$ we obtain:

$$\varphi\left(\mathbf{Z}^*X\widehat{\mathbf{V}}^{k^*}Y\widehat{\mathbf{V}}^k\mathbf{Z}\right)=\varphi\left(X_{1,1}\mathbf{V}^kY_{1,1}\mathbf{V}^k\right)+\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k\right)+\varphi\left(X_{1,2}\mathbf{W}^{k^*}Y_{2,1}\mathbf{V}^k\right)$$

and the proof of previous proposition is an easy consequence of the following lemma:

Lemma 1. *Let $X \in \mathcal{A}^*(\mathcal{S})$, the *-algebra generated by operator system $\mathcal{S}$ defined in 12 and $Y \in \widehat{\mathfrak{A}}$,*
a] if φ is an ergodic state we have:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k+X_{1,2}\mathbf{W}^{k^*}Y_{2,1}\mathbf{V}^k\right)=0, \tag{20}$$

b] if φ is weakly mixing we have:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k+X_{1,2}\mathbf{W}^{k^*}Y_{2,1}\mathbf{V}^k\right)\right|=0. \tag{21}$$

Proof. Since $X \in \mathcal{A}^*(\mathcal{S})$ we can assume that $X_{1,2} = A\Gamma(\gamma)\Delta_m(B,\alpha,\beta)$ with $A, B \in \mathfrak{A}$ and γ string of $\mathfrak{A}$. Then:

$$X_{1,2}=A(\gamma|\,\mathbf{F}\mathbf{\Pi}_{\dot{\gamma}-1}\Pi^*_{\dot{\alpha}+m}\mathbf{F}\,|\alpha)\,B\,(\beta|\,\mathbf{F}\Pi_{\dot{\beta}+m}=\begin{cases} A(\gamma|\,\mathbf{F}\,|\alpha)\,B\,(\beta|\,\mathbf{F}\Pi_{\dot{\beta}+m} & \dot{\gamma}-1=\dot{\alpha}+m \\ \mathbf{0} & \text{elsewhere}\end{cases} \tag{22}$$

Now we observe taht there is a natural number k_o such that for each $k > k_o$ we obtain:

$$X_{1,2}\mathbf{W}^{k^*}Y_{2,1}\mathbf{V}^k=0$$

In fact we have that

$$\mathbf{W}^{k^*}(\xi_0,\xi_1...\xi_n...) = \left(\overbrace{0,...0}^{k-time},\xi_0,\xi_1...\right),$$

for all $(\xi_0,\xi_1...\xi_n..) \in l^2(\mathbf{F}\mathcal{H})$ then $\Pi_{\beta+m}\mathbf{W}^{k^*} = \mathbf{0}$ for all $k > \dot{\beta} + m$.
It follows that:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k + X_{1,2}\mathbf{W}^{k^*}Y_{2,1}\mathbf{V}^k\right) = \lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k\right),$$

Then we compute only the term $\varphi\left(X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k\right)$ and by relationship 22 we can write that:

$$X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k = A(\gamma|\,\mathbf{F}\,|\alpha)\,B(\beta|\,\mathbf{F}\Pi_{\dot{\beta}+m}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k$$

moreover by relationship 6 for $k > \dot{\beta} + m$ we have:

$$\Pi_{\dot{\beta}+m}\mathbf{C}(k)^* = \mathbf{F}\mathbf{V}^{(k-\beta-m-1)^*},$$

it follows that

$$X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k = A(\gamma|\,\mathbf{F}\,|\alpha)\,B(\beta|\,\mathbf{F}\mathbf{V}^{(k-\beta-m-1)^*}Y_{1,1}\mathbf{V}^k = A(\gamma|\,\mathbf{F}\,|\alpha)\,B(\beta|\,\mathbf{F}\Phi^{(k-\beta-1)}(Y_{1,1})\,\mathbf{V}^{\beta+m+1}.$$

Since $\dot{\gamma} = \dot{\alpha} + m + 1$, by relationship 7 we obtain:

$$A(\gamma|\,\mathbf{F}\,|\alpha)\,B(\beta| \in \mathfrak{A}\left(\dot{\beta} + m + 1\right|,$$

it follows that there exists a ϑ string of $\mathfrak{A}$ with $\dot{\vartheta} = \dot{\beta} + m + 1$ and an operator $R \in \mathfrak{A}$, such that

$$A(\gamma|\,\mathbf{F}\,|\alpha)\,B(\beta| = R(\vartheta|\,.$$

Then

$$X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k = R(\vartheta|\,\mathbf{F}\Phi^{(k-\beta-1)}(Y_{1,1})\,\mathbf{V}^{\beta+m+1}.$$

If we set $\vartheta = (n_1, n_2, ...n_r, A_1, A_2, ...A_r)$. we have $n_1 + n_2 + ... + n_r = \dot{\beta} + m + 1$ and

$$R(\vartheta|\,\mathbf{F}\Phi^{\left(k-\dot{\beta}-1\right)}(Y_{1,1})\,\mathbf{V}^{\dot{\beta}+m+1} = R\mathbf{V}^{n_r^*}A_r\mathbf{V}^{n_{r-1}^*}A_{r-1}\cdots A_2\mathbf{V}^{n_1^*}A_1\mathbf{F}\Phi^{\left(k-\dot{\beta}-1\right)}(Y_{1,1})\,\mathbf{V}^{\dot{\beta}+m+1} =$$
$$= R\Phi^{n_r}\left(A_r\Phi^{n_{r-1}}\left(A_{r-1}\cdots\Phi^{n_2}\left(A_2\mathbf{R}_k\right)\right)\right),$$

where

$$\mathbf{R}_k = \Phi^{n_r}(A_r)\,\Phi^{(k-\beta-1)}(Y_{1,1}) - \Phi^{n_r-1}\left(\Phi(A_r)\,\Phi^{(k-\beta)}(Y_{1,1})\right).$$

We have:

$$\begin{aligned}
\varphi\left(X_{1,2}\mathbf{C}(k)^* Y_{1,1}\mathbf{V}^k\right) &= \varphi\left(R\Phi^{n_r}\left(A_r\Phi^{n_{r-1}}\left(A_{r-1}\cdots\Phi^{n_2}\left(A_2\mathbf{R}_k\right)\right)\right)\right) = \\
&= \varphi\left(\Phi_\natural^{n_r}(R)\,A_r\Phi^{n_{r-1}}\left(A_{r-1}\left(\cdots\Phi^{n_2}\left(A_2\mathbf{R}_k\right)\right)\right)\right) = \\
&= \varphi\left(\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)\,A_r\right)A_{r-1}\left(A_{r-2}\cdots A_3\Phi^{n_2}\left(A_2\mathbf{R}_k\right)\right)\right) = \\
&= \varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)\,A_r\right)\cdots A_3\right)A_2\mathbf{R}_k\right)
\end{aligned}$$

and replacing $\mathbf{R}_k$, we obtain:
$\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)\,A_r\right)\cdots A_3\right)A_2\mathbf{R}_k =$

$$= \Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})-$$
$$-\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})\right).$$

Then:

$$\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k\right)=$$
$$=\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})\right)-$$
$$-\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})\right)\right).$$

It follows that :

$$\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k\right)=$$
$$=\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})\right)-$$
$$-\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})\right)\right).$$

If the state φ is ergodic we have:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})\right)\right)=$$
$$=\varphi\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\Phi^{n_1}(A_1)\right)\varphi(Y_{1,1})=$$
$$=\varphi\left(\Phi_\natural^{n_1}\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\right)A_1\right)\varphi(Y_{1,1})$$

while

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\Phi_\natural^{n_1-1}\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\right)\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})\right)=$$
$$=\varphi\left(\Phi_\natural^{n_1-1}\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\right)\Phi(A_1)\right)\varphi(Y_{1,1})=$$
$$=\varphi\left(\Phi_\natural\left(\Phi_\natural^{n_1-1}\left(\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2\right)\right)A_1\right)\varphi(Y_{1,1}),$$

then we obtain

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(X_{1,2}\mathbf{C}(k)^*Y_{1,1}\mathbf{V}^k\right)=0.$$

In weakly mixing case, using the previous results, we obtain:

$$\left|\varphi\left(X_{1,2}\mathbf{C}_k^*Y_{1,1}\mathbf{V}^k\right)\right|=\left|\varphi\left(B\Phi^{n_1}(A_1)\Phi^{\left(k-\dot{\beta}-1\right)}(Y_{1,1})\right)-\varphi\left(B\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{\left(k-\dot{\beta}\right)}(Y_{1,1})\right)\right)\right|$$

where $B=\Phi_\natural^{n_2}\left(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}\left(\Phi_\natural^{n_r}(R)A_r\right)\cdots A_3\right)A_2$.

Adding and subtracting the element $\varphi\left(B\Phi^{n_1}(A_1)\right)\varphi(Y_{1,1})$ we can write:

$$\left|\varphi\left(B\Phi^{n_1}(A_1)\Phi^{\left(k-\dot{\beta}-1\right)}(Y_{1,1})\right)-\varphi\left(B\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{\left(k-\dot{\beta}\right)}(Y_{1,1})\right)\right)\right|\leq$$
$$\leq\left|\varphi\left(B\Phi^{n_1}(A_1)\Phi^{\left(k-\dot{\beta}-1\right)}(Y_{1,1})\right)-\varphi\left(B\Phi^{n_1}(A_1)\right)\varphi(Y_{1,1})\right|+$$
$$+\left|\varphi\left(B\Phi^{n_1-1}\left(\Phi(A_1)\Phi^{\left(k-\dot{\beta}\right)}(Y_{1,1})\right)\right)-\varphi\left(B\Phi^{n_1}(A_1)\right)\varphi(Y_{1,1})\right|.$$

Moreover

$$\left|\varphi\left(B\Phi^{n_1-1}\left(\Phi\left(A_1\right)\Phi^{\left(k-\dot{\beta}\right)}\left(Y_{1,1}\right)\right)\right)-\varphi\left(B\Phi^{n_1}\left(A_1\right)\right)\varphi\left(Y_{1,1}\right)\right|=$$

$$=\left|\varphi\left(\Phi_{\natural}^{n_1-1}\left(B\right)\Phi\left(A_1\right)\Phi^{\left(k-\dot{\beta}\right)}\left(Y_{1,1}\right)\right)-\varphi\left(\Phi_{\natural}^{n_1-1}\left(B\right)\Phi\left(A_1\right)\right)\varphi\left(Y_{1,1}\right)\right|,$$

and by the weakly mixing properties we obtain:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(B\Phi^{n_1}\left(A_1\right)\Phi^{\left(k-\dot{\beta}-1\right)}\left(Y_{1,1}\right)\right)-\varphi\left(B\Phi^{n_1}\left(A_1\right)\right)\varphi\left(Y_{1,1}\right)\right|=0,$$

and

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\Phi_{\natural}^{n_1-1}\left(B\right)\Phi\left(A_1\right)\Phi^{\left(k-\dot{\beta}\right)}\left(Y_{1,1}\right)\right)-\varphi\left(\Phi_{\natural}^{n_1-1}\left(B\right)\Phi\left(A_1\right)\right)\varphi\left(Y_{1,1}\right)\right|=0.$$

□

Finally, the proof of proposition 8 is a simple result of the previous lemma.

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