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Some Norm Estimates for Semimartingales
— Under Linear and Nonlinear Expectations

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Abstract

In this paper we introduce two types of norms for semimartingales, under both linear and nonlinear expectations. The first norm is motivated by quasimartingales, and characterizes square integrable semimartingales. The second norm characterizes the absolute continuity of the finite variation part of the semimartingale with respect to the Lebesgue measure. One typical example of nonlinear expectation is the G-expectation introduced by Peng [17]. By applying our estimates, we prove a Doob-Meyer type decomposition for G-submartingales, and obtain the component Γ in the G-martingale representation theorem, which improves the result of Soner-Touzi-Zhang [23].

Key words: Martingale, semimartingale, quasimartingale, G-expectation, G-martingale, G-semimartingale, Doob-Meyer decompostition, martingale representation, Backward SDEs, second order BSDEs.
AMS 2000 subject classifications: 60H10, 60H30.

1 Introduction

In recent years, the notions of G-expectation and Second Order Backward SDEs (2BSDEs, for short), proposed by Peng [16, 17, 18, 19, 20] and Soner-Touzi-Zhang [23, 24, 25, 26], respectively, have received strong attention in the literature, see, e.g. [2], [4], [9], [10], [14], [23], [24], [27], [28], [29], to mention a few. These two closely related notions have applications in many fields, notably providing a convenient tool for financial models with volatility uncertainty, see e.g. [1], [5], [7], [15]. In Markovian case, a 2BSDE provides a Feynman-Kac type representation for second order fully nonlinear PDEs, and thus opens

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the door to Monte Carlo methods for fully nonlinear PDEs. We refer to [3] for an earlier formulation of 2BSDEs and [8] for the corresponding Monte Carlo methods.

Roughly speaking, a G-expectation is a nonlinear expectation taking the following form: $\mathbb{E}^G := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}$, where $\mathcal{P}$ is a family of mutually singular probability measures $\mathbb{P}$ and in general the family $\mathcal{P}$ does not have a dominating probability measure. For a random variable ξ, the conditional G-expectation $\mathbb{E}^G_t[\xi]$ is a G-martingale. Soner-Touzi-Zhang [23] established the following G-martingale representation theorem: denoting $Y_t := \mathbb{E}^G_t[\xi]$,

$$Y_t = Y_0 + \int_0^t Z_s dB_s - K_t, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P}\in\mathcal{P}, \tag{1.1}$$

where B is the canonical process which is a martingale under all $\mathbb{P}\in\mathcal{P}$, and K is a nondcreasing process with $K_0 = 0$. In particular, a G-martingale is a supermartingale under each $\mathbb{P}\in\mathcal{P}$.

It is clear that a G-supermartingale is also a supermartingale under each $\mathbb{P}\in\mathcal{P}$. One natural and fundamental question is:

What is the structure of a G-submartingale?

Our first goal of this paper is to answer the above question. Given a G-submartingale Y, one may expect that $Y = M + L$, where M is a G-martingale and L is a nondecreasing process. Then by (1.1) one expects that

$$Y_t = Y_0 + \int_0^t Z_s dB_s + A_t, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P}\in\mathcal{P}, \tag{1.2}$$

where $A := L - K$ is a a semimartingale under each $\mathbb{P}\in\mathcal{P}$.

While the above analysis is intuitively clear, its rigorous proof is by no means easy, because it involves a priori estimates for total variations of A under each $\mathbb{P}\in\mathcal{P}$. We thus first turn our attention to norm estimates for semimartingales under a fixed probability measure $\mathbb{P}$. In the standard literature, the norm of a semimartingale is defined through its decomposition, see e.g. [22]. However, for our purpose it is important to have a norm defined through the semimartingale itself, without involving its decomposition. We shall introduce a norm $\|\cdot\|_{\mathbb{P}}$, see (2.11) below, such that a process Y is a square integrable semimartingale under $\mathbb{P}$ if and only if $\|Y\|_{\mathbb{P}} < \infty$. We remark that the norm $\|\cdot\|_{\mathbb{P}}$ is motivated from the definition of quasimartingales, and these estimates are interesting in their own rights.

Now in the G-framework, define $\|\cdot\|_{\mathcal{P}} := \sup_{\mathbb{P}\in\mathcal{P}} \|\cdot\|_{\mathbb{P}}$, we show that a process Y is a square integrable G-semimartingale if and only if $\|Y\|_{\mathcal{P}} < \infty$, and we obtain the desired estimates. As a special case, we prove the Doob-Meyer type decomposition (1.2) for G-submartingales. However, it still remains open whether or not we can write $A = L - K$ such that $M_t := \int_0^t Z_s dB_s - K_t$ is indeed a G-martingale.

As a by product of our estimates, we also obtain some a priori estimates for Doubly Reflected BSDEs without assuming the standard Mokobodski's condition directly, see our accompanying paper [21].

The second main object of this paper is to obtain the component Γ in the following G-martingale representation, an improved version of (1.1):

$$Y_t = Y_0 + \int_0^t Z_s dB_s - \int_0^t [2G(\Gamma_t)dt - \Gamma_t d\langle B\rangle_t], \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}. \tag{1.3}$$

Here G is a function used by Peng [17] to define G-expectation, and $\langle\cdot\rangle$ is the quadratic variation. This is an open problem proposed by Peng, and remained open in [23] as well as in [26] for solutions to 2BSDEs. In the Markovian case, the component Γ corresponds to the second order derivative of the solution to the associated PDE. In fact, Γ is part of the solution to the earlier formulation of 2BSDEs in [3], and plays a very important role in numerical methods for fully nonlinear PDEs in [8].

Clearly, the problem is more or less equivalent to when the increasing process K in (1.1) is absolutely continuous with respect to the Lebesgue measure dt. Again, we first study the problem under a fixed probability measure $\mathbb{P}$. For any $1 < p \le \infty$, we shall define a new norm $\|\cdot\|_{\mathbb{P},p}$, see (4.2) below. For any semimartingale Y under $\mathbb{P}$, if $\|Y\|_{\mathbb{P},p} < \infty$, then the finite variation part of Y can be written as $dA_t = a_t dt$ and $\mathbb{E}^{\mathbb{P}}[\int_0^T |a_t|^p dt] < \infty$. We then define $\|\cdot\|_{\mathcal{P},p} := \sup_{\mathbb{P}\in\mathcal{P}} \|\cdot\|_{\mathbb{P},p}$. For any G-semimartingale Y, if $\|Y\|_{\mathcal{P},p} < \infty$, then similarly we have $dA_t = a_t dt$ such that $\mathbb{E}^G[\int_0^T |a_t|^p dt] < \infty$. Finally, for a random variable ξ, if $\|\xi\|_{\mathcal{P},p} := \|\mathbb{E}^G_\cdot[\xi]\|_{\mathcal{P},p} < \infty$, we obtain the following decomposition in backward form:

$$\mathbb{E}^G_t[\xi] = \xi - \int_t^T Z_s dB_s + \int_t^T [G(\Gamma_s)ds - \Gamma_s d\langle B\rangle_s]. \tag{1.4}$$

However, the above analysis does not yield the uniqueness of Γ, not to mention the norm estimates for Γ. We thus introduce a much stronger metric for ξ, which will lead to the existence, uniqueness, as well as a priori norm estimates of Γ. We shall point out though, this metric is very strong and is in general not convenient to use. We hope to explore further properties of Γ in our future research.

The rest of the paper will be organized as follows. In next section we introduce the first new norm for semimartingales under a fixed probability measure and obtain the estimates. In Section 3 we introduce a variant of the G-framework proposed by Peng [17] and obtain the Doob-Meyer type decomposition for G-semimartingales and G-submartingales. In Section 4 we introduce the second new norm, under both linear and nonlinear expectations, and then in Section 5 we prove the new G-martingale representation theorem with the existence of the component Γ. Finally in Appendix we provide some additional results.

2 A Priori Estimates for Semimartingales

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space on a fixed finite time horizon $[0,T]$ such that $\mathbb{F}$ is right continuous. We note that the filtration $\mathbb{F}$ is not necessarily complete under $\mathbb{P}$. The removal of the completeness requirement will be important in next sections. However, the following simple lemma, see e.g. [23], shows that we may assume all the processes involved in this section are $\mathbb{F}$-progressively measurable.

Lemma 2.1 *Let $\bar{\mathbb{F}}^{\mathbb{P}}$ denote the augmented filtration of $\mathbb{F}$ under $\mathbb{P}$. For any $\bar{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process X, there exists a unique ($dt \times d\mathbb{P}$-a.s.) process $\tilde{X}$ such that $\tilde{X} = X$, $dt \times d\mathbb{P}$-a.s. Moreover, if X is càdlàg, $\mathbb{P}$-a.s., then so is $\tilde{X}$.*

We recall that an $\mathbb{F}$-progressively measurable càdlàg semimartingale Y has the following decomposition:

$$Y_t = Y_0 + M_t + A_t, \tag{2.1}$$

where M is a martingale, A has finite variation, and $M_0 = A_0 = 0$. Now given an $\mathbb{F}$-progressively measurable and càdlàg process Y, We are interested in the following questions:

(i) Is Y a semimartingale?

(ii) Do we have appropriate norm estimates for Y, M, and A?

(iii) When is dA_t absolutely continuous with respect to the Lebesgue measure dt?

The first question was answered by Bichteler-Dellacherie, see e.g. [22] and Appendix of this paper for some further discussion. The main goal of this section is to answer the second question, and the third question will be answered in Section 4 below. As explained in Introduction, the latter questions are natural and important for our study of semimartingales under nonlinear expectations.

In this section we will always assume:

$$\text{The augmented filtration } \bar{\mathbb{F}}^{\mathbb{P}} \text{ is a Brownian filtration.} \tag{2.2}$$

Consequently,

$$\text{any } \mathbb{F}\text{-martingale } M \text{ is continuous, } \mathbb{P}\text{-a.s.} \tag{2.3}$$

2.1 Some preliminary results

We first note that, when Y is a supermartingale or submartingale, it is well known that Y is a semimartingale and the following norm estimates hold. Since the arguments will be important for our general case, we provide the proof for completeness.

Lemma 2.2 *Let* (2.2) *hold. There exist universal constants* $0 < c < C$ *such that, for any* Y *in the form of* (2.1) *with monotone* A*, it holds*

$$c\|Y\|^2_{\mathbb{P},0} \le \mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + |A_T|^2\Big] \le C\|Y\|^2_{\mathbb{P},0}. \tag{2.4}$$

where, for any càdlàg process Y*,*

$$\|Y\|^2_{\mathbb{P},0} := \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|^2\Big]. \tag{2.5}$$

Proof. The first inequality is obvious. We shall only prove the second inequality. By otherwise using the standard stopping techniques, we may assume without loss of generality that

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|^2 + \langle M\rangle_T + |A_T|^2\Big] < \infty.$$

Apply Itô's formula, we have

$$Y_T^2 = Y_0^2 + \langle M\rangle_T + 2\int_0^T Y_t dM_t + 2\int_0^T Y_{t-}dA_t + \sum_{0\le t\le T}|\Delta Y_t|^2. \tag{2.6}$$

Note that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |Y_t|^2 d\langle M\rangle_t\Big)^{\frac{1}{2}}\Big] &\le \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|(\langle M\rangle_T)^{\frac{1}{2}}\Big]\\ &\le \frac{1}{2}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|^2 + \langle M\rangle_T\Big] < \infty.\end{aligned}$$

Then

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T Y_t dM_t\Big] = 0.$$

Thus, for any $\varepsilon > 0$, by (2.6) and the monotonicity of A we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T] \le \mathbb{E}^{\mathbb{P}}\Big[\langle M\rangle_T + \sum_{0\le t\le T}|\Delta Y_t|^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[Y_T^2 - Y_0^2 - 2\int_0^T Y_{t-}dA_t\Big]\\ \le\ &\mathbb{E}^{\mathbb{P}}\Big[|Y_T|^2 + |Y_0|^2 + 2\sup_{0\le t\le T}|Y_t||A_T|\Big] \le C\varepsilon^{-1}\|Y\|^2_{\mathbb{P},0} + \varepsilon\mathbb{E}^{\mathbb{P}}[|A_T|^2].\end{aligned} \tag{2.7}$$

Moreover, note that

$$A_T = Y_T - Y_0 - M_T.$$

Clearly we have

$$\mathbb{E}^{\mathbb{P}}[|A_T|^2] \le C\|Y\|^2_{\mathbb{P},0} + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T] \le C\varepsilon^{-1}\|Y\|^2_{\mathbb{P},0} + C\varepsilon\mathbb{E}^{\mathbb{P}}[|A_T|^2].$$

Set $\varepsilon := \frac{1}{2C}$ for the above C, we obtain

$$\mathbb{E}^{\mathbb{P}}[|A_T|^2] \le C\|Y\|^2_{\mathbb{P},0}.$$

This, together with (2.7), proves the second inequality. ∎

The next lemma is a discrete version of Lemma 2.2.

Lemma 2.3 *Let $0 = \tau_0 \le \cdots \le \tau_n = T$ be a sequence of stopping times. In the setting of Lemma 2.2, if $A_{\tau_i} \in \mathcal{F}_{\tau_{i-1}}$, then*

$$c\mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n}|Y_{\tau_i}|^2\Big] \le \mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + |A_T|^2\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n}|Y_{\tau_i}|^2\Big]. \tag{2.8}$$

Proof. Again we prove only the second inequality. Similar to the proof of Lemma 2.2, by otherwise using the standard stopping techniques, we may assume without loss of generality that

$$\mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n}|Y_{\tau_i}|^2 + \langle M\rangle_T + |A_T|^2\Big] < \infty.$$

Note that

$$Y_{\tau_{i+1}} = Y_{\tau_i} + A_{\tau_{j+1}} - A_{\tau_j} + M_{\tau_{j+1}} - M_{\tau_j}.$$

Then

$$\begin{aligned} Y^2_{\tau_{i+1}} - Y^2_{\tau_i} &= 2Y_{\tau_i}[A_{\tau_{j+1}} - A_{\tau_j}] + [A_{\tau_{j+1}} - A_{\tau_j}]^2 \\ &+ 2[Y_{\tau_i} + A_{\tau_{j+1}} - A_{\tau_j}][M_{\tau_{j+1}} - M_{\tau_j}] + [M_{\tau_{j+1}} - M_{\tau_j}]^2. \end{aligned}$$

Notice that $Y_{\tau_i} + A_{\tau_{j+1}} - A_{\tau_j} \in \mathcal{F}_{\tau_i}$. One can easily obtain

$$\mathbb{E}^{\mathbb{P}}\Big[Y^2_{\tau_{i+1}} - Y^2_{\tau_i}\Big] = \mathbb{E}^{\mathbb{P}}\Big[2Y_{\tau_i}[A_{\tau_{j+1}} - A_{\tau_j}] + [A_{\tau_{j+1}} - A_{\tau_j}]^2 + [M_{\tau_{j+1}} - M_{\tau_j}]^2\Big].$$

Then, since A is monotone,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\langle M\rangle_T] &= \mathbb{E}^{\mathbb{P}}[M_T^2] = \sum_{i=0}^{n}\mathbb{E}^{\mathbb{P}}\Big[[M_{\tau_{j+1}} - M_{\tau_j}]^2\Big] \\ &\le \sum_{i=0}^{n}\mathbb{E}^{\mathbb{P}}\Big[Y^2_{\tau_{i+1}} - Y^2_{\tau_i} - 2Y_{\tau_i}[A_{\tau_{j+1}} - A_{\tau_j}]\Big] \le \mathbb{E}^{\mathbb{P}}\Big[Y_T^2 + 2\sup_{0\le i\le n}|Y_{\tau_i}||A_T|\Big] \\ &\le \mathbb{E}^{\mathbb{P}}\Big[C\varepsilon^{-1}\max_{0\le i\le n}|Y_{\tau_i}|^2 + \varepsilon|A_T|^2\Big], \end{aligned} \tag{2.9}$$

for any $\varepsilon > 0$. Moreover, since $A_T = Y_T - Y_0 - M_T$, we have

$$\mathbb{E}^{\mathbb{P}}[|A_T|^2] \le C\mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n}|Y_{\tau_i}|^2 + |M_T|^2\Big] \le \mathbb{E}\Big[C\varepsilon^{-1}\max_{0\le i\le n}|Y_{\tau_i}|^2 + C\varepsilon|A_T|^2\Big]$$

Choose $\varepsilon = \frac{1}{2C}$ for the above C, we have

$$\mathbb{E}^{\mathbb{P}}[|A_T|^2] \le C\mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n}|Y_{\tau_i}|^2\Big].$$

This, together with (2.9), implies the second inequality. ∎

2.2 Square integrable semimartingales

In this subsection we characterize the norm for square integrable semimartingales. For $0 \le t_1 < t_2 \le T$, let $\bigvee_{t_1}^{t_2} A$ denote the total variation of A over the interval $(t_1, t_2]$.

Definition 2.4 *We say a semimartingale Y in the form of* (2.1) *is a square integrable semimartingale if*

$$\mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + \Big(\bigvee_0^T A\Big)^2\Big] < \infty. \tag{2.10}$$

We remark that (2.10) is the norm used in standard literature for semimartingales, see e.g. [22]. Clearly, for a square integrable semimartingale Y, we have $\|Y\|_{\mathbb{P},0} < \infty$. However, when A is not monotone, in general the left side of (2.10) cannot be dominated by $C\|Y\|^2_{\mathbb{P},0}$. See Example 6.1 below.

Our goal is to characterize square integrable semimartingales via the process Y itself, without involving M and A directly. In many situations, we may have a representation formula for the process Y, but in general it is difficult to obtain representation formulas for M and A. So it is much easier to verify conditions imposed on Y than those on M and A. We introduce the following norm:

$$\|Y\|^2_{\mathbb{P}} := \|Y\|^2_{\mathbb{P},0} + \sup_\pi \mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\big|\Big)^2\Big], \tag{2.11}$$

where the supremum is over all partitions $\pi : 0 = \tau_0 \le \cdots \le \tau_n = T$ for some stopping times $\tau_0, \cdots, \tau_n$.

Remark 2.5 The norm $\|\cdot\|_{\mathbb{P}}$ is motivated from the definition of *quasimartingale*, see e.g. [12]: A càdlàg process Y is a called a quasimartingale if

$$\sup_\pi \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big|E^{\mathbb{P}}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\big|\Big] < \infty. \tag{2.12}$$

■

Remark 2.6 The main reason that we assume $\mathbb{F}$ is the Brownian filtration in (2.2) is to ensure the martingale part M is continuous, see (2.3). When $\mathbb{F}$ is a general right continuous filtration, our results still hold true if M is continuous. If M is discontinuous, we shall modify the norm $\|\cdot\|$ as:

$$\|Y\|^2_{\mathbb{P}} := \|Y\|^2_{\mathbb{P},0} + \sup_\pi \mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\big|\Big)^2\Big] + \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}\big|Y_t - Y_{t-}|^2\Big]. \tag{2.13}$$

■

The following a priori estimate is the main technical result of the paper.

Theorem 2.7 *There exist universal constants $0 < c < C$ such that, for any square integrable semimartingale $Y_t = Y_0 + M_t + A_t$,*

$$c\|Y\|_{\mathbb{P}}^2 \le \mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + \Big(\bigvee_0^T A\Big)^2\Big] \le C\|Y\|_{\mathbb{P}}^2. \tag{2.14}$$

Proof. We first prove the left inequality. Let $\pi : 0 = \tau_0 \le \cdots \le \tau_n = T$ be an arbitrary partition, and denote $\Delta A_{\tau_{i+1}} := A_{\tau_{i+1}} - A_{\tau_i}$. Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\big|\Big)^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau_i}(A_{\tau_{i+1}}) - A_{\tau_i}\big|\Big)^2\Big]\\
\le\ &\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|)\Big)^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|] + \sum_{i=0}^{n-1}|\Delta A_{\tau_{i+1}}|\Big)^2\Big]\\
\le\ &C\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|]\Big)^2\Big] + C\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T A\Big)^2\Big].
\end{aligned} \tag{2.15}$$

Note that

$$\sum_{i=0}^{j}[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|], j = 0, \cdots, n-1, \quad \text{is a martingale.}$$

Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|]\Big)^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|\big]^2\Big]\\
\le\ &C\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big[\big(\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|)\big)^2 + |\Delta A_{\tau_{i+1}}|^2\big]\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big[\mathbb{E}^{\mathbb{P}}_{\tau_i}(|\Delta A_{\tau_{i+1}}|^2) + |\Delta A_{\tau_{i+1}}|^2\big]\Big]\\
\le\ &C\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}|\Delta A_{\tau_{i+1}}|^2\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\big(\sum_{i=0}^{n-1}|\Delta A_{\tau_{i+1}}|\big)^2\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T A\Big)^2\Big].
\end{aligned}$$

This, together with (2.15) and the left inequality of (2.4), proves the left inequality of (2.14).

We now prove the right inequality. First, for any $\varepsilon > 0$, following the arguments in Lemma 2.2 one can easily show that

$$\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T] \le C\varepsilon^{-1}\|Y\|_{\mathbb{P},0}^2 + \varepsilon\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A\big)^2\Big]. \tag{2.16}$$

We claim that

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A\big)^2\Big] \le C\|Y\|_{\mathbb{P}}^2 + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T]. \tag{2.17}$$

This, together with (2.16) and by choosing ε small enough, implies the right inequality of (2.14) immediately.

We prove (2.17) in several steps.

Step1. Let $\pi : 0 = \tau_0 \le \tau_1 \le ... \le \tau_n = T$ be an arbitrary partition. Note that

$$\mathbb{E}^{\mathbb{P}}_{\tau_i}[Y_{\tau_{i+1}}] - Y_{\tau_i} = \mathbb{E}^{\mathbb{P}}_{\tau_i}[A_{\tau_{i+1}}] - A_{\tau_i}.$$

Then

$$\begin{aligned}\sum_{i=0}^{n-1}\big[A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}[A_{\tau_{i+1}}]\big] &= A_T - \sum_{i=0}^{n-1}\big(\mathbb{E}^{\mathbb{P}}_{\tau_i}[A_{\tau_{i+1}}] - A_{\tau_i}\big)\\ &= Y_T - Y_0 - M_T - \sum_{i=0}^{n-1}\big(\mathbb{E}^{\mathbb{P}}_{\tau_i}[Y_{\tau_{i+1}}] - Y_{\tau_i}\big).\end{aligned}$$

By the definition of $\|Y\|_{\mathbb{P}}$, we see that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\big[A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}[A_{\tau_{i+1}}]\big]\Big)^2\Big] \le C\|Y\|^2_{\mathbb{P}} + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T].$$

Note that

$$\sum_{i=0}^{j-1}\big[A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}[A_{\tau_{i+1}}]\big], \quad j = 1, \cdots, n, \quad \text{is a martingale.}$$

Then

$$\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big[A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}[A_{\tau_{i+1}}]\big]^2\Big] \le C\|Y\|^2_{\mathbb{P}} + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T]. \tag{2.18}$$

Step 2. In this step we assume $A_t = \int_0^t a_s dK_s$, where K is a continuous nondecreasing process and a is a simple process. That is,

$$a = \sum_{i=0}^{n-1} a_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \quad \text{for some} \quad 0 = t_0 < \cdots < t_n = T.$$

Then, denoting $\alpha_i :=\text{sign}(a_{t_i})$,

$$\begin{aligned}V(A) &= \int_0^T |a_t| dK_t = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \alpha_i a_t dK_t = \sum_{i=0}^{n-1}\alpha_i[A_{t_{i+1}} - A_{t_i}]\\ &= \sum_{i=0}^{n-1}\alpha_i\big(A_{t_{i+1}} - \mathbb{E}^{\mathbb{P}}_{t_i}[A_{t_{i+1}}]\big) + \sum_{i=0}^{n-1}\alpha_i\big(\mathbb{E}^{\mathbb{P}}_{t_i}[A_{t_{i+1}}] - A_{t_i}\big).\end{aligned}$$

Note that

$$\sum_{i=0}^{j}\alpha_i\big(A_{t_{i+1}} - \mathbb{E}^{\mathbb{P}}_{t_i}[A_{t_{i+1}}]\big), \quad j = 0, \cdots, n-1, \quad \text{is a martingale.}$$

Then

$$\mathbb{E}^{\mathbb{P}}\Big[\big(V(A)\big)^2\Big] \;\le\; C\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n-1}\big|A_{t_{i+1}}-\mathbb{E}^{\mathbb{P}}_{t_i}[A_{t_{i+1}}]\big|^2+\Big(\sum_{i=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{t_i}[A_{t_{i+1}}]-A_{t_i}\big|\Big)^2\Big].$$

By (2.18) and the definition of $\|Y\|_{\mathbb{P}}$ we obtain (2.17).

Step 3. We now prove (2.17) for general continuous A . Denote $K_t := \bigvee_0^t A$. Since A is continuous, K is also continuous. Moreover dA_t is absolutely continuous with respect to dK_t and thus $dA_t = a_t dK_t$ for some a. By [11] Chapter 3 Lemma 2.7, for every $\varepsilon > 0$ there exists a simple process $\{a^\varepsilon\}$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |a^\varepsilon_t - a_t| dK_t\Big)^2\Big] \le \varepsilon. \tag{2.19}$$

Denote

$$A^\varepsilon_t := \int_0^t a^\varepsilon_s dK_s, \quad Y^\varepsilon_t := Y_0 + M_t + A^\varepsilon_t.$$

Then by Step 2 we see that

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A^\varepsilon\big)^2\Big] \le C\|Y^\varepsilon\|^2_{\mathbb{P}} + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T]. \tag{2.20}$$

Note that

$$\bigvee_0^T A \le \bigvee_0^T A^\varepsilon + \bigvee_0^T [A^\varepsilon - A] \le \bigvee_0^T A^\varepsilon + \int_0^T |a^\varepsilon_t - a_t| dK_t.$$

Then

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A\big)^2\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A^\varepsilon\big)^2\Big] + C\varepsilon. \tag{2.21}$$

On the other hand, apply the left inequality of (2.14) on $Y^\varepsilon - Y = A^\varepsilon - A$, we get

$$\|Y^\varepsilon - Y\|^2_{\mathbb{P}} \le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T (A^\varepsilon - A)\Big)^2\Big] \le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |a^\varepsilon_t - a_t| dK_t\Big)^2\Big] \le C\varepsilon.$$

Then

$$\|Y^\varepsilon\|^2_{\mathbb{P}} \le C\|Y\|^2_{\mathbb{P}} + C\varepsilon.$$

Plug this and (2.21) into (2.20), we get

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A\big)^2\Big] \le C\|Y\|^2_{\mathbb{P}} + C\mathbb{E}^{\mathbb{P}}[\langle M\rangle_T] + C\varepsilon.$$

Since ε is arbitrary, we obtain (2.17).

Step 4. We now prove (2.17) for the general case. Since A is of bounded variation, we can decompose $A = A^c + A^d$, where A^c is the continuous part and A^d is the part with pure jumps. Since Y is càdlàg and M is continuous, A and A^d are càdlàg. We denote $Y^c_t = Y_0 + M_t + A^c_t$. From step 3 we have

$$\mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + \Big(\bigvee_0^T A^c\Big)^2\Big] \le C\|Y^c\|^2_{\mathbb{P}}.$$

Note that

$$\|Y^c\|_{\mathbb{P}} \le \|Y\|_{\mathbb{P}} + \|A^d\|_{\mathbb{P}}$$

and apply the left inequality of (2.14) on A^d we see that

$$\|A^d\|^2_{\mathbb{P}} \le C\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T A^d\Big)^2\Big].$$

Then

$$\mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + \Big(\bigvee_0^T A^c\Big)^2\Big] \le C\|Y\|^2_{\mathbb{P}} + C\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T A^d\Big)^2\Big],$$

and thus it suffices to show that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\bigvee_0^T A^d\Big)^2\Big] \le C\|Y\|^2_{\mathbb{P}}. \tag{2.22}$$

To this end, we first note that

$$\bigvee_0^T A^d = \sum_{0\le t\le T} |\Delta A_t| = \sum_{0\le t\le T} |\Delta Y_t|. \tag{2.23}$$

Define, for each n,

$$D_n := \sum_{0\le t\le T} |\Delta Y_t| \mathbf{1}_{\{|\Delta Y_t|\ge \frac{1}{n}\}},$$

and, $\tau^n_0 := 0$, and for $m \ge 0$, by denoting $Y_t := Y_T$ for $t \ge T$,

$$\tau^n_{m+1} := \inf\Big\{t > \tau^n_m : |\Delta Y_t| \ge \frac{1}{n}\Big\} \wedge (T+1).$$

We remark that we use $T+1$ instead of T here so that ΔY_T will not be counted repeatedly at below. Then it is clear that

$$D_n \uparrow \sum_{0\le t\le T} |\Delta Y_t| \;\text{ as }\; n\to\infty, \quad\text{and}\quad \sum_{i=1}^m |\Delta Y_{\tau^n_i}| \uparrow D_n \;\text{ as }\; m\to\infty.$$

Therefore, to obtain (2.22) it suffices to show that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=1}^{m}|\Delta Y_{\tau_i^n}|\Big)^2\Big] \le \|Y\|_{\mathbb{P}}^2 \quad \text{for all} \quad n, m. \tag{2.24}$$

We now fix n, m. Notice that the $\mathbb{F}$ is quasi-left continuous. Then for each τ_i^n, there exist $\{\tau_{i,j}^n, j \ge 1\}$ such that $\tau_{i,j}^n < \tau_i^n$ and $\tau_{i,j}^n \uparrow \tau_i^n$ as $j \to \infty$. By definition of $\|Y\|_{\mathbb{P}}$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=1}^{m}|\mathbb{E}^{\mathbb{P}}_{\tau_{i-1}^n \vee \tau_{i,j}^n}[Y_{\tau_i^n}] - Y_{\tau_{i-1}^n \vee \tau_{i,j}^n}|\Big)^2\Big] \le \|Y\|_{\mathbb{P}}^2. \tag{2.25}$$

Send $j \to \infty$, since $\mathbb{F}$ is continuous, we see that

$$\lim_{j\to\infty}[\mathbb{E}^{\mathbb{P}}_{\tau_{i-1}^n \vee \tau_{i,j}^n}[Y_{\tau_i^n}] - Y_{\tau_{i-1}^n \vee \tau_{i,j}^n}] = Y_{\tau_i^n} - Y_{\tau_i^n -} = \Delta Y_{\tau_i^n}.$$

Then by applying the Dominated Convergence Theorem we obtain (2.24) from (2.25). ■

Theorem 2.8 *Let Y be an $\mathbb{F}$-progressively measurable càdlàg process. Then Y is a square integrable semimartingale if and only if $\|Y\|_{\mathbb{P}} < \infty$.*

Proof. By Theorem 2.7, it suffices to prove the if part. Assume $\|Y\|_{\mathbb{P}} < \infty$. For each n, let $t_i^n := \frac{i}{n}T$, $i = 0, \cdots, n$. Denote, for $i = 0, \cdots, n$,

$$\begin{aligned}
M_{t_i^n}^n &:= \sum_{j=1}^{i}\big(Y_{t_j^n} - \mathbb{E}^{\mathbb{P}}_{t_{j-1}^n}[Y_{t_j^n}]\big),\\
K_{t_i^n}^{+,n} &:= \sum_{j=1}^{i}\big(\mathbb{E}^{\mathbb{P}}_{t_{j-1}^n}[Y_{t_j^n}] - Y_{t_{j-1}^n}\big)^+,\\
K_{t_i^n}^{-,n} &:= \sum_{j=1}^{i}\big(\mathbb{E}^{\mathbb{P}}_{t_{j-1}^n}[Y_{t_j^n}] - Y_{t_{j-1}^n}\big)^-.
\end{aligned}$$

Then M^n is a martingale, $K^{+,n}, K^{-,n}$ are nondecreasing, and

$$Y_{t_i^n} = Y_0 + M_{t_i^n}^n + A_{t_i^n}^n, \quad \text{where} \quad A_{t_i^n}^n := K_{t_i^n}^{+,n} - K_{t_i^n}^{-,n}.$$

Moreover,

$$\mathbb{E}^{\mathbb{P}}\Big[(K_T^{+,n})^2 + (K_T^{-,n})^2\Big] \le \|Y\|_{\mathbb{P}}^2 < \infty.$$

Then following the arguments for the standard Doob-Meyer decomposition, see e.g. [11], one can easily prove the result. ■

3 Semimartingales under G-expectation

In this section we introduce a nonlinear expectation, which is a variant of the G-expectation proposed by Peng [17]. Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space such that $\mathbb{F}$ is right continuous, $\mathcal{P}$ a family of probability measures. For each $\mathbb{P} \in \mathcal{P}$ and $\mathbb{F}$-stopping time τ, denote

$$\mathcal{P}(\tau, \mathbb{P}) := \big\{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \ \text{ on } \ \mathcal{F}_\tau\big\}. \tag{3.1}$$

We shall assume

Assumption 3.1 *(i) $\mathcal{N}_\mathcal{P} \subset \mathcal{F}_0$, where $\mathcal{N}_\mathcal{P}$ is the set of all $\mathcal{P}$-polar sets, that is, all $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ for all $\mathbb{P} \in \mathcal{P}$.*

(ii) For any $\mathbb{P} \in \mathcal{P}$, $\mathbb{F}$-stopping time τ, $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\tau, \mathbb{P})$, and any partition $E_1, E_2 \in \mathcal{F}_\tau$ of Ω , the probability measure $\bar{\mathbb{P}}$ defined below also belongs to $\mathcal{P}(\tau, \mathbb{P})$:

$$\bar{\mathbb{P}}(E) := \mathbb{P}_1(E \cap E_1) + \mathbb{P}_2(E \cap E_2), \quad \forall E \in \mathcal{F}. \tag{3.2}$$

3.1 Definitions

We first define

Definition 3.2 *We say an $\mathbb{F}$-progressively measurable process Y is a $\mathcal{P}$-martingale (resp. $\mathcal{P}$-supermartingale, $\mathcal{P}$-submartingale, $\mathcal{P}$-semimartingale) if it is a $\mathbb{P}$-martingale (resp. $\mathbb{P}$-supermartingale, $\mathbb{P}$-submartingale, $\mathbb{P}$-semimartingale) for all $\mathbb{P} \in \mathcal{P}$.*

We next define the conditional G-expectation. For any $\mathcal{F}$-measurable random variable ξ such that $\mathbb{E}^\mathbb{P}[|\xi|] < \infty$ for all $\mathbb{P} \in \mathcal{P}$, and any $\mathbb{F}$-stopping time τ, denote

$$\mathbb{E}^{G,\mathbb{P}}_\tau[\xi] := \underset{\mathbb{P}' \in \mathcal{P}(\tau,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_\tau[\xi], \quad \mathbb{P} - \text{a.s.} \tag{3.3}$$

We note that, by Lemma 2.1, $\mathbb{E}^{G,\mathbb{P}}_\tau[\xi]$ is $\mathcal{F}_\tau$-measurable. When the family $\{\mathbb{E}^{G,\mathbb{P}}_\tau[\xi], \mathbb{P} \in \mathcal{P}\}$ can be aggregated, that is, there exists an $\mathcal{F}_\tau$-measurable random variable, denoted as $\mathbb{E}^G_\tau[\xi]$, such that

$$\mathbb{E}^G_\tau[\xi] = \mathbb{E}^{G,\mathbb{P}}_\tau[\xi], \quad \mathbb{P} - \text{a.s. for all } \ \mathbb{P} \in \mathcal{P}, \tag{3.4}$$

we call $\mathbb{E}^G_\tau[\xi]$ the conditional G-expectation of ξ. Following standard arguments, we have the following Dynamic Programming Principle, whose proof is provided in the Appendix for completeness:

Lemma 3.3 *Under Assumption 3.1, for any $\tau_1 \le \tau_2$ and any $\mathbb{P} \in \mathcal{P}$, we have*

$$\mathbb{E}^{G,\mathbb{P}}_{\tau_1}[\xi] = \underset{\mathbb{P}' \in \mathcal{P}(\tau_1,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_{\tau_1}\big[\mathbb{E}^{G,\mathbb{P}'}_{\tau_2}[\xi]\big], \quad \mathbb{P} - a.s.$$

Definition 3.4 *We say an $\mathbb{F}$-progressively measurable process Y is a G-martingale (resp. G-supermartingale, G-submartingale) if, for any $\mathbb{P} \in \mathcal{P}$ and any $\mathbb{F}$-stopping times $\tau_1 \leq \tau_2$,*

$$Y_{\tau_1} = (\textit{resp. } \geq, \leq) \mathbb{E}^{G,\mathbb{P}}_{\tau_1}[Y_{\tau_2}], \quad \mathbb{P} - a.s.$$

We remark that a $\mathcal{P}$-martingale is also called a symmetric G-martingale in the literature, see e.g. [29].

3.2 Characterization of $\mathcal{P}$-semimartingales

The following result is immediate:

Proposition 3.5 *Let Assumption 3.1 hold.*

(i) A $\mathcal{P}$-martingale (resp. $\mathcal{P}$-supermartingale, $\mathcal{P}$-submartingale) must be a G-martingale (resp. G-supermartingale, G-submartingale).

(ii) If Y is a G-martingale (resp. G-supermartingale, G-submartingale) and M is a $\mathcal{P}$-martingale, then $Y + M$ is a G-martingale (resp. G-supermartingale, G-submartingale).

(iii) A G-supermartingale is a $\mathcal{P}$-supermartinagle. In particular, a G-martingale is a $\mathcal{P}$-supermartinagle.

Proof. (i) and (ii) are obvious. To prove (iii), let Y be a G-supermartingale. Then for any $\tau_1 \leq \tau_2$ and any $\mathbb{P} \in \mathcal{P}$,

$$Y_{\tau_1} \geq \mathbb{E}^{G,\mathbb{P}}_{\tau_1}[Y_{\tau_2}] \geq \mathbb{E}^{\mathbb{P}}_{\tau_1}[Y_{\tau_2}], \quad \mathbb{P}\text{-a.s.}$$

That is, Y is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$, and thus is a $\mathcal{P}$-supermartingale. ∎

We next study $\mathcal{P}$-semimartingales. In light of Theorem 2.8, we define a new norm:

$$\|Y\|_{\mathcal{P}} := \sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P}}. \tag{3.5}$$

The following result is a direct consequence of Theorems 2.7 and 2.8.

Theorem 3.6 *Assume Assumption 3.1 holds and* (2.2) *holds for all $\mathbb{P} \in \mathcal{P}$. If $\|Y\|_{\mathcal{P}} < \infty$, then Y is a $\mathcal{P}$-semimartingale. Moreover, for any $\mathbb{P} \in \mathcal{P}$ and for the decomposition*

$$Y_t = Y_0 + M^{\mathbb{P}}_t + A^{\mathbb{P}}_t, \quad \mathbb{P}\text{-}a.s. \tag{3.6}$$

we have

$$\mathbb{E}^{\mathbb{P}}\Big[\langle M^{\mathbb{P}}\rangle_T + \big(\bigvee_0^T A^{\mathbb{P}}\big)^2\Big] \leq C\|Y\|^2_{\mathcal{P}}.$$

The norm $\|\cdot\|_{\mathcal{P}}$ is defined through each $\mathbb{P}\in\mathcal{P}$. The following definition relies on the G-expectation directly:

$$\|Y\|_G^2 := \mathbb{E}^G\Big[\sup_{0\le t\le T}|Y_t|^2\Big] + \sup_\pi \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n-1}\Big|\mathbb{E}^{G,\mathbb{P}}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\Big|\Big)^2\Big]. \tag{3.7}$$

Remark 3.7 (i) If the involved conditional G-expectations exist, then we may simplify the definition of $\|Y\|_G$:

$$\|Y\|_G^2 := \mathbb{E}^G\Big[\sup_{0\le t\le T}|Y_t|^2\Big] + \sup_\pi \mathbb{E}^{G}\Big[\Big(\sum_{i=0}^{n-1}\Big|\mathbb{E}^{G}_{\tau_i}(Y_{\tau_{i+1}}) - Y_{\tau_i}\Big|\Big)^2\Big].$$

(ii) In general $\|\cdot\|_G$ does not satisfy the triangle inequality and thus is not a norm.

(iii) For G-submartingales Y^1, Y^2, the triangle inequality holds:

$$\|Y^1+Y^2\|_G \le \|Y^1\|_G + \|Y^2\|_G.$$

However, in general Y^1+Y^2 may not be a G-submartingale anymore. ■

Nevertheless, $\|Y\|_G$ involves the process Y only, and we have the following estimate.

Theorem 3.8 *Assume Assumption 3.1 holds and* (2.2) *holds for all $\mathbb{P}\in\mathcal{P}$. Then there exists a universal constant C such that $\|Y\|_{\mathcal{P}} \le C\|Y\|_G$.*

Proof. Without loss of generality, we assume $\|Y\|_G < \infty$. For any $\mathbb{P}\in\mathcal{P}$ and any partition $\pi: 0=\tau_0\le\cdots\le\tau_n = T$, denote

$$N_{\tau_i} := \sum_{j=0}^{i-1}\Big[\mathbb{E}^{G,\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}}) - Y_{\tau_j}\Big].$$

Then

$$\begin{aligned} Y_{\tau_i} - N_{\tau_i} &= Y_0 + \sum_{j=0}^{i-1}\Big[Y_{\tau_{j+1}} - \mathbb{E}^{G,\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\Big] \\ &= Y_0 + \sum_{j=0}^{i-1}\Big[Y_{\tau_{j+1}} - \mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\Big] - \sum_{j=0}^{i-1}\Big[\mathbb{E}^{G,\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}}) - \mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\Big]. \end{aligned}$$

Note that

$$\sum_{j=0}^{i-1}\Big[Y_{\tau_{j+1}} - \mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\Big] \quad \text{is a } \mathbb{P}\text{-martingale,}$$

$$\sum_{j=0}^{i-1}\Big[\mathbb{E}^{G,\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}}) - \mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\Big] \quad \text{is nondecreasing and is } \mathcal{F}_{\tau_{i-1}}\text{-measurable.}$$

Applying Lemma 2.3 we obtain

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{j=0}^{n-1}\big[\mathbb{E}^{G,\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})-\mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})\big]\Big)^2\Big]\le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le i\le n}[|Y_{\tau_i}|^2+|N_{\tau_i}|^2]\Big]\le C\|Y\|_G^2.$$

This, together with the definition of $\|\cdot\|_G$, implies that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{j=0}^{n-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau_j}(Y_{\tau_{j+1}})-Y_{\tau_j}\big|\Big)^2\Big]\le C\|Y\|_G^2.$$

Since π is arbitrary, we get $\|Y\|_{\mathbb{P}}\le C\|Y\|_G$. Finally, since $\mathbb{P}\in\mathcal{P}$ is arbitrary, we prove the result. ∎

3.3 Doob-Meyer Decomposition for G-submartingales

As a special case of Theorem 3.6, we have the following decomposition for G submartingales.

Proposition 3.9 *Assume Assumption 3.1 holds and* (2.2) *holds for all $\mathbb{P}\in\mathcal{P}$. If Y is a G-submartingale satisfying $\|Y\|_{\mathcal{P}}<\infty$, then all the results in Theorem 3.6 hold.*

Remark 3.10 Unlike Lemma 2.2, for G-submartingales in general we do not have $\|Y\|_{\mathcal{P}}\le C\sup_{\mathbb{P}\in\mathcal{P}}\|Y\|_{\mathbb{P},0}$. See Example 6.2 below. ∎

Now let Y be as in Proposition 3.9, and consider its decomposition (3.6). Let $A^{\mathbb{P}}=L^{\mathbb{P}}-K^{\mathbb{P}}$ be the orthogonal decomposition. We have the following conjecture:

Conjecture: *The family $\{K^{\mathbb{P}},\mathbb{P}\in\mathcal{P}\}$ satisfies the following property:*

$$-K^{\mathbb{P}}_t=\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}_t\Big[-K^{\mathbb{P}'}_T\Big].\tag{3.8}$$

In particular, if the families $\{M^{\mathbb{P}},K^{\mathbb{P}},L^{\mathbb{P}},\mathbb{P}\in\mathcal{P}\}$ can be aggregated into $\{M,K,L\}$ (e.g. if $\mathcal{P}$ is separable, in the sense of [24]), then $-K$ is a G-martingale, and we have the following desired Doob-Meyer decomposition for G-submartingales:

$$\begin{array}{c}Y_t=Y_0+[M_t-K_t]+L_t,\\ \text{where } M-K \text{ is a } G\text{-martingale and } L \text{ is nondecreasing.}\end{array}\tag{3.9}$$

∎

4 Absolute continuity of the finite variational processes

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P})$ be as in Section 3, where $\mathbb{F}$ is right continuous and Assumption 3.1 holds. But we do not require (2.2) in this section. Let Y be a $\mathcal{P}$-semimartingale. In this section we investigate when its finite variation part is absolutely continuous with respect to the Lebesgue measure dt.

For this purpose, we let $\mathbb{L}^*$ denote the space of $\mathbb{F}$-progressively measurable processes η such that η is bounded and piecewise constant. For an $\mathbb{F}$-progressively measurable càdlàg process Y, define the Daniel integral as a linear operator on $\mathbb{L}^*$:

$$I_Y(\eta) := \sum_{i=0}^{n-1} \eta_{\tau_i}(Y_{\tau_{i+1}} - Y_{\tau_i}), \quad \text{for all} \quad \eta := \sum_{i=0}^{n-1} \eta_{\tau_i} \mathbf{1}_{[\tau_i, \tau_{i+1})} \in \mathbb{L}^* \tag{4.1}$$

4.1 The absolute continuity of $\mathbb{P}$-semimartingales

We first fix $\mathbb{P} \in \mathcal{P}$. For $1 \leq p \leq \infty$, let $\mathbb{L}^p_{\mathbb{P}}$ denote the space of $\mathbb{F}$-progressively measurable processes η such that $\|\eta\|^p_{\mathbb{L}^p_{\mathbb{P}}} := \mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\eta_t|^p dt\Big] < \infty$. Now for an $\mathbb{F}$-progressively measurable càdlàg process Y which is uniformly integrable under $\mathbb{P}$, define

$$\|Y\|_{\mathbb{P},p} := \sup\Big\{\frac{|\mathbb{E}^{\mathbb{P}}[I_Y(\eta)]|}{\|\eta\|_{\mathbb{L}^q_{\mathbb{P}}}} : 0 \neq \eta \in L^*\Big\}, \tag{4.2}$$

where $1 \leq q < \infty$ is the conjugate of p.

Theorem 4.1 *If $\|Y\|_{\mathbb{P},p} < \infty$, then $dY_t = dM_t + a_t dt$ where M is a martingale and $a \in L^p_{\mathbb{P}}$ with $\|a\|_{\mathbb{L}^p_{\mathbb{P}}} \leq \|Y\|_{\mathbb{P},p}$.*

Proof. Note that

$$|\mathbb{E}^{\mathbb{P}}[I_Y(\eta)]| \leq \|Y\|_{\mathbb{P},p}\|\eta\|_{\mathbb{L}^q_{\mathbb{P}}} \quad \text{for all} \quad \eta \in \mathbb{L}^*.$$

Since $\mathbb{L}^*$ is dense in $\mathbb{L}^q_{\mathbb{P}}$ under norm $\|\cdot\|_{\mathbb{L}^q_{\mathbb{P}}}$, we can extend I_Y to $\mathbb{L}^q_{\mathbb{P}}$ such that

$$|\mathbb{E}^{\mathbb{P}}[I_Y(\eta)]| \leq \|Y\|_{\mathbb{P},p}\|\eta\|_{\mathbb{L}^q_{\mathbb{P}}} \quad \text{for all} \quad \eta \in \mathbb{L}^q_{\mathbb{P}}.$$

By the Riesz's representation theorem, there is $a \in \mathbb{L}^p_{\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{P}}[I_Y(\eta)] = \mathbb{E}^{\mathbb{P}}\Big(\int_0^T \eta_t a_t dt\Big) \quad \text{for all} \quad \eta \in \mathbb{L}^q_{\mathbb{P}}, \quad \text{and} \quad \|a\|_{\mathbb{L}^p_{\mathbb{P}}} \leq \|Y\|_{\mathbb{P},p}.$$

Define

$$M_t := Y_t - Y_0 - \int_0^t a_s ds.$$

We see that, for any stopping times $\tau_1 \le \tau_2$ and any $\eta_{\tau_1} \in \mathcal{F}_{\tau_1}$, by denoting $\eta := \eta_{\tau_1}\mathbf{1}_{[\tau_1,\tau_2)} \in \mathbb{L}^*$,

$$\mathbb{E}^{\mathbb{P}}\Big[\eta_{\tau_1}[M_{\tau_2} - M_{\tau_1}]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\eta_{\tau_1}[Y_{\tau_2} - Y_{\tau_1}] - \int_{\tau_1}^{\tau_2} \eta_t a_t dt\Big] = \mathbb{E}^{\mathbb{P}}\Big[I_Y(\eta) - \int_0^T \eta_t a_t dt\Big] = 0.$$

This implies that M is a martingale. ■

Corollary 4.2 *Assume* (2.2) *holds. There exists a constant* C *such that, for any* $\mathbb{F}$-*progressively measurable uniformly integrable process* Y*,*

$$\|Y\|_{\mathbb{P}} \le C\Big[\|Y\|_{\mathbb{P},0} + \|Y\|_{\mathbb{P},2}\Big].$$

Proof. Without loss of generality we assume $\|Y\|_{\mathbb{P},0} + \|Y\|_{\mathbb{P},2} < \infty$. By Theorem 4.1 we have $dY_t = dM_t + a_t dt$ where M is a martingale and $a \in L^2_{\mathbb{P}}$. Denote $A_t := \int_0^t a_s ds$. Then

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T A\big)^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\big(\int_0^T |a_t| dt\big)^2\Big] \le T\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |a_t|^2 dt\Big] = T\|Y\|^2_{\mathbb{P},2}. \tag{4.3}$$

Note that

$$dY_t^2 = 2Y_t dM_t + 2Y_t a_t dt + d\langle M\rangle_t.$$

Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\langle M\rangle_T\Big] &= \mathbb{E}^{\mathbb{P}}\Big[|Y_T|^2 - |Y_0|^2 - 2\int_0^T Y_t a_t dt\Big] \\ &\le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |a_t|^2 dt\Big] \le C\Big[\|Y\|^2_{\mathbb{P},0} + \|Y\|^2_{\mathbb{P},2}\Big].\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) we obtain

$$\mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \langle M\rangle_T + \big(\bigvee_0^T A\big)^2\Big] \le C\Big[\|Y\|^2_{\mathbb{P},0} + \|Y\|^2_{\mathbb{P},2}\Big].$$

Then by applying Theorem 2.7 we prove the result. ■

4.2 Absolute continuity of $\mathcal{P}$-semimartingales

We now let Y be an $\mathbb{F}$-progressively measurable càdlàg process such that Y is uniformly integrable under $\mathbb{P}$ for all $\mathbb{P} \in \mathcal{P}$. For $1 < p \le \infty$, define

$$\|Y\|_{\mathcal{P},p} := \sup_{\mathbb{P}\in\mathcal{P}} \|Y\|_{\mathbb{P},p}. \tag{4.5}$$

The following result is a direct consequence of Theorem 4.1

Proposition 4.3 *Assume Assumption 3.1 holds. If $\|Y\|_{\mathcal{P},p} < \infty$ for some $1 < p \le \infty$, then Y is a $\mathcal{P}$-semimartingale with decomposition $dY_t = dM^{\mathbb{P}}_t + a^{\mathbb{P}}_t dt$, where $M^{\mathbb{P}}$ is a $\mathbb{P}$-martingale and*

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\int_0^T |a^{\mathbb{P}}_t|^p dt\Big] < \infty.$$

Moreover, if $\mathcal{P}$ is separable in the sense of [24], then $dY_t = dM_t + a_t dt$, where M is a $\mathcal{P}$-martingale and $\mathbb{E}^G\Big[\int_0^T |a_t|^p dt\Big] < \infty$.

5 G-martingale representation with component Γ

We now consider the framework in [23]. Let $\Omega := \big\{\omega \in C([0,T]) : \omega_0 = 0\big\}$, B the canonical process, and $0 \le \underline{\sigma} < \overline{\sigma}$ be two constants. Let $\mathcal{P}$ be the set of all probability measures $\mathbb{P}$ such that B is a $\mathbb{P}$-martingale, and there exists a constant $0 < \varepsilon_{\mathbb{P}} \le \overline{\sigma}^2$ such that $[\varepsilon_{\mathbb{P}} \vee \underline{\sigma}^2]dt \le d\langle B\rangle_t \le \overline{\sigma}^2 dt$. By [23], there exists a process $\hat{a}$ such that

$$d\langle B\rangle_t = \hat{a}_t dt, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}. \tag{5.1}$$

We use the filtration $\mathbb{F} = \{\mathcal{F}_t\}$:

$$\mathcal{F}_t := \mathcal{F}^B_{t+} \vee \mathcal{N}_{\mathcal{P}} \ \text{ where } \mathcal{N}_{\mathcal{P}} \text{ is as in Assumption 3.1 (i).} \tag{5.2}$$

Then one can easily see that Assumption 3.1 holds.

Peng [17] introduced the following function:

$$G(\gamma) := \frac{1}{2} \sup_{\underline{\sigma}\le\sigma\le\overline{\sigma}} \sigma^2\gamma = \frac{1}{2}\Big[\overline{\sigma}^2\gamma^+ - \underline{\sigma}^2\gamma^-\Big]. \tag{5.3}$$

It is known that, see e.g. [23], for $\xi = g(B_T)$ where g is a Lipschitz continuous function, we have $\mathbb{E}^G_t[\xi] = u(t, B_t)$ where u is the unique viscosity solution to the following PDE:

$$u_t + G(u_{xx}) = 0, \quad u(T,x) = g(x). \tag{5.4}$$

Let $\mathcal{L}_{ip}$ denote the space of all random variables $\varphi(B_{t_1}, \cdots, B_{t_n})$ where φ is a Lipschitz continuous function. For $\xi \in \mathcal{L}_{ip}$, define

$$\|\xi\|^2_G := \mathbb{E}^G\Big[\sup_{0\le t\le T} \big(\mathbb{E}^G_t[|\xi|]\big)^2\Big], \tag{5.5}$$

and let $\mathcal{L}_G$ be the closure of $\mathcal{L}_{ip}$ under the norm $\|\cdot\|_G$. By [23], for any $\xi \in \mathcal{L}_G$, the conditional G-expectation $\mathbb{E}^G_t[\xi]$ exists and is a continuous G-martingale. Moreover, we have the decomposition (1.1). Our goal of this section is to study the further decomposition (1.3), conjectured by Peng.

5.1 Existence of Γ

Theorem 5.1 *Let $\xi \in \mathcal{L}_G$. If $\|\mathbb{E}^G_\cdot[\xi]\|_{\mathcal{P},p} < \infty$ for some $1 < p \le \infty$, then we have the following decomposition:*

$$\begin{aligned}\mathbb{E}^G_t[\xi] &= \mathbb{E}^G[\xi] + \int_0^t Z_s dB_s - \int_0^t 2G(\Gamma_s)ds + \int_0^t \Gamma_s d\langle B\rangle_s \\ &= \mathbb{E}^G[\xi] + \int_0^t Z_s dB_s - \int_0^t [2G(\Gamma_s) - \Gamma_s \hat{a}_s]ds,\end{aligned} \tag{5.6}$$

where Z, Γ are $\mathbb{F}$-progressively measurable such that

$$\mathbb{E}^G\Big[\int_0^T Z_t^2 dt + \int_0^T k_t^p dt\Big] < \infty \quad \textit{where} \quad k := 2G(\Gamma) - \Gamma\hat{a} \ge 0. \tag{5.7}$$

Proof. For $\xi \in \mathcal{L}_G$, by [23] there exist Z and nondecreasing process K such that

$$\mathbb{E}^G_t[\xi] = \mathbb{E}^G[\xi] + \int_0^t Z_s dB_s - K_t \quad \text{and} \quad \mathbb{E}^G\Big[\int_0^T Z_t^2 dt + |K_T|^2\Big] < \infty$$

Since $\|\mathbb{E}^G_\cdot[\xi]\|_{\mathcal{P},p} < \infty$, by Proposition 4.3 we see that

$$dK_t = k_t dt \quad \text{and} \quad \mathbb{E}^G\Big[\int_0^T k_t^p dt\Big] < \infty.$$

Note that $\underline{\sigma}^2 \le \hat{a} \le \overline{\sigma}^2$, and the k in (5.7) is equivalent to

$$k = [\overline{\sigma}^2 - \hat{a}]\Gamma^+ - [\hat{a} - \underline{\sigma}^2]\Gamma^-. \tag{5.8}$$

Set

$$\Gamma := \begin{cases} -\frac{k}{\hat{a} - \underline{\sigma}^2}, & \text{on } \{\hat{a} = \overline{\sigma}^2\}; \\ \frac{k}{\overline{\sigma}^2 - \hat{a}}, & \text{on } \{\hat{a} = \underline{\sigma}^2\}; \\ \frac{k}{\overline{\sigma}^2 - \hat{a}} \text{ or } -\frac{k}{\hat{a} - \underline{\sigma}^2}, & \text{on } \{\underline{\sigma}^2 < \hat{a} < \overline{\sigma}^2\}. \end{cases} \tag{5.9}$$

One can check straightforwardly that Γ satisfies all the requirements. ■

Remark 5.2 The above martingale representation theorem holds true without assuming B has martingale representation property under each $\mathbb{P} \in \mathcal{P}$. The main reason is that in this framework we may start from the PDE (5.4) and apply the Itô's formula. ■

Remark 5.3 Denote

$$\|\xi\|_{G,p} := \|\xi\|_G + \|\mathbb{E}^G_\cdot(|\xi|)\|_{\mathcal{P},p}. \tag{5.10}$$

We shall note that $\|\cdot\|_{G,p}$ does not satisfy the triangle inequality and thus is not a norm. We can define instead: for any $1 < p \le \infty$,

$$\rho_p(\xi_1,\xi_2) := \|\xi_1-\xi_2\|_G + \|\mathbb{E}^G_\cdot(\xi_1) - \mathbb{E}^G_\cdot(\xi_2)\|_{\mathcal{P},p}. \tag{5.11}$$

Then ρ_p defines a metric. Let $\mathcal{L}_{G,p}$ denote the closure of $\mathcal{L}_{ip}$ under ρ_p. Then clearly we have the decomposition (5.6) for all $\xi \in \mathcal{L}_{G,p}$. ■

Remark 5.4 Hu-Peng [10] considers the following metric: for some $\alpha \in (1,2)$,

$$\rho_0(\xi_1,\xi_2) := \|\xi_1-\xi_2\|_{\bar G} + \Big(\mathbb{E}^{\bar G}\Big[\sup_\pi \big|[K^1_{t_{i+1}} - K^1_{t_i}] - [K^2_{t_{i+1}} - K^2_{t_i}]\big|^\alpha\Big]\Big)^{\frac{1}{\alpha}}, \tag{5.12}$$

where $\bar G$ is a modification the G, and K^i is the increasing process in the (unique) decomposition of the G-martingale $\mathbb{E}^G_t[\xi_i]$. They also proved (5.6) when ξ is in the closure of $\mathcal{L}_{ip}$ under ρ. We note that the above metric depends on the process K, while our metric ρ_p involves only $\mathbb{E}^G_t[\xi]$. Moreover, in (5.12) the supremum over the partitions is inside the G-expectation, while in (5.11) which depends on (4.5) and (4.2), essentially the supremum over the partitions are outside of the expectations and thus is weaker. ■

5.2 Uniqueness of Γ

From (5.9), clearly Γ is not unique unless $k = 0$, that is, $\mathbb{E}^G_t[\xi]$ is a $\mathcal{P}$-martingale. Song [28] proved that there is at most one Γ in the space $\mathcal{M}^2_G$ as defined below.

Let $\mathcal{M}^0_G$ denote the space of $\mathbb{F}$-progressively measurable and piecewise constant processes η such that $\eta_t \in \mathcal{L}_{ip}$ for all t, and $\mathcal{M}^2_G$ be the closure of $\mathcal{M}^0_G$ under the norm:

$$\|\eta\|^2_{\mathcal{H}^2_G} := \mathbb{E}^G\Big[\int_0^T |\eta_t|^2 dt\Big]. \tag{5.13}$$

We next introduce another space of ξ for which we shall have existence of Γ in $\mathcal{M}^2_G$. For this purpose, we assume

$$\underline{\sigma} > 0. \tag{5.14}$$

For $\xi = \varphi(B_{t_1},\cdots,B_{t_n}) \in \mathcal{L}_{ip}$, by Peng [17] we know there exist $Z, \Gamma \in \mathcal{M}^2_G$ such that (5.6) holds. Now for $\xi_i \in \mathcal{L}_{ip}$ and for the corresponding $Z^i, \Gamma^i \in \mathcal{M}^2_G$, $i = 1,2$, we define:

$$\rho(\xi_1,\xi_2) := \|\xi_1-\xi_2\|_G + \|Z^1-Z^2\|_{\mathcal{H}^2_G} + \|\Gamma^1-\Gamma^2\|_{\mathcal{H}^2_G}. \tag{5.15}$$

Let

$$\mathcal{L} := \text{the closure of } \mathcal{L}_{ip} \text{ under the above metric } \rho. \tag{5.16}$$

We then have

Theorem 5.5 *Assume Assumption 3.1 and* (5.14) *hold. Then for any* $\xi \in \mathcal{L}$*, there exist unique* $Z, \Gamma \in \mathcal{M}_G^2$ *such that* (5.6) *holds.*

Proof. Let $\xi \in \mathcal{L}$ and $\xi_n \in \mathcal{L}_{ip}$ such that $\lim_{n\to\infty} \rho(\xi, \xi_n) = 0$. Let $(Z^n, \Gamma^n) \in \mathcal{M}_G^2 \times \mathcal{M}_G^2$ be corresponding to ξ_n. Then by definition of ρ we see that $\{(Z^n, \Gamma^n), n \geq 1\}$ are Cauchy sequence under the norm $\|\cdot\|_{\mathcal{H}_G^2}$. Thus there exist $(Z, \Gamma) \in \mathcal{M}_G^2 \times \mathcal{M}_G^2$ such that

$$\lim_{n\to\infty} \Big[\|Z^n - Z\|_{\mathcal{H}_G^2} + \|\Gamma^n - \Gamma\|_{\mathcal{H}_G^2}\Big] = 0.$$

Then it is straightforward to check that (Z, Γ) satisfy (5.6) for ξ. The uniqueness of Z and Γ follow from [23] and [28], respectively. ■

Remark 5.6 While the conclusion of Theorem 5.5 looks nice, the metric ρ is rather strong and consequently the space $\mathcal{L}$ could be small. It is not clear to us how large $\mathcal{L}$ is. Moreover, in (5.15), we use the same norm for Z and Γ. This is not desirable, because in the Markovian case Z and Γ correspond to the first and second derivatives of the PDE, respectively. Intuitively, the norm for Γ should be weaker than that for Z. We hope to explore further properties of Γ in our future research. ■

6 Appendix

We first provide an example such that $\|Y\|_{\mathbb{P},0} < \infty$ but $\|Y\|_{\mathbb{P}} = \infty$.

Example 6.1 *Fix* $\mathbb{P}$*. Let* K *be an* $\mathbb{F}$*-progressively measurable continuous increasing process such that* $K_0 = 0$ *and* $\mathbb{E}^{\mathbb{P}}[K_T^2] = \infty$*. Define the sequence of stopping times:* $\tau_0 := 0$ *and, for* $n \geq 1$, $\tau_n := \inf\{t \geq 0 : K_t = n\} \wedge T$*. Since* $K_T < \infty$, $\tau_n = T$ *for* n *large enough, a.s. We now define the process* Y_t *as follows:* $Y_0 := 0$*, and for* $n \geq 0$*,*

$$Y_t := \begin{cases} Y_{\tau_{2n}} - K_t + K_{\tau_{2n}}, & t \in (\tau_{2n}, \tau_{2n+1}]; \\ Y_{\tau_{2n+1}} + K_t - K_{\tau_{2n+1}}, & t \in (\tau_{2n+1}, \tau_{2n+2}]. \end{cases} \tag{6.1}$$

Then $\|Y\|_{\mathbb{P},0} < \infty$ *but* $\|Y\|_{\mathbb{P}} = \infty$*.*

Proof. It is easy to check that $-1 \leq Y_t \leq 0$ and $\bigvee_0^T Y = K_T$. Then $\|Y\|_{\mathbb{P},0} \leq 1$ and $\mathbb{E}^{\mathbb{P}}\Big[\big(\bigvee_0^T Y\big)^2\Big] = \infty$. By Theorem 2.8, we get $\|Y\|_{\mathbb{P}} = \infty$. ■

We next provide a G-submartingale such that $\sup_{\mathbb{P}\in\mathcal{P}} \|Y\|_{\mathbb{P},0} < \infty$, but $\|Y\|_{\mathcal{P}} = \infty$.

Example 6.2 *Fix* $\mathcal{P}$*. Let* K *be as in Example 6.1 such that* $-K$ *is a* G*-martingale and* $\mathbb{E}^G[K_T^2] = \infty$*, instead of* $\mathbb{E}^{\mathbb{P}}[K_T^2] = \infty$*. Then the process* Y *defined in Example 6.1 satisfies all the requirements.*

Proof. By the proof of Example 6.1, clearly $\sup_{\mathbb{P}\in\mathcal{P}} \|Y\|_{\mathbb{P},0} < \infty$, but $\|Y\|_{\mathcal{P}} = \infty$. Moreover, on $(\tau_{2n}, \tau_{2n+1}]$, $dY_t = -dK_t$ and thus is a G-martingale; and on $(\tau_{2n+1}, \tau_{2n+2}]$, $dY_t = dK_t$, then Y is increasing and thus is a G-submartingale. So Y is a G-submartingale on $[0, T]$. ■

We now prove Lemma 3.3.

Proof of Lemma 3.3. We have

$$\mathbb{E}^{G,\mathbb{P}}_{\tau_1}[\xi] \;=\; \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}(\tau_1,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_{\tau_1}[\xi] = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}(\tau_1,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_{\tau_1}[\mathbb{E}^{\mathbb{P}'}_{\tau_2}[\xi]] \le \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}(\tau_1,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_{\tau_1}[\mathbb{E}^{G,\mathbb{P}'}_{\tau_2}[\xi]].$$

To prove the other inequality, we fix $\mathbb{P}' \in \mathcal{P}(\tau, \mathbb{P})$. By [13], there exist a sequence $\mathbb{P}^n \in \mathcal{P}(\tau_2, \mathbb{P}')$ such that

$$\sup_{n\ge 1} \mathbb{E}^{\mathbb{P}^n}_{\tau_2}[\xi] = \operatorname*{ess\,sup^{\mathbb{P}'}}_{\tilde{\mathbb{P}}\in\mathcal{P}(\tau_2,\mathbb{P}')} \mathbb{E}^{\tilde{\mathbb{P}}}_{\tau_2}[\xi] = \mathbb{E}^{G,\mathbb{P}'}_{\tau_2}[\xi], \quad \mathbb{P}\text{-a.s.}.$$

We claim that there exists $\tilde{\mathbb{P}}^n \in \mathcal{P}(\tau_2, \mathbb{P}')$ such that

$$\mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_2}[\xi] \;\uparrow\; \sup_{n\ge 1} \mathbb{E}^{\mathbb{P}^n}_{\tau_2}[\xi], \quad \mathbb{P}'\text{-a.s. as } n \uparrow \infty. \tag{6.2}$$

Then, since $\tilde{\mathbb{P}}^n \in \mathcal{P}(\tau_2, \mathbb{P}') \subset \mathcal{P}(\tau_1, \mathbb{P})$

$$\mathbb{E}^{\mathbb{P}'}_{\tau_1}\Big[\mathbb{E}^{G,\mathbb{P}'}_{\tau_2}[\xi]\Big] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}'}_{\tau_1}\Big[\mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_2}[\xi]\Big] = \lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_1}\Big[\mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_2}[\xi]\Big] = \lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_1}[\xi] \le \mathbb{E}^{G,\mathbb{P}}_{\tau_1}[\xi], \quad \mathbb{P}\text{-a.s.}$$

Since $\mathbb{P}' \in \mathcal{P}(\tau_1, \mathbb{P})$ is arbitrary, we obtain the second inequality.

It remains to prove (6.2). We proceed by induction. Let $\tilde{\mathbb{P}}^1 := \mathbb{P}^1$. For $n = 2$, denote

$$E^+ := \Big\{\mathbb{E}^{\mathbb{P}^1}_{\tau_2}[\xi] \ge \mathbb{E}^{\mathbb{P}^2}_{\tau_2}[\xi]\Big\} \quad \text{and} \quad E^- := \Big\{\mathbb{E}^{\mathbb{P}^1}_{\tau_2}[\xi] < \mathbb{E}^{\mathbb{P}^2}_{\tau_2}[\xi]\Big\},$$

and define

$$\tilde{\mathbb{P}}^2(E) := \mathbb{P}^1(E \cap E^+) + \mathbb{P}^2(E \cap E^-), \quad \text{for all } E \in \mathcal{F}.$$

Then clearly $E^+, E^- \in \mathcal{F}_{\tau_2}$ and thus, by Assumption 3.1, $\tilde{\mathbb{P}}^2 \in \mathcal{P}(\tau_2, \mathbb{P}')$. One can easily check that

$$\mathbb{E}^{\tilde{\mathbb{P}}^2}_{\tau_2}[\xi] = \mathbb{E}^{\mathbb{P}^1}[\xi]\mathbf{1}_{E^+} + \mathbb{E}^{\mathbb{P}^2}[\xi]\mathbf{1}_{E^-} = \mathbb{E}^{\mathbb{P}^1}_{\tau_2}[\xi] \vee \mathbb{E}^{\mathbb{P}^2}_{\tau_2}[\xi].$$

By induction one can construct $\tilde{\mathbb{P}}^n \in \mathcal{P}(\tau_2, \mathbb{P}')$ such that

$$\mathbb{E}^{\tilde{\mathbb{P}}^n}_{\tau_2}[\xi] = \max_{1\le i\le n} \mathbb{E}^{\mathbb{P}^i}_{\tau_2}[\xi].$$

This implies (6.2) and completes the proof. ■

We conclude the paper by providing the connection with the Bichteler-Dellacherie's theorem. Fix $\mathbb{P}$ and recall (4.1). We call Y a *good integrator* if: for any $\{\eta^k\} \subset \mathbb{L}^*$,

$$\lim_{k\to\infty} \|\eta^k\|_{\mathbb{L}^\infty(\mathbb{P})} = 0 \ \text{ implies that } I_Y(\eta^k) \text{ converges to 0 in probability } \mathbb{P}. \tag{6.3}$$

Bichteler-Dellacherie's Theorem states that, see e.g. [22],

$$Y \text{ is a semimartingale if and only if } Y \text{ is a good indicator.} \tag{6.4}$$

Clearly, if $\|Y\|_{\mathbb{P}} < \infty$, by Theorem 2.8 and (6.4) we know that Y must be a good integrator. At below we provide a direct proof of this.

Proposition 6.3 *If $\|Y\|_{\mathbb{P}} < \infty$, then Y is a good integrator.*

Proof. Let $\eta^k = \sum_{i=0}^{n_k-1} \alpha_i^k \mathbf{1}_{[\tau_i^k, \tau_{i+1}^k)} \in \mathbb{L}^*$ such that $\lim_{k\to\infty} \|\eta^k\|_{\mathbb{L}^\infty(\mathbb{P})} = 0$. Denote $\Delta Y_{i+1}^k := Y_{\tau_{i+1}^k} - Y_{\tau_i^k}$. Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\big(I_Y(\eta^k)\big)^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\big(\sum_{i=0}^{n_k-1} \alpha_i^k \Delta Y_{i+1}^k\big)^2\Big] \\
\le\;& C\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n_k-1} \alpha_i^k \big[\Delta Y_{i+1}^k - \mathbb{E}^{\mathbb{P}}_{\tau_i^k}[\Delta Y_{i+1}^k]\big]\Big)^2\Big] + C\mathbb{E}^{\mathbb{P}}\Big[\big(\sum_{i=0}^{n_k-1} \alpha_i^k \mathbb{E}^{\mathbb{P}}_{\tau_i^k}[\Delta Y_{i+1}^k]\big)^2\Big].
\end{aligned}$$

Since

$$\sum_{i=0}^{j-1} \alpha_i^k \big[\Delta Y_{i+1}^k - \mathbb{E}^{\mathbb{P}}_{\tau_i^k}[\Delta Y_{i+1}^k]\big], \quad j = 1, \cdots, n_k, \quad \text{is a martingale,}$$

we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\big(I_Y(\eta^k)\big)^2\Big] \\
\le\;& C\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n_k-1} |\alpha_i^k|^2 \big[\Delta Y_{i+1}^k - \mathbb{E}^{\mathbb{P}}_{\tau_i^k}[\Delta Y_{i+1}^k]\big]^2\Big] + C\mathbb{E}^{\mathbb{P}}\Big[\Big(\sum_{i=0}^{n_k-1} \alpha_i^k \big[\mathbb{E}^{\mathbb{P}}_{\tau_i^k}[Y_{\tau_{i+1}^k}] - Y_{\tau_i^k}\big]\Big)^2\Big] \\
\le\;& C\|\eta^k\|^2_{\mathbb{L}^\infty(\mathbb{P})} \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n_k-1} \big[\Delta Y_{i+1}^k - \mathbb{E}^{\mathbb{P}}_{\tau_i^k}[\Delta Y_{i+1}^k]\big]^2\Big] + C\|\eta^k\|^2_{\mathbb{L}^\infty(\mathbb{P})} \|Y\|_{\mathbb{P}} \\
\le\;& C\|\eta^k\|^2_{\mathbb{L}^\infty(\mathbb{P})} \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n_k-1} |\Delta Y_{i+1}^k|^2\Big] + C\|\eta^k\|^2_{\mathbb{L}^\infty(\mathbb{P})} \|Y\|^2_{\mathbb{P}}.
\end{aligned} \tag{6.5}$$

Note that

$$|\Delta Y_{i+1}^k|^2 = |Y_{\tau_{i+1}^k}|^2 - |Y_{\tau_i^k}|^2 - 2Y_{\tau_i^k} \Delta Y_{i+1}^k.$$

Then

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n_k-1}|\Delta Y^k_{i+1}|^2\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=0}^{n_k-1}\Big(|Y_{\tau^k_{i+1}}|^2-|Y_{\tau^k_i}|^2-2Y_{\tau^k_i}\big[\mathbb{E}^{\mathbb{P}}_{\tau^k_i}[Y_{\tau^k_{i+1}}]-Y_{\tau^k_i}\big]\Big)\Big]\\
&= \mathbb{E}^{\mathbb{P}}\Big[|Y_T|^2-|Y_0|^2-2\sum_{i=0}^{n_k-1}Y_{\tau^k_i}\big[\mathbb{E}^{\mathbb{P}}_{\tau^k_i}[Y_{\tau^k_{i+1}}]-Y_{\tau^k_i}\big]\Big]\\
&\le \mathbb{E}^{\mathbb{P}}\Big[|Y_T|^2+2\sup_{0\le t\le T}|Y_t|\sum_{i=0}^{n_k-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau^k_i}[Y_{\tau^k_{i+1}}]-Y_{\tau^k_i}\big|\Big]\\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|Y_t|^2+\Big(\sum_{i=0}^{n_k-1}\big|\mathbb{E}^{\mathbb{P}}_{\tau^k_i}[Y_{\tau^k_{i+1}}]-Y_{\tau^k_i}\big|\Big)^2\Big]\le C\|Y\|^2_{\mathbb{P}}.
\end{aligned}
$$

Pluging into (6.5) we obtain

$$
\mathbb{E}^{\mathbb{P}}\Big[\big(I_Y(\eta^k)\big)^2\Big]\le C\|Y\|^2_{\mathbb{P}}\|\eta^k\|^2_{\mathbb{L}^\infty(\mathbb{P})}\to 0,\quad \text{as } k\to\infty.
$$

This implies that $I_Y(\eta^k)$ converges to 0 in probability $\mathbb{P}$, and thus Y is a good integrator. ■

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