THE BURGERS EQUATION AND THE KORTEWEG-DE VRIES EQUATION WITH QUADRATIC NONLINEARITY

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Abstract. We study generalized variants of the Burgers equation and the KdV equation on the circle. The main goal of the paper is to show that both extensions can be recast as geodesic equations on a suitable diffeomorphism group of the circle and the corresponding Bott-Virasoro group respectively. As a consequence we obtain that the initial value problem for the Burgers equation with an additional quadratic term is well-posed on a scale of Sobolev spaces on the circle.

Contents

1. Introduction and main results

In the present paper, the following generalized versions of the Burgers equation

$$u_t + 3uu_x = \nu u_{xx} + f(u), \quad \nu \in \mathbb{R}, \tag{1}$$

and the Korteweg-de Vries (KdV) equation

$$u_t + 3uu_x = g(x)u_{xxx} + f(u), \tag{2}$$

are of interest. Here $u = u(t, x)$ is a periodic function of a space variable $x \in \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}$, where $\mathbb{S}$ denotes the unit circle, and time $t \geq 0$. We write $u(0, x) = u_0(x)$ for all $x \in \mathbb{S}$. For $f \equiv 0$ and $g \equiv 1$, Eqs. (1) and (2) reduce to two well-known equations which come up in the mathematical theory of fluids and water waves and which have been studied in great detail.

The KdV equation first appeared in a paper of Boussinesq in 1877 and has later been named for Diederik Korteweg and Gustav de Vries. This equation is appealing from the mathematical point of view for several reasons: it admits for solitary wave

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solutions which can be specified explicitly, it is solvable via the inverse scattering approach, it possesses two compatible Hamiltonian structures and is related to Lagrange's variational principle. For a brief overview about the most important results we refer to [14].

The Burgers equation can be derived from Euler's equations for the motion of an ideal fluid by a double asymptotic expansion of the relevant physical variables, cf. [7]. The non-viscous Burgers equation ($\nu = 0$) can be solved by the method of characteristics, the viscous Burgers equation can be linearized by the Cole-Hopf transformation. The non-viscous equation is also of great importance in the study of gas flow in one dimension since the associated Riemann problem (a flow problem with particular discontinuous initial data) admits for rarefaction waves and shocks.

Generalized versions of type (1) and (2) are introduced in, e.g., [15–17]. We will choose $\nu = 0$, $f(u) = \alpha u^2$ and $g(x) = \beta e^{\alpha x}$, $\alpha, \beta \in \mathbb{R}$, in the following; precisely, we discuss the equations

$$u_t + 3uu_x = \alpha u^2 \tag{3}$$

and

$$u_t + 3uu_x = \beta e^{\alpha x} u_{xxx} + \alpha u^2. \tag{4}$$

In this paper we will focus on the geometric aspects of Eqs. (3) and (4): The more recent papers [11, 12] show that the classical Burgers and KdV equations are Euler equations on the diffeomorphism group of the circle and the Bott-Virasoro group respectively. In particular, these equations re-express a geodesic flow which is related to the L_2 metric on the diffeomorphism group and its central extension respectively. We show that a similar geometric interpretation for Eqs. (3) and (4) is possible using appropriate weighted L_2 spaces. Our main results read as follows.

Theorem 1. *The Euler equation describing the geodesic flow on the diffeomorphism group of the circle and the associated Bott-Virasoro group with respect to the weighted L_2 metric with weight $\omega_\alpha(x) = e^{-\alpha x}$, $x \in \mathbb{S}$, is the modified Burgers equation* (3) *and the modified KdV equation* (4) *respectively.*

The geometric approach will result in the following local well-posedness result for Eq. (3).

Corollary 2. *The Cauchy problem for the modified Burgers equation* (3) *is well-posed in the Sobolev spaces $H^s(\mathbb{S})$ for any $s > 3/2$, i.e., for an arbitrary $u_0 \in H^s(\mathbb{S})$ there exists a time interval $(-T, T)$ and a unique solution*

$$u = u(t; u_0) \in C((-T, T); H^s(\mathbb{S})) \times C^1((-T, T); H^{s-1}(\mathbb{S}))$$

to Eq. (3)*, with $u(0; u_0) = u_0$, such that the mapping $(t, u_0) \mapsto u(t; u_0)$ is continuous.*

The paper is organized as follows: In a preliminary section we recall some basic facts about the diffeomorphism group of the circle and the Bott-Virasoro group and explain the notion of Euler equations in this context. In the following section we give a proof of our main theorem and its corollary. Finally, we explain some open problems and further tasks.

2. Euler equations on the diffeomorphism group of the circle and the Bott-Virasoro group

Let $s \geq 0$. We denote by $H^s = H^s(\mathbb{S})$ the Sobolev space of order s on the circle, i.e.,

$$H^s(\mathbb{S}) = \left\{ u \in L_2(\mathbb{S});\ \left\| \mathcal{F}^{-1} M_{(1+4\pi^2 n^2)^{s/2}} \mathcal{F} u \right\|_{L_2(\mathbb{S})} < \infty \right\};$$

here, $\mathcal{F}\colon L_2(\mathbb{S}) \to \ell_2(\mathbb{Z})$ denotes the Fourier transform and $M_{(1+4\pi^2 n^2)^{s/2}}$ is the multiplication operator associated with the symbol $(1+4\pi^2 n^2)^{s/2}$, $n \in \mathbb{Z}$, of the elliptic pseudo-differential operator $(1-\partial_x^2)^{s/2}$. Endowed with the norms

$$\|u\|_s^2 = \left\| \mathcal{F}^{-1} M_{(1+4\pi^2 n^2)^{s/2}} \mathcal{F} u \right\|_{L_2(\mathbb{S})}^2 = \left\langle (1-\partial_x^2)^s u, u \right\rangle_{L_2(\mathbb{S})}$$

the spaces H^s become Hilbert spaces.

Let $H^s\mathrm{Diff}(\mathbb{S})$ denote the set of orientation-preserving diffeomorphisms $\mathbb{S} \to \mathbb{S}$ in H^s. It is well-known that $H^s\mathrm{Diff}(\mathbb{S})$ is a topological group (with respect to composition) and a Hilbert manifold for any $s > 3/2$, cf. [4]; an atlas is given by the charts (U_i, Φ_i), $i = 1, 2$, where

$$\begin{aligned} U_1 &= \left\{ u \in H^s;\ u_x > -1, -\tfrac{1}{2} < u(0) < \tfrac{1}{2} \right\}, \\ U_2 &= \left\{ u \in H^s;\ u_x > -1, 0 < u(0) < 1 \right\} \end{aligned}$$

and

$$\Phi_i\colon U_i \to H^s\mathrm{Diff}(\mathbb{S}), \quad \Phi_i(u) = \mathrm{id} + u,$$

for $i = 1, 2$, cf. [5]. The tangent space of $H^s\mathrm{Diff}(\mathbb{S})$ at the identity can be identified with the H^s vector fields on the circle, $H^s\mathfrak{vect} = \{u(x)\partial_x;\ u \in H^s\} \simeq H^s$. Furthermore $H^s\mathrm{Diff}(\mathbb{S})$ is parallelizable, i.e., one has the trivialization

$$TH^s\mathrm{Diff}(\mathbb{S}) \simeq H^s\mathrm{Diff}(\mathbb{S}) \times H^s\mathfrak{vect} \simeq H^s\mathrm{Diff}(\mathbb{S}) \times H^s,$$

and the derivative of the right translation map $R_\varphi\colon \psi \mapsto \psi \circ \varphi$ on $H^s\mathrm{Diff}(\mathbb{S})$ is an automorphism of H^s. Further details can be found in the introductory section of Lenells' paper [9].

The *Bott-Virasoro group* is the manifold $H^s\mathrm{Diff}(\mathbb{S}) \times \mathbb{R}$ with the group product defined by

$$(\psi, a)(\varphi, b) = (\psi \circ \varphi, a + b + B(\psi, \varphi))$$

where

$$B(\psi, \varphi) = \int_{\mathbb{S}} \log((\psi \circ \varphi)_x)\, \mathrm{d} \log \varphi_x$$

is the Bott cocycle. Note that $H^s\mathrm{Diff}(\mathbb{S}) \times \mathbb{R}$ is a non-trivial one-dimensional central extension of $H^s\mathrm{Diff}(\mathbb{S})$. The vector space $\mathfrak{vir} = H^s\mathfrak{vect} \oplus \mathbb{R}$ is called the *Virasoro algebra* and is equipped with the commutator

$$[(u(x)\partial_x, a), (v(x)\partial_x, b)] = ((u_x v - v_x u)(x)\partial_x, C(u\partial_x, v\partial_x))$$

where

$$C(u\partial_x, v\partial_x) = \frac{1}{2} \int_{\mathbb{S}} \det \begin{pmatrix} u_x & v_x \\ u_{xx} & v_{xx} \end{pmatrix} \mathrm{d}x = \int_{\mathbb{S}} u_x v_{xx}\, \mathrm{d}x$$

is the Gelfand-Fuchs cocycle and $u_x v - v_x u$ is the Lie bracket of u and v thought of as elements of $H^s\mathfrak{vect}$. We refer the reader to [8, 11, 12] for related material.

It goes back to the works of Arnold [1] and Ebin and Marsden [4] that, defining suitable right-invariant metrics on $H^s\mathrm{Diff}(\mathbb{S})$ and the Bott-Virasoro group respectively, one can recast equations arising in fluid mechanics as geodesic equations on these configuration manifolds: Defining the positive definite bilinear forms

$$\langle u,v\rangle = \int_{\mathbb{S}} u(x)v(x)\,\mathrm{d}x, \quad \langle (u,a),(v,b)\rangle = \int_{\mathbb{S}} u(x)v(x)\,\mathrm{d}x + ab \tag{5}$$

on $H^s\mathfrak{vect}$ and $\mathfrak{vir}$, and extending them by right invariance to (weak) Riemannian metrics on the diffeomorphism group and the Bott-Virasoro group, one observes that the inviscid Burgers equation $u_t+3uu_x=0$ and the KdV equation $u_t+3uu_x-u_{xxx}=0$ are the associated *Euler equations* on $H^s\mathfrak{vect}$ and $\mathfrak{vir}$ respectively: Let ad_u^* and $\mathrm{ad}_{(u,a)}^*$ denote the dual operators of the adjoint actions $\mathrm{ad}_u\colon H^s\mathfrak{vect}\to H^s\mathfrak{vect}$ and $\mathrm{ad}_{(u,a)}\colon \mathfrak{vir}\to\mathfrak{vir}$ with respect to the metrics defined in (5). The geodesic equation for these metrics on both groups reads

$$X_t = -\mathrm{ad}_X^* X \tag{6}$$

and explicit calculations of the right-hand side $\mathrm{ad}_X^* X$ in both cases show that (6) reduces to the Burgers equation and the KdV equation respectively.

Theorem 3 (see, e.g., [8]). *The Burgers equation and the KdV equation describe geodesic motion on the diffeomorphism group of the circle and the Bott-Virasoro group respectively and they are the Euler equations for the right-invariant L_2 metric on these groups.*

3. A proof of the main result

Let ω be a positive function on $\mathbb{S}$ with $\inf_{x\in[0,1)}\omega(x)>0$ and denote by M_ω the associated multiplication operator, i.e., $(M_\omega u)(x)=\omega(x)u(x)$. For any solution u of the Burgers equation (3) we let the function φ be the solution in $H^s\mathrm{Diff}(\mathbb{S})$ of the initial value problem

$$\begin{cases} \varphi_t(t,x) = u(t,\varphi(t,x)), & (t,x)\in\mathbb{R}_+\times\mathbb{S},\\ \varphi(0,x) = x, & x\in\mathbb{S}.\end{cases}$$

We will use the short hand notation $\varphi_t = u\circ\varphi$, i.e., $\circ$ denotes composition with respect to the space variable. Since $\varphi_{tt}=(u_t+uu_x)\circ\varphi$ we find that the equation

$$m_t = -m_x u - 2u_x m, \quad m = M_\omega u, \tag{7}$$

is equivalent to

$$\begin{cases} \varphi_{tt} = \Gamma_\varphi(\varphi_t,\varphi_t),\\ \varphi_t(0) = u_0,\\ \varphi(0) = \mathrm{id}\end{cases} \tag{8}$$

where

$$\Gamma_\varphi(U,V) = \Gamma(U\circ\varphi^{-1}, V\circ\varphi^{-1})\circ\varphi,$$

for $U,V\in T_\varphi H^s\mathrm{Diff}(\mathbb{S})\simeq H^s$, and

$$\Gamma(u,v) = -\frac{1}{2}M_{\omega^{-1}}[(M_\omega u)_x v + (M_\omega v)_x u + 2u_x M_\omega v + 2v_x M_\omega u],$$

for $u,v \in H^s$. Eq. (8) is the geodesic equation associated with Eq. (7) for the right-invariant weighted L_2 metric

$$\langle m,u\rangle = \langle M_\omega u,u\rangle = \int_{\mathbb{S}} (M_\omega u)(x)u(x)\,\mathrm{d}x = \int_{\mathbb{S}} u(x)^2\omega(x)\,\mathrm{d}x \tag{9}$$

on the diffeomorphism group in terms of the *Lagrangian* coordinate φ; observe that the corresponding equation (6) (or (7), equivalently) in terms of the *Euclidean* variable u is the geodesic equation on the tangent plane at the identity of the diffeomorphism group. For this purpose, we call Γ the *Christoffel map* for Eq. (7), since it generalizes the Christoffel symbols from finite-dimensional Riemannian geometry.

It is easy to see that

$$\Gamma_\varphi(U,V) = -\frac{3}{2\varphi_x}(U_xV+V_xU) - \left[\left(\frac{\omega_x}{\omega}\right)\circ\varphi\right]UV.$$

Letting $\omega(x) = e^{-\alpha x}$, we see that the second term on the right hand side equals αUV. Since H^s is a Banach algebra for any $s > 1/2$, the map $\varphi \mapsto \Gamma_\varphi(U,V)$, $H^s\mathrm{Diff}(\mathbb{S}) \to H^{s-1}$, is smooth for any $U,V \in H^s$. By the Picard-Lindelöf Theorem, the problem (8) possesses a unique short-time solution $\varphi(t;u_0) \in H^s\mathrm{Diff}(\mathbb{S})$, for t in some open interval $(-T,T)$ containing zero, depending smoothly on $(t,u_0) \in (-T,T)\times H^s$. Since $H^s\mathrm{Diff}(\mathbb{S})$ is a topological group, we find that the function $u(t;u_0) := \varphi_t\circ\varphi^{-1} = DR_{\varphi^{-1}}\varphi_t$ depends continuously on time and on the initial value u_0. Furthermore, u solves the initial value problem

$$u_t + 3uu_x = \alpha u^2, \quad u(0) = u_0,$$

and this is Eq. (3). This proves the first part of Theorem 1 and Corollary 2.

Let $H^s\mathrm{Diff}(\mathbb{S})\times\mathbb{R}$ denote the Bott-Virasoro group associated with $H^s\mathrm{Diff}(\mathbb{S})$ and define the weighted inner product

$$\langle (u,a),(v,b)\rangle = \int_{\mathbb{S}} u(x)v(x)\omega(x)\,\mathrm{d}x + ab\int_{\mathbb{S}}\omega(x)\,\mathrm{d}x$$

on the Virasoro algebra $\mathfrak{vir} = H^s\oplus\mathbb{R}$. We conclude from

$$\begin{aligned}
\left\langle \mathrm{ad}_{(u,a)}(v,b),(w,c)\right\rangle &= \langle [(u,a),(v,b)],(w,c)\rangle \\
&= \left\langle \left(u_xv - v_xu, \int_{\mathbb{S}} u_xv_{xx}\,\mathrm{d}x\right),(w,c)\right\rangle \\
&= \int_{\mathbb{S}} [(u_xvw - v_xuw)\omega + c\tilde{\omega}u_xv_{xx}]\,\mathrm{d}x \\
&= \int_{\mathbb{S}} [(2u_xw + uw_x)\omega + c\tilde{\omega}u_{xxx} + uw\omega_x]v\,\mathrm{d}x \\
&= \left\langle (v,b), \mathrm{ad}^*_{(u,a)}(w,c)\right\rangle
\end{aligned}$$

where $\tilde{\omega} = \int_{\mathbb{S}}\omega(x)\,\mathrm{d}x$, that

$$\mathrm{ad}^*_{(u,a)}(w,c) = \left(2u_xw + uw_x + c\frac{\tilde{\omega}}{\omega}u_{xxx} + uw\frac{\omega_x}{\omega}, 0\right).$$

The geodesic equation

$$(u_t,a_t) = -\mathrm{ad}^*_{(u,a)}(u,a)$$

reads

$$\begin{cases} u_t &= -3uu_x - a\frac{\tilde{\omega}}{\omega}u_{xxx} - u^2\frac{\omega_x}{\omega}, \\ a_t &= 0. \end{cases}$$

The second of these equations shows that a is constant and the first reduces to Eq. (4) for the choice $w(x) = e^{-\alpha x}$ and with $a = \frac{\beta\alpha}{e^{-\alpha}-1}$. This proves the second part of Theorem 1.

Remark 4. Note that $\omega_\alpha(x) = e^{-\alpha(x-[x])}$ for $x \in \mathbb{R}$ so that ω_α is in fact a periodic function on the real line for any $\alpha \in \mathbb{R}$. Moreover, ω_α is a smooth function on $[0,1)$, i.e., on the standard representative system for $\mathbb{S}$.

4. Outlook

It is an open problem whether the idea to consider a weighted metric can be generalized, e.g., to the family

$$m_t = -m_x u - b u_x m, \quad m = Au, \tag{10}$$

where b is a real number and A is an invertible linear operator. Observe that the family (10) reduces to (7) for $b = 2$ and $A = M_\omega$. It is also well-known that the famous Camassa-Holm (CH) equation [3, 13]

$$u_t + 3uu_x = 2u_x u_{xx} + uu_{xxx} + u_{txx} \tag{11}$$

and the Hunter-Saxton (HS) equation [6]

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0 \tag{12}$$

emerge from Eq. (10) for the choice $b = 2$ and $A_k = k - \partial_x^2$, $k \in \{0,1\}$. As shown in [2, 10], Eqs. (11) and (12) re-express geodesic motion on the diffeomorphism group with respect to the right invariant H_1 metric and the $\dot{H}_1$ metric respectively; these metrics are defined at the identity element by

$$\langle u, v\rangle_k = \int_{\mathbb{S}} u(x)(A_k v)(x)\,\mathrm{d}x, \quad k \in \{0,1\}.$$

Following the approach presented in the main body of the paper, we define, for $\omega > 0$, the isomorphisms $A_{k,\omega} = k\omega - \omega_x\partial_x - \omega\partial_x^2$, $k \in \{0,1\}$. Then the associated geodesic equation reads

$$ku_t - u_{txx} = -3kuu_x + uu_{xxx} + 2u_x u_{xx} + \frac{\omega_x}{\omega}(2uu_{xx} + 2u_x^2 - ku^2 + u_{tx}) + \frac{\omega_{xx}}{\omega}uu_x. \tag{13}$$

On the Bott-Virasoro group equipped with the right-invariant metrics

$$\langle (u,a),(v,b)\rangle_k = \int_{\mathbb{S}} u(x)(A_{k,\omega}v)(x)\,\mathrm{d}x + kab\int_S \omega(x)\,\mathrm{d}x$$

we obtain Eq. (13) with an additional term $ck\frac{\tilde{\omega}}{\omega}u_{xxx}$, $c \in \mathbb{R}$. Choosing $\omega = e^{-\alpha x}$, we obtain from (13) the identity

$$ku_t - u_{txx} = -3kuu_x + uu_{xxx} + 2u_x u_{xx} - \alpha(2uu_{xx} + 2u_x^2 - ku^2 + u_{tx}) + \alpha^2 uu_x. \tag{14}$$

Clearly, (14) reduces to (11) and (12) for $\alpha = 0$. It is an open question to determine the physical meaning of the terms for $\alpha \neq 0$ in the modified equation (14) since it has not appeared in the literature up to now.

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