Convergence results in continuous-time quantized consensus

Paolo Frasca *

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Abstract

This note studies a network of agents having continuous-time dynamics with quantized interactions and time-varying topology. We show that under a mild connectivity condition, the agents reach consensus (up to the quantizer precision) after a finite time, which can be precisely estimated when the topology is independent of time.

1 Introduction

In this note we study a coordination task for a network of agents having a scalar continuous-time dynamics. The interaction between agents is weighted by time-dependent coefficients which represent a time-varying topology, and connected agents can exchange information about their states only through a quantizer. Due to the quantization constraint, the goal of consensus between states can only be approximated up to the quantizer precision. Our contribution consists in a sufficient condition for finite-time convergence to the best achievable approximation, in terms of the connectivity of a suitable limit graph. We first consider uniform quantizers, and then extend our analysis to non-uniform quantizers. Additionally, assuming uniform quantizers and time-invariant topology, we derive and discuss an upper bound on the convergence time, which is inversely proportional to the quantizer precision.

1.1 Related works

Many papers have studied consensus and coordination problems in time-depending networks: we refer the reader to the books [2][14] for an introduction and to [11] for recent related results. Results about coordination and consensus of systems subject to quantization have been presented in a number of papers in the last few years: most authors have focused on discrete-time systems –a non-exhaustive list includes [13][3][1][9][4][12]– while continuous-time systems have been considered in [8][5][15]. In the latter case, the inherent discontinuity of the system right-hand side entails some additional mathematical difficulties, which are discussed in [6] and [5]. The latter paper considers a simple continuous-time average consensus dynamics with time-invariant topology and uniform static quantizers, and shows that a suitable definition of solution is essential to ensure that solutions are defined for all times and thus to permit a significant convergence analysis. A natural and effective choice is taking Krasowskii solutions, which indeed are complete for every initial condition and converge to approximate consensus conditions under mild assumptions. In the present paper, we consider the quantized dynamics

*P. Frasca is with the Dipartimento di Matematica, Politecnico di Torino, Torino, Italy. paolo.frasca@polito.it.

in [5] and we extend the analysis of convergence for Krasowskii solutions to time-dependent topologies and to more general quantizers.

1.2 Preliminary definitions

In this section we provide some background in differential equations and graph theory. For our analysis it is necessary to recall from [10, 7] a certain definition of solution to a –possibly discontinuous– differential equation, which is based on considering a suitable differential inclusion. Given[1] $f : \mathbb{R}_{>0} \times \mathbb{R}^N \to \mathbb{R}^N$ and the differential equation $\dot{x} = f(t, x)$, we say that $x : J \to \mathbb{R}^N$ solves this differential equation in the Krasowskii sense if $x(\cdot)$ is absolutely continuous and for almost every time $t \in J \subset \mathbb{R}_{>0}$ satisfies the diffential inclusion $\dot{x}(t) \in \mathcal{K}f(t, x(t))$, where

$$\mathcal{K}f(t, x) = \bigcap_{\delta>0} \overline{\mathrm{co}} f(t, B(x, \delta)),$$

with $\overline{\mathrm{co}}$ denoting the convex closure and $B(y, r)$ the Euclidean ball of radius r centered in y. Here and elsewhere in the paper, "almost every" means "except in a set of zero Lebesque measure". A solution is said to be complete if $J = (0, +\infty)$. Note that we will also apply the Krasowskii operator $\mathcal{K}$ to autonomous functions $f(x)$.

Our analysis also involves graphs and weighted graphs. Given a finite set of vertices (or nodes) V, a (directed) graph G is a pair (V, E) where $E \subset V \times V$ is the set of edges (or arcs). A weighted graph is triple (V, E, A) which includes a weighted adjacency matrix $A \in \mathbb{R}_{\geq 0}^{V \times V}$ with the consistency condition that $A_{uv} > 0$ if and only if $(u, v) \in E$. We also assume that $A_{uu} = 0$ for all $u \in V$. The Laplacian matrix associated to A is a matrix $L \in \mathbb{R}^{V \times V}$ such that $L_{uv} = -A_{uv}$ if $u \neq v$ and $L_{uu} = \sum_{v \in V} A_{uv}$. A sink is a node u with no outgoing edge –that is, such that E does not contain any edge of the form (u, v). A path (of length l) from u to v in G is an ordered list of edges $(e_1, \ldots, e_l)$ in the form $((u, w_1), (w_1, w_2), (w_2, w_3), \ldots, (w_{l-1}, v))$. If such a path exists, u is said to be connected to v. A cycle is a path from a node to itself. A graph is said to be strongly connected if every two nodes are connected to each other. Instead, if every two nodes can be connected by a path when we permit to reverse the order of nodes in the edges, then the graph is said to be weakly connected. Given any directed graph $G = (V, E)$ we can consider its strongly connected components, namely maximal strongly connected subgraphs G_k, $k \in \{1, \ldots, s\}$ with set of vertices $V_k \subset V$ and set of arcs $E_k = E \cap (V_k \times V_k)$ such that the sets V_k form a partition of V. The various components may have connections among each other. We define another directed graph $\mathcal{T}(G)$ with set of vertices $\{1, \ldots, s\}$ such that there is an arc from h to k if there is an arc in G from a vertex in V_k to a vertex in V_h. It can be shown that $\mathcal{T}(G)$ has no cycle and is weakly connected (i.e., a tree) if G is weakly connected. We refer to $\mathcal{T}(G)$ as the tree of the connected components of G. The reader is referred to the above referenced literature or to a book like [14] for a more complete introduction.

2 Problem and results

In this section we introduce the dynamics of interest, and we state and prove our convergence results. Let there be N agents, indexed in a set I, and for any pair $(i, j) \in I \times I$, let $a_{ij}(\cdot) : \mathbb{R}_{>0} \to [0, 1]$ be a measurable function. These interaction functions naturally lead to the following definitions. For every time t, we consider a weighted interaction graph $\mathcal{G}(t) =$

[1] The symbols $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ $\mathbb{R}_{>0}$ denote the sets of integer, real, nonnegative and positive numbers, respectively. $\mathbb{R}^N$ denotes an N-dimensional Euclidean space. Given $a \in \mathbb{R}$, the set of the (integer) multiples of a is denoted by $a\mathbb{Z}$.

$(I, \mathcal{E}(t), A(t))$, such that the i,j-th component of the matrix $A(t)$ is the value $a_{ij}(t)$, and $(i,j) \in \mathcal{E}(t)$ if and only if $a_{ij}(t) > 0$. We also consider a limit graph defined as $\mathcal{G}_\infty = (I, \mathcal{E}_\infty)$ is such that

$$\mathcal{E}_\infty = \{(i,j) \in I \times I \, : \, \lim_{t \to +\infty} \int_{t_0}^{t} a_{ij}(s) ds = +\infty \quad \forall \, t_0 \geq 0\}.$$

For $i \in I$, let $x_i(t)$ be a real variable and consider the dynamics

$$\dot{x}_i = \sum_{j \in I} a_{ij}(t)(q(x_j) - q(x_i)) \tag{1}$$

where $q : \mathbb{R} \to \Delta\mathbb{Z}$ is the uniform quantizer with precision Δ, that is

$$q(z) = \left\lfloor \frac{z}{\Delta} + \frac{1}{2} \right\rfloor \Delta.$$

System (1) can also be rewritten in vector form as

$$\dot{x} = -L(t)q(x),$$

where $x(t) \in \mathbb{R}^I$ is the state vector, $L(t)$ is the Laplacian matrix associated to the weighted adjacency matrix $A(t)$ and by a slight notational abuse, q is defined to operate componentwise on vectors.

We consider for (1) solutions in the sense of Krasowskii, which we have defined above, and thanks to the linearity of the Krasowskii operator $\mathcal{K}$, we have that a Krasowskii solution to (1) is for almost every time a solution to

$$\dot{x} \in -L(t)\mathcal{K}q(x).$$

The multivalued function $\mathcal{K}q(x)$ is illustrated in Figure 1. By the current assumptions of boundedness on the functions a_{ij}, for any $\bar{x} \in \mathbb{R}^I$ there exists a complete Krasowskii solution $x(t)$ to (1), such that $x(0) = \bar{x}$. The solution, however, needs not to be unique.

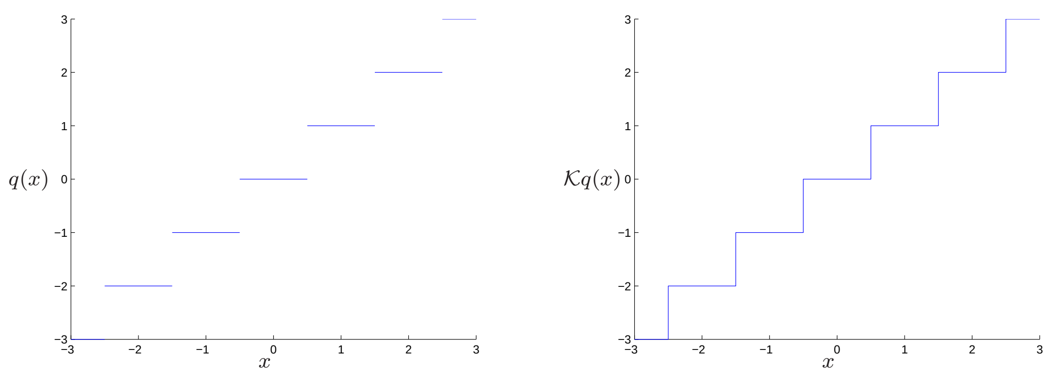

Figure 1: Visualization of the map $q(x)$ and the set-valued map $\mathcal{K}q(x)$, when $\Delta = 1$.

After these preliminary observations, we are ready to state and prove that the system (1) reaches consensus in finite time, up to the quantizer precision.

Proposition 1 (Finite-time quantized consensus)**.** *Let $x(t)$ be Krasowskii solution to* (1)*. If $\mathcal{G}_\infty$ is strongly connected, then there exist $T_{\rm con} \in \mathbb{Z}_{\geq 0}$ and $q_\infty \in \Delta\mathbb{Z}$ such that for all $t \geq T_{\rm con}$,*

$$x_i(t) \in \left[q_\infty - \frac{\Delta}{2}, q_\infty + \frac{\Delta}{2}\right] \qquad \textit{for all } i \in I.$$

Proof. Let $m(t) = \min_{i\in I} \min \mathcal{K}q(x_i(t))$ and $M(t) = \max_{i\in I} \max \mathcal{K}q(x_i(t))$ and notice that both $M(t)$ and $m(t)$ are multiples of Δ and that $m(t) \geq m(0)$ and $M(t) \leq M(0)$. The proof is based on showing that $m(t)$ is actually increasing, as long as the system is not at the equilibrium: an analogous argument can be developed using $M(t)$.

The thesis can be rewritten as follows: there exist a nonnegative integer $T_{\rm con}$ and an integer k such that for all $t \geq T_{\rm con}$, it holds

$$k\Delta \in \mathcal{K}q(x_i(t)) \qquad \text{for all } i \in I.$$

Following this interpretation we define, given the solution $x(\cdot)$ and $a \in \mathbb{R}$, the set

$$I_a(t) = \{i \in I \,:\, a \in \mathcal{K}q(x_i(t))\}.$$

Given $h \in \mathbb{Z}$ and $t \geq 0$, note that $I_{h\Delta}(t) = \{i \in I \,:\, x_i(t) \in [\Delta(h - \frac{1}{2}), \Delta(h + \frac{1}{2})]\}$. Then, the set $I_{h\Delta}(t) \cap I_{(h+1)\Delta}(t)$ needs not to be empty. In view of this remark and in order to show that $m(t)$ is increasing, we consider the set $I_{m(0)}(t) \setminus I_{m(0)+\Delta}(t)$ and we note that by the dynamics (1),

$$I_{m(0)}(t_1) \setminus I_{m(0)+\Delta}(t_1) \supset I_{m(0)}(t_2) \setminus I_{m(0)+\Delta}(t_2)$$

for all $t_2 \geq t_1 \geq 0$. Then, we define

$$I^+_{m(0)}(t) = \left(\bigcup_{h=m(0)+\Delta}^{M(0)} I_h(t)\right) \setminus I_{m(0)}(t)$$

and $T_{\rm empty} = \inf\{t \geq 0 \,:\, I^+_{m(0)}(t) = \emptyset\}$. If $T_{\rm empty}$ is finite, then $T_{\rm con} = T_{\rm empty}$ and $q_\infty = m(0)$, and we have concluded the proof. Otherwise, thanks to the strong connectivity of $\mathcal{G}_\infty$ we choose $(i,j) \in \mathcal{E}_\infty$ such that

$$i \in I_{m(0)}(0) \setminus I_{m(0)+\Delta}(0)$$

and

$$j \in \bigcap_{t\geq 0} I^+_{m(0)}(t).$$

Notice that $\dot{x}_i(t) \geq a_{ij}(t)\Delta$ for almost every $t \geq 0$. Then,

$$x_i(t) \geq x_i(0) + \Delta \int_0^t a_{ij}(s)ds \geq m(0) - \frac{1}{2}\Delta + \Delta \int_0^t a_{ij}(s)ds.$$

Since the integral of a_{ij} is divergent, there exists $T' > 0$ such that $\int_0^{T'} a_{ij}(s)ds$ is so large that $x_i(T') = m(0) + \frac{1}{2}\Delta$, implying that $i \notin I_{m(0)}(T') \setminus I_{m(0)+\Delta}(T')$. By repeatedly choosing suitable pairs $(i,j) \in \mathcal{E}_\infty$, we obtain that $I_{m(0)}(T_0) \setminus I_{m(0)+\Delta}(T_0) = \emptyset$ for some finite $T_0 > 0$.

The same argument can then be applied, with straightforward modifications, to $m(0)+\Delta$, $m(0) + 2\Delta$, ..., showing that there exists a sequence of times T_k such that $I_{m(0)+k\Delta}(T_k) \setminus I_{m(0)+(k+1)\Delta}(T_k) = \emptyset$. Since $M(t) \leq M(0)$, then the sequence of T_k's must be finite. This implies that there exist $T_{\rm con}$ and q_∞ such that $I_{q_\infty}(T_{\rm con}) = I$, and concludes the proof. □

In many applications one is concerned, rather than with mere convergence, with convergence to a certain target value, which is a function of the initial condition. For instance, the value can be the average of the initial states: this problem is referred to as the average consensus problem, and is studied in the next result.

Corollary 2 (Average-preserving dynamics)**.** *Let $x(t)$ be Krasowskii solution to* (1)*, and define $x_{\mathrm{ave}}(t) = \frac{1}{N}\sum_{j\in I} x_j(t)$. If $\mathcal{G}_\infty$ is strongly connected and $\sum_{j\in I} a_{ij}(t) = \sum_{i\in I} a_{ij}(t)$ for almost every $t \geq 0$, then $x_{\mathrm{ave}}(t) = x_{\mathrm{ave}}(0)$ for every $t > 0$ and the conclusion of Proposition 1 holds. Moreover, if $x_{\mathrm{ave}}(0) \neq (k + \frac{1}{2})\Delta$ for every $k \in \mathbb{Z}$, then $q_\infty = q(x_{\mathrm{ave}}(0))$, whereas if $x_{\mathrm{ave}}(0) = (h + \frac{1}{2})\Delta$ for some $h \in \mathbb{Z}$, then $x_i(T_{\mathrm{con}}) = x_{\mathrm{ave}}(0)$ for every $i \in I$.*

Proof. By linearity, for almost every $t > 0$

$$\frac{d}{dt}x_{\mathrm{ave}}(t) \in \mathcal{K}\left(\frac{1}{N}\sum_{i\in I}\sum_{j\in I} L_{ij}q(x_j(t))\right).$$

By the assumptions on the a_{ij}'s and the definition of Laplacian, this implies that $\frac{d}{dt}x_{\mathrm{ave}}(t) = 0$ for almost every $t > 0$, so that the average is preserved. Proposition 1 then implies that $x_{\mathrm{ave}}(0) \in \left[q_\infty - \frac{\Delta}{2}, q_\infty + \frac{\Delta}{2}\right]$. If in particular $x_{\mathrm{ave}}(0) \in \left(q_\infty - \frac{\Delta}{2}, q_\infty + \frac{\Delta}{2}\right)$, then it is clear that $q(x_{\mathrm{ave}}(0)) = q_\infty$. Otherwise, being $x_{\mathrm{ave}}(T_{\mathrm{con}})$ at the border of the interval, necessarily all $x_i(T_{\mathrm{con}})$ must coincide. □

Note that Corollary 2 provides a formula for the limit value and also a sufficient condition to achieve exact consensus between the states. Next, we prove a simple extension of Proposition 1 allowing for a weaker requirement on the topology.

Corollary 3 (Weak connectivity)**.** *If $\mathcal{G}_\infty$ is weakly connected and the associated tree of connected components $\mathcal{T}(\mathcal{G}_\infty)$ has only one sink, then the conclusion of Proposition 1 holds.*

Proof. We proceed as in the proof of Proposition 1, with one modification. If at any time the sink component is included in the set of minima under consideration –for instance, it is contained in $I_{m(0)}(0) \setminus I_{m(0)+\Delta}(0)$–, then the argument can not be concluded as above. However, since there is only one sink, it is still possible to conclude by applying the analogous argument based on the maxima $M(t)$. □

2.1 Convergence time

This section is devoted to estimate the convergence time in Proposition 1. In order to obtain precise and significant results, we restrict ourselves to consider time-invariant topologies. Namely, we assume that $a_{ij}(t)$ are constant in time, so that $\mathcal{G}(t) \equiv \mathcal{G}_\infty$ and we do not need to write the dependence on t any longer.

Corollary 4 (Estimate of T_{con})**.** *If $\mathcal{G}$ is time-invariant and strongly connected, then*

$$T_{\mathrm{con}} \leq \frac{N}{\Delta\,\underline{a}} \max_{i,j\in I} |q(x_i(0)) - q(x_j(0))|, \tag{2}$$

where $\underline{a} = \min\{a_{ij} \,:\, (i,j) \in \mathcal{E}\}$.

Proof. The proof is based on specializing the proof of Proposition 1 to the case at hand: for simplicity and clarity, the same notation is used. Reviewing our argument, when it comes to choosing the pairs (i,j), the assumption on a_{ij} implies that if $(i,j) \in \mathcal{E}$, then $\int_{t_1}^{t_2} a_{ij}ds \geq 1$ when $t_2 - t_1 \geq \frac{1}{\underline{a}}$. Then, considering the sequence of T_k's, we argue that $T_k - T_{k-1} \leq \frac{N}{\underline{a}}$ for every $k \geq 1$, as every quantization interval contains at most N agents. On the other hand, k needs not to be larger than $(M(0) - m(0))/\Delta$. These remarks prove the statement. □

Next, we claim that the bound (2) is tight in the sense that, for every N, we can find a weighted graph $\mathcal{G}$ and an initial condition $\bar{x}$ such that for a certain solution such that $x(0) = \bar{x}$,

$$T_{\text{con}} \geq \frac{1}{8} \frac{N}{\underline{a}\,\Delta} \max_{i,j \in I} |q(x_i(0)) - q(x_j(0))|.$$

The construction proving this claim is given in the following example.

Example 1 (Slow convergence)**.** We let $I = \{1, \ldots, N\}$ and we assume the topology to be a line graph, namely

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2 \\ 1 & \text{if } 2 \leq i \leq N-1 \text{ and } j = i-1, i+1 \\ 1 & \text{if } i = N \text{ and } j = N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the resulting dynamics (1) preserves the average of the states. Regarding the initial condition, we assume $x_i(0) = \Delta(i-1)$ for all $i \in I$. In the analysis of the resulting system, we think of the agents as arranged on a line and we only describe the evolution of the leftmost agents (1,2,…,$\lfloor N/2 \rfloor$), the evolution of the others being symmetrical. For early positive times, all agents are still, except agent 1 which moves to the right with constant speed Δ. Then, at time $T' = \frac{1}{2}$ we have that $x_1(T') = \Delta/2$, that is agent 1 reaches the border of the first quantization interval. Since $\mathcal{K}q(x_1(T')) \ni \Delta$, there is one Krasowskii solution such that for $t \in (T', 2T')$, $x_1(t)$ is constant while agent 2 moves to the right until it reaches $x_2(2T') = 3\Delta/2$, so that $\mathcal{K}q(x_2(2T')) \ni 2\Delta$ and $\mathcal{K}q(x_1(2T')) \ni \Delta$. Then, for $t \in (2T', 4T')$ the only agent on the move is again agent 1, until $x_1(4T') = 3\Delta/2$. At time $t = 4T'$, the two agents have the same state $x_2(4T') = x_1(4T')$. Later, $x_3(t)$, $x_2(t)$, $x_1(t)$ move to the right during subsequent intervals, so that at $t = 9T'$ they are all collocated as $x_1(t) = x_2(t) = x_3(t) = 5\Delta/2$. By repeating this reasoning, we observe that the solution $x(\cdot)$ is such that the condition of Proposition 1 is reached in a time

$$T_{\text{con}} = \frac{1}{2} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} (1 + 2k) = \frac{1}{2} \left(\left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right)^2 \geq \frac{1}{8} N(N-1).$$

Since $(N-1)\Delta = q(x_N(0) - q(x_1(0)))$, then

$$T_{\text{con}} \geq \frac{1}{8} N \frac{q(x_N(0) - q(x_1(0)))}{\Delta}.$$

□

We remark that (2) yields a convergence time which is polynomial in the required precision Δ, namely $T_{\text{con}} \propto \Delta^{-1}$. Such a qualitative behavior should be contrasted with the logarithmical convergence time of nonquantized consensus dynamics. Indeed, considering convergence up to a precision ε in a suitable norm, consensus dynamics without quantization yield

$$T^{\varepsilon}_{\text{con}} \leq C(x(0))\, \underline{\lambda} \log \varepsilon^{-1},$$

where $C(x(0))$ is a constant depending on the initial condition and $\underline{\lambda}$ is the algebraic connectivity of the interaction graph at hand. We conclude that theoretical results imply a qualitative loss of performance due to quantization. However, we wish to point out that Corollary 4 is, by nature, a worst-case result, and typical solutions need not to achieve the performance bound. Indeed, it is argued in [5, Remark 5] that, far from the equilibria, the quantized dynamics converges exponentially fast and has the same rate of convergence as the nonquantized linear consensus dynamics. This is confirmed by simulations, which typically show logarithmical convergence times in both cases. These remarks entail no contradiction: far away from the equilibria the quantized dynamics is well approximated by the nominal linear dynamics, while in a neighborhood of the equilibria the approximation is no longer good and the intrinsic limits due to quantization may come out.

2.2 Non-uniform quantizers

So far, we have considered a uniform quantizer. However, one could be interested in more general quantizers, mapping the real line into a discrete (possibly finite) set. Our arguments can be promptly extended to dynamics based on such quantizers, proving finite-time convergence to quantized consensus equilibria.

Theorem 5 (Finite-time quantized consensus – Extended)**.** *Let $\mathcal{S}$ be a subset of $\mathbb{R}$ with no limit point, and the quantizer $q:\mathbb{R}\to\mathcal{S}$ a non-decreasing function. Let $x(t)$ be a Krasowskii solution to the corresponding dynamics* (1)*. If $\mathcal{G}_\infty$ is weakly connected and $\mathcal{T}(\mathcal{G}_\infty)$ has only one sink, then there exist $T_{\rm con}\in\mathbb{Z}_{\geq 0}$ and $s^*\in\mathcal{S}$ such that, for every $t\geq T_{\rm con}$,*

$$s^*\in\mathcal{K}q(x_i(t)) \quad \text{for every } i\in I.$$

Proof. Without loss of generality, we may think of the elements of $\mathcal{S}$ as indexed in a set A of consecutive integers, in such a way that $\mathcal{S}=\{s_a\ :\ a\in A\}$ and $s_a<s_b$ if and only if $a<b$. Let $\Delta_{\min}=\inf\{|s_a-s_b|\ :\ a,b\in A\}$. As there is no limit point of $\mathcal{S}$, then $\Delta_{\min}>0$. Given the solution $x(\cdot)$ and $a\in\mathbb{R}$, we define $I_a(t)=\{i\in I\ :\ a\in\mathcal{K}q(x_i(t))\}$ and let $m(t)=\min_{i\in I}\min\mathcal{K}q(x_i(t))$ and $M(t)=\max_{i\in I}\max\mathcal{K}q(x_i(t))$. By definition, $M(t)$ and $m(t)$ belong to $\mathcal{S}$ and we denote $m(0)=s_m$ and $M(0)=s_M$. The dynamics (1) implies that $m(t)\geq m(0)$ and $M(t)\leq M(0)$. The proof, similarly to the proof of Proposition 1, is based on showing that $m(t)$ increases until the system reaches an equilibrium. In this proof, we assume that $\mathcal{G}_\infty$ is strongly connected and we refer to the proof of Corollary 3 for the extension to weak connectivity. We consider the set $I_{s_m}(t)\setminus I_{s_{m+1}}(t)$ and we note that by the dynamics (1),

$$I_{s_m}(t_1)\setminus I_{s_{m+1}}(t_1)\supset I_{s_m}(t_2)\setminus I_{s_{m+1}}(t_2)$$

for all $t_2\geq t_1\geq 0$. Then, we define

$$I^+_{s_m}(t)=\left(\bigcup_{h=m+1}^{M} I_{s_h}(t)\right)\setminus I_{s_m}(t)$$

and $T_{\rm empty}=\inf\{t\geq 0\ :\ I^+_{s_m}(t)=\emptyset\}$. If $T_{\rm empty}$ is finite, then $T_{\rm con}=T_{\rm empty}$ and $s^*=s_m$, and we have concluded the proof. Otherwise, thanks to the strong connectivity of $\mathcal{G}_\infty$, we choose $(i,j)\in\mathcal{E}_\infty$ such that $i\in I_{s_m}(0)\setminus I_{s_{m+1}}(0)$ and $j\in\bigcap_{t\geq 0}I^+_{s_m}(t)$. Notice that $\dot{x}_i(t)\geq a_{ij}(t)\,\Delta_{\min}$ for almost every $t\geq 0$. Then,

$$x_i(t)\geq x_i(0)+\Delta_{\min}\int_0^t a_{ij}(s)ds.$$

Let $q^{-1}(s_m)$ denote the pre-image of s_m under q. If $\sup q^{-1}(s_m)=+\infty$, then necessarily $s_m=\max\mathcal{S}$ and the proof is completed. Otherwise, since the integral of a_{ij} is divergent, there exists $T'>0$ such that $\int_0^{T'}a_{ij}(s)ds$ is so large that $\mathcal{K}q(x_i(T'))\ni s_{m+1}$ and $i\notin I_{s_m}(T')\setminus I_{s_{m+1}}(T')$. By repeatedly choosing suitable pairs $(i,j)\in\mathcal{E}_\infty$, we obtain that $I_{s_m}(T_0)\setminus I_{s_{m+1}}(T_0)=$

$\emptyset$ for some finite $T_0 > 0$. The same argument can then be applied, with straightforward modifications, to $s_{m+1}, s_{m+2}, \ldots$, showing that there exists a sequence of times T_k such that $I_{s_{m+k}}(T_k) \setminus I_{s_{m+k+1}}(T_k) = \emptyset$. Since $M(t) \leq s_M$, then the sequence of T_k's must be finite. This implies that there exist T_{con} and s^* such that $I_{s^*}(T_{\text{con}}) = I$, and concludes the proof. $\square$

Note that the above assumptions on $\mathcal{S}$ are fulfilled, for instance, when $\mathcal{S}$ is a finite set or when $\mathcal{S} = \Delta\mathbb{Z}$. Then Proposition 1 can be seen as a corollary of Theorem 5. However, the former has been stated and proved independently because of the importance of the uniform quantizer and in order to familiarize the reader with the proof method.

3 Conclusion

This paper has demonstrated that a mathematical framework combining graph theory and Krasowskii differential inclusions can be useful in problems of distributed control. In particular, the convergence conditions provided here are general enough to be used in most applications. A few natural generalizations of the present work would be of interest. For instance, one can wonder what happens if $\mathcal{G}_\infty$ is not connected, or what are the convergence properties of a state-dependent network described by interaction functions of type $a_{ij}(t, x)$. Such studies may have broad applications, including rendez-vous and coordination problems in robotic networks where the ability to communicate depends on the robot locations [2], and modeling opinion dynamics with limited verbalization capabilities [16] in social networks.

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