Hamiltonian study for Chern-Simons and Pontryagin theories

⋆Alberto Escalante and Leopoldo Carbajal

⋆ *Instituto de Física Luis Rivera Terrazas, Benemérita Universidad Autónoma de Puebla, (IFUAP). Apartado postal J-48 72570 Puebla. Pue., México,*

⋆ *LUTh, Observatoire de Paris, Meudon, France,*

Universidad Autónoma Benito Juárez de Oaxaca, Oax., México,

(⋆ aescalan@sirio.ifuap.buap.mx, alberto.escalante@obspm.fr)

Abstract

The Hamiltonian analysis for the Chern-Simons theory and Pontryagin invariant, which depends of a connection valued in the Lie algebra of $SO(3,1)$, is performed. By applying a pure Dirac's method we find for both theories the extended Hamiltonian, the extended action, the constraint algebra, the gauge transformations and we carry out the counting of degrees of freedom. From the results obtained in the present analysis, we will conclude that the theories under study have a closed relation among its constraints and defines a topological field theory. In addition, we extends the configuration space for the Pontryagin theory and we develop the Hamitonian analysis for this modified version, finding a best description of the results previously obtained.

I. INTRODUCTION

Presently, the study of topological field theories is a topic of great interest in physics. The importance to study those theories rise because shares a closed relation with General Relativity. Topological field theories are characterized due to they are devoid of local physical degrees of freedom, background independent and invariant under diffeomorphisms [1]. Relevant examples of topological field theories with closed symmetries to General Relativity are the called BF theories. BF theories were introduced as generalizations of three dimensional Chern-Simons actions or in other cases, can also be consider as a zero coupling limit of Yang-Mills theories [2, 3]. We can find in the literature several examples where BF theories comes to be relevant models, for instance in alternative formulations of gravity such as the Plebański or Macdowell-Mansoury. Plebański's formulation consists in to obtain General Relativity by imposing extra constraints on a BF theory with the gauge group $SO(3,1)$ or $SO(4)$ [4]. On the other hand, MacDowell-Maunsouri formulation of gravity consists in braking down the $SO(5)$ symmetry of a BF-theory from $SO(5)$ group to $SO(4)$, to obtain the Palatini action plus the sum of second Chern (or Pontryagin class) and Euler topological invariants [5]. Because those topological classes have trivial local variations that do not contribute classically to the dynamics, we thus obtain essentially general relativity [6].

Other interesting theories reported in the literature with a closed relation to BF theories, can be found in the next relation among two functionals by means of [7]

$$\int_{\partial M_4} d(A^{IJ} \wedge dA_{IJ} + \frac{2}{3} A^{IK} \wedge A_{KL} \wedge A^{L}{}_{I}) = \frac{1}{2} \int_{M_4} R[A] \wedge R[A], \tag{1}$$

where the left hand side can be identified with the Chern-Simons functional and the right hand side with the Pontryagin class. Here, A^{IJ} is a one-form valued in the Lie algebra of $SO(3,1)$ and R^{IJ} is the two-form curvature (see below). As we can see, both the Chern-Simos and Pontryagin actions are related since the exterior derivative of the former generates the latter [7]. We can observe in the relation (1), that the Chern-Simons functional is defined on the boundary of a four dimensional manifold M_4, while Pontryagin class is defined on M_4. The study of the Chern-Simons functional has been a topic of several works because basically describes General Relativity in 3 dimensions and its quantization has been worked out [8]. Furthermore, by using the Chern-Simons functional we can construct a wave function that corresponds to an exact state of the Schrodinger equation for Yang-Mills theory in four dimensions [9]. In addition, we can find a recent work where the Chern-Simons state describes a topological state with unbroken diffeomorphism invariance in Yang-Mills and General Relativity [10]. In the loop quantum gravity context, that state is called the Kodama state and has been studied in interesting works by Smolin, arguing that the Kodama state at least for the Sitter spacetime, loop quantum gravity does have a good low energy limit [11]. On the other hand, the Pontryagin invariant is another interesting topological field theory [12, 13] and has been topic of study in recently works because is expected to be related to physical observables, as for instance in the case of anomalies [14, 15, 16, 17, 18, 19].

With these antecedents in mind, the purpose of this paper is to report a pure Dirac's method of constrained systems applied for the actions involved in the relation (1), which is absent in the literature. There are several reasons to develop this work. The first one, we will perform a pure Dirac analysis which means that we will work with the full configuration space and therefore with the full phase space. In other words, we will consider all the set of one-forms "A^{IJ}" that defines our theories as dynamical ones. Thus, with the present study we will be able to known the relation among the actions at Lagrangian level as well as at Hamiltonian level. With the analysis at hand, we will can identify the relevant symmetries for both theories for example, the constraints, the extended action, the extended Hamiltonian, the constrained algebra and the gauge transformations. In particular, with all the constraints classified as first or second class, we will be able to carry out the counting of the physical degrees of freedom. The second one, with the present analysis we wish to report a complete study of the relation among the constraints that there exists in the Chern-Simons theory and the constraints for the Pontryagin invariant. We can find in recent results by developing Dirac's quantization or covariant canonical program for the Pontryagin invariant, that the Chern-Simons wave function represents a quantum state for theory [12, 13], but the analysis reported in [12, 13] has been developed on a smaller phase space and the full constraints program was not performed. Therefore, the results of this paper intend to extend and complete these results by performing a complete Dirac analysis where we shall work with the full phase space reproducing in particular the results found in [12]. It is important to remark, that usually can be found in the literature the Dirac's analysis applied to several theories [20] , but generally the way to perform the study is on a smaller phase space context, this means that the dynamical variables are considered as those variables that occurs with temporal derivative in the Lagrangian density [21]. However, is not common to find a pure Dirac's method (working with the complete phase space) for field theories [22]. The principal reasons for studying the Hamiltonian formalism under a smaller phase space context and not carried out on the complete phase space, is because the separation of the constraints into first or second class is not easy to carry out. In this manner, in the literature we find that the people prefer to work on a smaller phase space context because generally there are present only first class constraints and is common to avoid the difficult part of the separation among the constraints. The price that we pay by work on a smaller phase space context is that we can not neither know the complete form of the constraints, the complete form of gauge transformations defined on the full space phase nor the complete algebra among the constraints for the theory under study. Of course, by working with the full configuration space we can reproduce the results obtained by working on a smaller configuration space.

In this manner, because of the previous explanation in this work we will perform a pure Dirac method for the theories expressed in (1), obtaining as relevant results the complete identification of its symmetries. All this part will be clarified along the present work .

The paper is organized as follows: In the Section II we will perform by using a pure Dirac method the Hamiltonian analysis for the Chern-Simons action. We will identify the full constraints for the theory, the extended action, the extended Hamiltonian, the gauge transformations and we will carry out the counting of degrees of freedom, concluding that this theory is devoid of degrees of freedom as is expected. In particular, we will show the way to identify the first and second class constraints and then compute the algebra among them. In Section II we will develop the Hamiltonian analysis for the Pontryagin invariant expressed as in (1). We will find the extended action, the extended Hamiltonian, the full constraints program, the gauge transformations and the counting of degrees of freedom, allow us to conclude that the theory is a topological field theory too. As important part of this section, we will find that contrary to Chern-Simons theory the Pontryagin invariant presents only a set of first class constraints. In Section III we will extend the configuration space for the Pontryagin invariant and we will perform the Hamiltonian analysis for this modified theory. As important result that we will find in this section is that we will have a best description than the results obtained above, but the price to pay for this description is that contrary to Section II, now we will have the presence of first and second class constraints. In particular, we will reproduce the results found previously considering the second class constraints as strong equations.

I. Hamiltonian dynamics for the Chern-Simons term

In this section, we will perform the Hamiltonian dynamics for the Chern-Simons term which will be expressed by [12]

$$S[A]_{C-S} = \frac{\alpha}{2}\int_M A^{IJ}\wedge dA_{IJ} + \frac{2}{3}A^{IK}\wedge A_{KL}\wedge A^{L}{}_{I}, \tag{2}$$

here, $A^{IJ} = A_\mu{}^{IJ} dx^\mu$ is the Lorentz connection valued in the Lie algebra of $SO(3,1)$, $\mu, \nu = 0, 1, 2$ are spacetime indices, x^μ are the coordinates that label the points for the 3-dimensional Minkowski manifold M and $I, J = 0, 1.., 3$ are internal indices that can be raised and lowered by the internal Lorentzian metric $\eta_{IJ} = (-1, 1, 1, 1)$.

We start computing the Euler-Lagrange equations obtained from the variation of the action (2), which are given by

$$\epsilon^{\alpha\beta\mu} F_{\beta\mu IJ} = 0, \tag{3}$$

where, $F_{\beta\mu IJ} = \partial_\beta A_{\mu IJ} - \partial_\mu A_{\beta IJ} + A_{\mu IK} A_\beta{}^K{}_J - A_{\beta IK} A_\mu{}^K{}_J$. The equations of motion (3) whose solutions corresponds to the space of flat connections, will be useful to identify the gauge transformations for the theory, work that will be developed below.

Now, we will consider that the manifold M has a topology $\Sigma \times R$, where Σ corresponds to a Cauchy's surface. By using this fact, we perform the 2+1 decomposition in the action (2) obtaining

$$S[A]_{C-S} = \int_M \left[\frac{\alpha}{2} \epsilon^{0ab} A_0{}^{IJ} F_{abIJ} + \frac{\alpha}{2} \epsilon^{0ab} A_b{}^{IJ} \dot{A}_{aIJ} \right] dx^3, \tag{4}$$

where $F_{abIJ} = \partial_a A_{bIJ} - \partial_b A_{aIJ} + A_{aI}{}^L A_{bLJ} - A_{bI}{}^L A_{aLJ}$, with $a, b, c = 1, 2$. From (4) we can identify the next Lagrangan density for the Chern-Simons theory

$$\mathcal{L} = \frac{\alpha}{2} \epsilon^{0ab} A_0{}^{IJ} F_{abIJ} + \frac{\alpha}{2} \epsilon^{0ab} A_b{}^{IJ} \dot{A}_{aIJ}. \tag{5}$$

At this step, it is common to find in the literature that the Hamiltonian analysis for the action (4) is performed on a smaller phase space context. This means that the dynamical variables are considered those one-forms A^{IJ}'s that occurs in the action with temporal derivative; in others words, the follow 12 one-forms $\rightarrow A_{aIJ}$ are identified as dynamical variables for the action (4), and the rest 6 one-forms $\rightarrow A_0{}^{IJ}$ are identified as Lagrange multipliers. Nevertheless, in this work we will develop a pure Dirac method which means that we will consider our dynamical variables the set of A^{IJ}'s$= (A_{aIJ}, A_0{}^{IJ})$ that defines our theory. Therefore, a pure Dirac's method calls for the definition of the momenta $(\Pi^\alpha{}_{IJ})$ canonically conjugate to $(A_\alpha{}^{IJ})$

$$\Pi^\alpha{}_{IJ} = \frac{\delta \mathcal{L}}{\delta \dot{A}_\alpha{}^{IJ}}. \tag{6}$$

The matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial \partial_\mu (A_\alpha{}^{IJ}) \partial \partial_\mu (A_\beta{}^{IJ})}, \tag{7}$$

are identically zero, the rank of the Hessian is zero, thus, we expect 18 primary constraints. From the definition of the momenta (6) we identify the next 18 primary constraints

$$\begin{aligned} \phi^0{}_{IJ} &:= \Pi^0{}_{IJ} \approx 0, \\ \phi^a{}_{IJ} &:= \Pi^a{}_{IJ} - \frac{\alpha}{2}\epsilon^{0ab}A_{bIJ} \approx 0. \end{aligned} \tag{8}$$

We can observe that by working on a smaller phase space context (the dimension of this smaller space is 24, 12$\rightarrow \dot{A}_{aIJ}$ and its respective momenta) the first constraint related with $\phi^0{}_{IJ}$ is not taken in to account. However, the purpose of this paper is to work with the full phase space and therefore with the 18 primary constraints (8). May be for the lector is not relevant this part, but once finished the analysis for the Chern-Simons and Pontryagin theory, we will be able to appreciate the advantage to perform a pure Dirac method, because we will can identify the extended action, the extended Hamiltonian, the complete form of the constrains and the algebra among them. The correct identification of the constrains is very important because can be used to carry out the counting of the physical degrees of freedom. On the other hand, constraints are the guideline to make the best progress for the quantization of the theory. We need to remember that the quantization scheme for theories as Maxwell or Yang-Mills can not be directly applied to theories with the symmetry of invariance under diffeomorphisms (as for instance topological field theories) because we can lose relevant physical information [12].

By following with the method, the canonical Hamiltonian for the Chern-Simons system is given by

$$H_c = \int dx^2 \left[\dot{A}_\alpha{}^{IJ}\Pi^\alpha{}_{IJ} - \mathcal{L}\right] = -\int dx^2 \left[\frac{\alpha}{2}A_0{}^{IJ}\epsilon^{0ab}F_{abIJ}\right]. \tag{9}$$

In this manner, the primary Hamiltonian will be constructed by adding the primary constraints (8) to (9), this is

$$H_P = H_c + \int dx^2 \left[\lambda^{IJ}{}_0\phi^0{}_{IJ} + \lambda^{IJ}{}_a\phi^a{}_{IJ}\right], \tag{10}$$

where $\lambda^{IJ}{}_0$ and $\lambda^{IJ}{}_a$ are Lagrange multipliers enforcing the constraints. The non-vanishing fundamental brackets for our theory are given by

$$\{A_\alpha{}^{IJ}(x), \Pi^\beta{}_{KL}(y)\} = \frac{1}{2}\delta^\beta{}_\alpha\left(\delta^I{}_K\delta^J{}_L - \delta^I{}_L\delta^J{}_K\right)\delta^2(x-y). \tag{11}$$

Now, we compute the 18 $\times$ 18 matrix whose entries are the Poisson brackets among the constraints

(8)

$$\begin{aligned}
\{\phi^0{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^b{}_{KL}(y)\} &= \frac{\alpha}{2}\epsilon^{0ab}\left(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL}\right)\delta^2(x-y),
\end{aligned} \tag{12}$$

we can appreciate that this matrix has rank=12 and 6 linearly independent null-vectors. By using the 6 null-vectors and consistency conditions we arrive to the next 6 secondary constraints

$$\dot{\phi}^0{}_{IJ} = \{\phi^0{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi_{IJ} := \frac{\alpha}{2}\epsilon^{0ab}F_{abIJ} \approx 0. \tag{13}$$

Consistency requires that their conservation in the time vanish as well. For this theory there no, third constraints. Now, we need to identify from the primary and secondary constrains which ones corresponds to first and second class. For this aim, we need to calculate the rank and the null-vectors of the 24× 24 matrix whose entries will be the Poisson brackets among primary and secondary constraints, this is

$$\begin{aligned}
\{\phi^0{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \Psi_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^b{}_{KL}(y)\} &= \frac{\alpha}{2}\epsilon^{0ab}\left(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL}\right)\delta^2(x-y), \\
\{\phi^a{}_{IJ}(x), \Psi_{KL}(y)\} &= \frac{\alpha}{2}\epsilon^{0ab}\Big\{\left(\eta_{KI}\eta_{LJ} - \eta_{KJ}\eta_{LI}\right)\partial_b\delta^2(x-y) + (\eta_{KJ}A_{bIL} - \eta_{KI}A_{bJL})\delta^2(x-y) \\
&- (\eta_{LI}A_{bKJ} - \eta_{LJ}A_{bKI})\delta^2(x-y)\Big\},
\end{aligned} \tag{14}$$

this matrix has rank=12 and 12 null-vectors. From the null vectors we can identify the next 12 first class constraints

$$\begin{aligned}
\gamma^0{}_{IJ} &:= \phi^0{}_{IJ} \approx 0, \\
\gamma_{IJ} &:= \Psi_{IJ} + D_a\phi^a{}_{IJ} \approx 0,
\end{aligned} \tag{15}$$

Here, we can identify that γ_{IJ} takes the role of Gauss constraint for the Chern-Simons theory. On the other hand, the rank yields to identify the next 12 second class constraints

$$\chi^a{}_{IJ} := \phi^a{}_{IJ} \approx 0. \tag{16}$$

The correct identification of first and second class constraints allow us to carry out the counting of degrees of freedom in the next form; we have 36 canonical variables $(A_\alpha{}^{IJ}, \Pi^\alpha{}_{IJ})$, 12 first class constraints $(\gamma^0{}_{IJ}, \gamma_{IJ})$ and 12 second class constraints $(\chi^a{}_{IJ})$ which yields to conclude that Chern-Simons theory is devoid of degrees of freedom. Therefore, defines a topological field theory.
To compute the algebra of constraints is convenient to smear them

$$\begin{aligned}
\phi_1 &:= \gamma^0{}_{IJ}[A] = \int dx^2 A^{IJ} \Pi^0{}_{IJ}, \\
\phi_2 &:= \gamma_{IJ}[B] = \int dx^2 B^{IJ} \left[\Psi_{IJ} + D_a \phi^a{}_{IJ}\right], \\
\phi_3 &:= \chi^a{}_{IJ}[C] = \int dx^2 C_a{}^{IJ} \left[\Pi^a{}_{IJ} - \frac{\alpha}{2}\epsilon^{0ab} A_{bIJ}\right],
\end{aligned} \tag{17}$$

In this manner, the algebra is

$$\begin{aligned}
\left\{\phi_1\left[B^{IJ}\right], \phi_1\left[C^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[B_{IJ}\right], \phi_2\left[G^{IJ}\right]\right\} &= 0, \\
\left\{\phi_1\left[B_{IJ}\right], \phi_3\left[G_a{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B^{IJ}\right], \phi_2\left[G^{KL}\right]\right\} &= \int dx^2 \left[B^I{}_K G^{KJ} - B^J{}_K G^{KI}\right] \gamma_{IJ} \approx 0, \\
\left\{\phi_2\left[B^{IJ}\right], \phi_3\left[C_a{}^{KL}\right]\right\} &= \int dx^2 \left[B^I{}_K C_a{}^{KJ} - B^J{}_K C_a{}^{KI}\right] \chi^a{}_{IJ} \approx 0, \\
\left\{\phi_3\left[C_a{}^{IJ}\right], \phi_3\left[G_b{}^{KL}\right]\right\} &= -\frac{\alpha}{2}\int dx^2 \epsilon^{0ab}\left[C_{aIJ} G_b{}^{IJ} - C_{bIJ} G_a{}^{IJ}\right],
\end{aligned} \tag{18}$$

where we can see that the algebra is closed.
By identifying the first class and second class constraints, we can find the extended action given by

$$\begin{aligned}
S_E[A_\alpha{}^{IJ}, \Pi^\alpha{}_{IJ}, \lambda_0{}^{IJ}, \lambda^{IJ}] &= \int dx^3 \Big[\Pi^0{}_{IJ}\dot{A}_0{}^{IJ} + \Pi^a{}_{IJ}\dot{A}_a{}^{IJ} - H - \lambda_0{}^{IJ}\gamma^0{}_{IJ} - \lambda^{IJ}\gamma_{IJ} \\
&- \upsilon_a{}^{IJ}\chi^a{}_{IJ}\Big]
\end{aligned} \tag{19}$$

where $H = -A_0{}^{IJ}\gamma_{IJ}$, which is proportional to Gauss first class constraint, and

$$H_E = H + \lambda_0{}^{IJ}\gamma^0{}_{IJ} + \lambda^{IJ}\gamma_{IJ}, \tag{20}$$

being the extended Hamiltonian, which a linear combination of first class constraints. As we known, the equations of motion obtained from the extended Hamiltonian are not equivalent with Euler-Lagrange equations, but the difference is unphysical [7].

Now, we shall compute the equations of motion obtained from the extended action (19), which are given by

$$\begin{aligned}
\delta A_0{}^{IJ} : \dot{\Pi}^0{}_{IJ} &= \gamma_{IJ}, \\
\delta \Pi^0{}_{IJ} : \dot{A}_0{}^{IJ} &= \lambda_0{}^{IJ}, \\
\delta \Pi^a{}_{IJ} : \dot{A}_a{}^{IJ} &= D_a(A_0{}^{IJ} - \lambda^{IJ}) + \upsilon_a{}^{IJ}, \\
\delta A_a{}^{IJ} : \dot{\Pi}^a{}_{IJ} &= \frac{1}{2}\epsilon^{0ba}\upsilon_{bIJ} - \frac{1}{2}\epsilon^{0ba}\partial_b(A_{0IJ} - \lambda_{IJ}) - (A_{0I}{}^L - \lambda_I{}^L)\Pi^a{}_{LJ}, \\
&+ (A_{0J}{}^L - \lambda_J{}^L)\Pi^a{}_{LI}, \\
\delta \lambda_0{}^{IJ} : \gamma^0{}_{IJ} &= 0, \\
\delta \lambda^{IJ} : \gamma_{IJ} &= 0, \\
\delta \upsilon_a{}^{IJ} : \chi^a{}_{IJ} &= 0.
\end{aligned} \tag{21}$$

I.I Gauge generator

One of the most important symmetries that we can study by using the Hamiltonian method, are the gauge transformations. Gauge transformations are an important symmetry, because they can help us to identify physical observables [20]. Thus, we need to know explicitly the gauge transformations for our theory. For this aim, we will apply the Castellani's algorithm [20] to construct the gauge generator. We define the generator of gauge transformations as

$$G = \int dx^2 \left(D_0\varepsilon_0{}^{IJ}\gamma^0{}_{IJ} + \varepsilon^{IJ}\gamma_{IJ} \right), \tag{22}$$

thus, we can identify the next gauge transformations on the phase space

$$\delta A_0{}^{IJ} = D_0\varepsilon_0{}^{IJ}, \tag{23}$$

$$\delta A_b{}^{IJ} = -D_b \varepsilon^{IJ}, \tag{24}$$

$$\delta \Pi^0{}_{IJ} = -\varepsilon_I{}^L \Pi^0{}_{LJ} + \varepsilon_J{}^L \Pi^0{}_{LI}, \tag{25}$$

$$\delta \Pi^a{}_{IJ} = \frac{1}{2}\epsilon^{0ba}\partial_b \varepsilon_{IJ} + \Pi^a{}_J{}^L \varepsilon_{LI} - \Pi^a{}_I{}^L \varepsilon_{LJ}. \tag{26}$$

On the other hand, we know that Chern-Simons theory shares the symmetries of general relativity [8] namely, background independence and diffeomorphisms. So, we can formulate the next question; what about the diffeomorphisms in our theory?. Apparently diffeomorphisms are not an internal symmetry, but that is not true at all because we can take $\varepsilon_0{}^{IJ} = -\varepsilon^{IJ}$ and introducing the new gauge parameters as [7]

$$\varepsilon^{IJ} = -\xi^\alpha A_\alpha{}^{IJ}, \tag{27}$$

we obtain

$$A_\mu{}^{IJ} \rightarrow A_\mu{}^{IJ} + \mathcal{L}_\xi A_\mu{}^{IJ} + \xi^\alpha F_{\mu\alpha}{}^{IJ}. \tag{28}$$

Therefore, diffeomorphisms corresponds to an internal symmetry of the theory.
As conclusion of this part, we have performed the Hamiltonian analysis for the Chern-Simons theory by working with the complete configuration space. With the present analysis, we have obtained the extended action, the extended Hamiltonian, the full constraints program and the algebra among them. With all these results at hand, we could confirm that Cher-Simons action is a topological field theory and shares symmetries with General Relativity as for instance, diffeormorphisms as gauge transformations. It is important to note that this theory presents a set of first and second class constraints. However, we will see in the next section that Pontryagin theory presents only a set of first class constraints and reducibility conditions among them. This fact will be important because Pontryagin theory is defined in four dimensions. Nevertheless, we do not lose the symmetries of Chern-Simons theory which is defined in three dimensions. This fact will be clarified below.

II. Hamiltonian dynamics for the Pontryagin invariant

In this section, we will perform a pure Hamiltonian dynamics for the Pontryagin invariant [12, 23] which is absent in the literature.

We start with the Pontryagin action expressed as the action (1)

$$S[A] = \alpha \int_M R^{IJ}[A] \wedge R_{IJ}[A], \tag{29}$$

where $R^{IJ}[A] = \frac{1}{2} R_{\mu\nu}{}^{IJ} dx^\mu \wedge dx^\nu$ is the curvature of the $SO(3,1)$ 1-form connection $A_\nu{}^{IJ}$ with $R_{\mu\nu}{}^{IJ} = \partial_\mu A_\nu{}^{IJ} - \partial\nu A_\mu{}^{IJ} + A_\mu{}^{IK} A_{\nu K}{}^J - A_\nu{}^{IK} A_{\mu K}{}^J$. Here, $\mu, \nu = 0, 1, .., 3$ are spacetime indices, x^μ are the coordinates that label the points for the 4-dimensional Minkowski manifold M and $I, J = 0, 1.., 3$ are internal indices that can be raised and lowered by the internal Lorentzian metric $\eta_{IJ} = (-1, 1, 1, 1)$.

The equations of motion obtained from the variation of the action (29) are given by

$$DR = 0, \tag{30}$$

where we can see that these equations corresponds to Bianchi identities.

By performing the $3 + 1$ decomposition of (29) we find

$$S[A] = \alpha \int dt \int dx^3 \eta^{abc} R_{bcIJ} \left(\dot{A}_a{}^{IJ} - D_a A_0{}^{IJ} \right), \tag{31}$$

here, $a, b, c = 1, .., 3$, $R_{abIJ} = \partial_a A_{bIJ} - \partial_b A_{aIJ} + A_{aI}{}^L A_{bLJ} - A_{bI}{}^L A_{aLJ}$ and $D_a A_b{}^{IJ} = \partial_a A_b^{IJ} + A_a^{IK} A_{bK}{}^J + A_a{}^{JK} A_b{}^I{}_K$.

From (18) we can identify the next Lagrangian density

$$\mathcal{L} = \alpha \eta^{abc} R_{bcIJ} \left(\dot{A}_a{}^{IJ} - D_a A_0{}^{IJ} \right). \tag{32}$$

Just as in the last section, a pure Dirac's method calls for the definition of the momenta $(\Pi^\alpha{}_{IJ})$ canonically conjugate to $(A_\alpha{}^{IJ})$

$$\Pi^\alpha{}_{IJ} = \frac{\delta \mathcal{L}}{\delta \dot{A}_\alpha{}^{IJ}}. \tag{33}$$

The matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial \partial_\mu (A_\alpha{}^{IJ}) \partial \partial_\mu (A_\beta{}^{IJ})}, \tag{34}$$

are identically zero, the rank of the Hessian is zero, thus, we expect 24 primary constraints. From the definition of the momenta (33) we identify the next 24 primary constraints

$$\begin{aligned} \phi^0{}_{IJ} &:= \Pi^0{}_{IJ} \approx 0, \\ \phi^a{}_{IJ} &:= \Pi^a{}_{IJ} - \alpha \eta^{abc} R_{bcIJ} \approx 0. \end{aligned} \tag{35}$$

Neglecting terms on the frontier, the canonical Hamiltonian for the second Chern class system is given by

$$H_c = -\int dx^3 A_0{}^{IJ} D_a \Pi^a{}_{IJ}. \tag{36}$$

In this manner, we add the primary constraints to identify the primary Hamiltonian, given by

$$H_P = H_c + \int dx^3 \left[\lambda^{IJ}{}_0 \phi^0{}_{IJ} + \lambda^{IJ}{}_a \phi^a{}_{IJ}\right], \tag{37}$$

where $\lambda^{IJ}{}_0$ and $\lambda^{IJ}{}_a$ are Lagrange multipliers enforcing the constraints. The non-vanishing fundamental brackets are

$$\{A_\alpha{}^{IJ}(x), \Pi^\beta{}_{KL}(y)\} = \frac{1}{2}\delta^\beta{}_\alpha \left(\delta^I{}_K \delta^J{}_L - \delta^I{}_L \delta^J{}_K\right) \delta^3(x-y). \tag{38}$$

Now we compute the 24×24 matrix whose entries are the Poisson brackets among the constraints (35)

$$\begin{aligned}
\{\phi^0{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^b{}_{KL}(y)\} &= 0,
\end{aligned} \tag{39}$$

we can observe that this part is quite different respect to Chern-simons theory because the entries of the matrix (39) are all equal to zero. This means that we can determine all the values of the Lagrange multipliers al most weakly [20]. However, consistency allow us to identify the next 6 reducibility conditions

$$\dot{\phi}^0{}_{IJ} = \{\phi^0{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \Psi_{IJ} := D_a \Pi^a{}_{IJ} \approx 0, \tag{40}$$

where can be identified as the Gauss constraint for the theory. In addition, for this theory there no, third constraints.

To compute the algebra among the constraints is convenient rewrite them as

$$\begin{aligned}
\phi_1 &:= \gamma^0{}_{IJ}[A] = \int dx^3 A^{IJ} \Pi^0{}_{IJ}, \\
\phi_2 &:= \gamma_{IJ}[B] = \int dx^3 B^{IJ} \left[D_a \Pi^a{}_{IJ}\right], \\
\phi_3 &:= \gamma^a{}_{IJ}[C] = \int dx^3 C_a{}^{IJ} \left[\Pi^a{}_{IJ} - \alpha \eta^{abc} R_{bcIJ}\right],
\end{aligned} \tag{41}$$

In this manner, the algebra is

$$\begin{aligned}
\Big\{\phi_1\left[B^{IJ}\right],\phi_1\left[C^{KL}\right]\Big\} &= 0,\\
\Big\{\phi_1\left[BIJ\right],\phi_2\left[G^{IJ}\right]\Big\} &= 0,\\
\Big\{\phi_1\left[B_{IJ}\right],\phi_3\left[G_a{}^{KL}\right]\Big\} &= 0,\\
\Big\{\phi_2\left[B^{IJ}\right],\phi_2\left[G^{KL}\right]\Big\} &= \int dx^3\left[B^I{}_K G^{KJ}-B^J{}_K G^{KI}\right]\gamma_{IJ}\approx 0,\\
\Big\{\phi_2\left[B^{IJ}\right],\phi_3\left[C_a{}^{KL}\right]\Big\} &= \int dx^3\left[B^I{}_K C_a{}^{KJ}-B^J{}_K C_a{}^{KI}\right]\gamma^a{}_{IJ}\approx 0,\\
\Big\{\phi_3\left[C_a{}^{IJ}\right],\phi_3\left[G_b{}^{KL}\right]\Big\} &= 0,
\end{aligned}\tag{42}$$

where we can see that the constraints form a first class set. The identification of the constraints allow us carry out the counting of degrees of freedom as follows: We have 48 canonical variables $(A_\alpha{}^{IJ},\Pi^\alpha{}_{IJ})$ and 30 first class constraints $(\gamma^0{}_{IJ},\gamma^a{}_{IJ},\gamma_{IJ})$. However, Bianchi's identities $DR=0$ implies 6 reducibility conditions among the constraints given by $D_a\gamma^a{}_{IJ}=\gamma_{IJ}$. Therefore, there are 24 independent first class constraints, this allow us to conclude that the Second Chern invariant is devoid of degrees of freedom and defines a topological field theory too.

It is important to note that while in Chern-Simons theory there are present second class constraints in Pontryagin there are not. Thus, Pontryagin theory preserves the topological symmetry with only first class constraints and the reducibility condition (40).

With all these results at hand, we can identify the extended action which is given by

$$\begin{aligned}
S_E[A_\alpha{}^{IJ},\Pi^\alpha{}_{IJ},\lambda_0{}^{IJ},\lambda_a{}^{IJ},\lambda^{IJ}] &= \int dx^4\Big[\Pi^0{}_{IJ}\dot{A}_0{}^{IJ}+\Pi^0{}_{IJ}\dot{A}_0{}^{IJ}-H\\
&- \lambda_0{}^{IJ}\gamma^0{}_{IJ}-\lambda_a{}^{IJ}\gamma^a{}_{IJ}-\lambda^{IJ}\gamma_{IJ}\Big],
\end{aligned}\tag{43}$$

where $H=-A_0{}^{IJ}D_a\Pi^a{}_{IJ}=-A_0{}^{IJ}\gamma_{IJ}$, and is a linear combination of Gauss constraint. From the extended action we can identify the extended Hamiltonian given by

$$H_E=H+\lambda_0{}^{IJ}\gamma^0{}_{IJ}+\lambda_a{}^{IJ}\gamma^a{}_{IJ}+\lambda^{IJ}\gamma_{IJ}.\tag{44}$$

where is a linear combination of first class constraints. Now, we shall compute the equations of motion obtained from the extended action (43), which are given by

$$\delta A_0{}^{IJ}:\dot{\Pi}^0{}_{IJ} = \gamma_{IJ},$$

$$\begin{aligned}
\delta\Pi^0{}_{IJ} : \dot{A}_0{}^{IJ} &= \lambda_0{}^{IJ}, \\
\delta A_a{}^{IJ} : \dot{\Pi}^a{}_{IJ} &= (A_{0I}{}^K - \lambda_I{}^K)\Pi^a{}_{JK} - (A_{0J}{}^K - \lambda_J{}^K)\Pi^a{}_{IK} + 2\alpha\eta^{abc}D_b\lambda_{cIJ}, \\
\delta\Pi^a{}_{IJ} : \dot{A}_a{}^{IJ} &= D_a\left(A_0{}^{IJ} - \lambda^{IJ}\right) + \lambda_a{}^{IJ}, \\
\delta\lambda_0{}^{IJ} : \gamma^0{}_{IJ} &= 0, \\
\delta\lambda_a{}^{IJ} : \gamma^a{}_{IJ} &= 0, \\
\delta\lambda^{IJ} : \gamma_{IJ} &= 0.
\end{aligned} \tag{45}$$

II.I Gauge generator

As we have showed, our theory presents a set of first class constraints. In this manner, we will have the presence of gauge transformations. We will proceed to identify the gauge transformations for the system by applying the Castellani's algorithm, constructing the follow gauge generator

$$G = \int dx^3 \left[D_0\varepsilon_0{}^{IJ}\gamma^0{}_{IJ} + \varepsilon_a{}^{IJ}\gamma^a{}_{IJ} + \varepsilon^{IJ}\gamma_{IJ}\right]. \tag{46}$$

Thus, the gauge transformations on the phase space are given by

$$\begin{aligned}
\delta A_0{}^{IJ} &= D_0\varepsilon_0{}^{IJ}, \\
\delta A_a{}^{IJ} &= \varepsilon_a{}^{IJ} - D_a\varepsilon^{IJ}, \\
\delta\Pi^0{}_{IJ} &= -\varepsilon_I{}^L\Pi^0{}_{LJ} + \varepsilon_J{}^L\Pi^0{}_{LI}, \\
\delta\Pi^a{}_{IJ} &= \alpha\eta^{abc}D_b\varepsilon_{cIJ} + \Pi^a{}_{IK}\varepsilon_J{}^K - \Pi^a_{JK}\varepsilon_I{}^K.
\end{aligned} \tag{47}$$

We can see that in correspondence with the Chern-Simons theory, diffeomorphisms are not present in these gauge transformations. However, we introduce a set of new gauge parameters $\varepsilon_0{}^{IJ} = \varepsilon^{IJ} = -\xi^\rho A_\rho{}^{IJ}$ and $\varepsilon_\mu{}^{IJ} = -\xi^\rho F^{IJ}_{\rho\mu}$, allowing us rewrite the gauge transformations as

$$A'_\mu{}^{IJ} \to A_\mu{}^{IJ} + \mathcal{L}_\xi A_\mu{}^{IJ}, \tag{48}$$

which corresponds to diffeomorphisms. Therefore diffeomorphisms corresponds to an internal symmetry of the theory. It is important to observe, that the Pontryagin invariant which is defined in four dimensions inherit the principal symmetries of Chern-Simons theory defined in three dimensions, as for instance the invariance under diffeomorphisms. In this manner, because of Pontryagin is defined in four dimensions its quantization study could be a good attempt to understand the constrained gravitational field because we have at hand similar symmetries. However, we need to be careful because Pontryagin invariant is a topological field theory such as has been showed in this section while general relativity is not, because there exists two degrees of freedom per point of the space [23].

III. Hamiltonian dynamics for modified Pontryagin invariant

We will complete the analysis of this work by performing a pure Dirac analysis for a modified version of Pontryagin theory. In particular, we shall reproduce the results discussed above.
For our purposes, we will work with the next action [12]

$$S[A,R] = \alpha \int_M \frac{1}{2} R^{IJ} \wedge R_{IJ} - R_{IJ} \wedge (dA^{IJ} + A^I{}_K \wedge A^{KJ}). \tag{49}$$

Now, we will consider that the 1-form A^{IJ} and the two-form R^{IJ} represents our new independent set of dynamical variables. We can see with this election of variables, that we have extended the configuration space respect to Pontryagin theory and therefore, by performing the Hamiltonian analysis we will extend the phase space. The equations of motion obtained from the action (49) are given by

$$\begin{aligned} R^{IJ} &= dA^{IJ} + A^I{}_K \wedge A^{KJ}, \\ DR^{IJ} &= 0. \end{aligned} \tag{50}$$

By using the equation of motion (50) in (49) we can eliminate R, obtaining the same action (29) and the equations of motion (30) [23]. In this manner, the following question rise; will be the same symmetries for the action (49) those found above for the action (29)?. Our answer at Lagrangian level can be yes. However, at Hamiltonian level we need to be careful because of two systems sharing the same equations of motion, not necessary yields to the same symmetries and symplectic structures [12] (see [24] as well). Therefore, we will answer the question by performing a pure Dirac method for the action (49) and then, compare the results with those obtained above for Pontryagin theory.

From now on, to avoid confusion with Pontryagin action, we will refer to the action (49) as modified Pontryagin theory.

By performing the 3+1 decomposition for the modified action (49) we find

$$S[A,R] = \int_{\Sigma} dx^3 \int dt \left[\frac{\alpha}{2}\eta^{abc} R_{0aIJ}\left(F^{IJ}{}_{bc} - R_{bc}{}^{IJ}\right) + \frac{\alpha}{2}\eta^{abc} R_{bc}{}^{IJ}\left(\dot{A}_a{}^{IJ} - D_a A_0{}^{IJ}\right)\right], \qquad (51)$$

where we identify with $F_{abIJ} = \partial_a A_{bIJ} - \partial_b A_{aIJ} + A_{aI}{}^L A_{bLJ} - A_{bI}{}^L A_{aLJ}$ the two-form curvature. For this modified theory we have a set of $(A_\alpha{}^{IJ}, R^{IJ}{}_{\alpha\beta}) = 60$ dynamical variables, so Dirac's method calls for the definition of the momenta $(\Pi^\alpha{}_{IJ}, \Pi^{\mu\nu}{}_{IJ})$ canonically conjugate to $(A_\alpha{}^{IJ}, R^{IJ}{}_{\mu\nu})$

$$\begin{aligned} \Pi^\alpha{}_{IJ} &= \frac{\delta\mathcal{L}}{\delta \dot{A}_\alpha{}^{IJ}}, \\ \Pi^{\mu\nu}{}_{IJ} &= \frac{\delta\mathcal{L}}{\delta \dot{R}_{\mu\nu}{}^{IJ}}. \end{aligned} \qquad (52)$$

The matrix elements of the Hessian

$$\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(A_\beta{}^{IJ}))}, \qquad \frac{\partial^2\mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(R_{\rho\nu}{}^{IJ}))}, \qquad \frac{\partial^2\mathcal{L}}{\partial(\partial_\mu(R_{\rho\nu}{}^{IJ}))\partial(\partial_\mu(R_{\gamma\sigma}{}^{IJ}))}, \quad (53)$$

are identically zero, the rank of the Hessian is zero, thus, we expect 60 primary constraints. From the definition of the momenta (52) we identify the next 60 primary constraints

$$\begin{aligned} \phi^0{}_{IJ} &:= \Pi^0{}_{IJ} \approx 0, \\ \phi^a{}_{IJ} &:= \Pi^a{}_{IJ} - \frac{\alpha}{2}\eta^{abc} R_{bcIJ} \approx 0, \\ \phi^{0a}{}_{IJ} &:= \Pi^{0a}{}_{IJ} \approx 0, \\ \phi^{ab}{}_{IJ} &:= \Pi^{ab}{}_{IJ} \approx 0. \end{aligned} \qquad (54)$$

For the system under study, the canonical Hamiltonian is given by

$$H_c = \int dx^3 \left[-\frac{1}{2} A_0{}^{IJ} D_a \Pi^a{}_{IJ} + R_{0a}{}^{IJ}\left(\Pi^a{}_{IJ} - \frac{\alpha}{2}\eta^{abc} F_{bcIJ}\right)\right]. \qquad (55)$$

In this manner, with the canonical Hamiltonian and the primary constraints at hand, can be identify the primary Hamiltonian expressed by

$$H_P = H_c + \int dx^3 \left[\lambda^{IJ}{}_0 \phi_{IJ}{}^0 + \lambda^{IJ}{}_a \phi_{IJ}{}^a + \lambda_{0a}{}^{IJ} \phi^{0a}{}_{IJ} + \lambda_{ab}{}^{IJ} \phi^{ab}{}_{IJ}\right], \qquad (56)$$

where $\lambda^{IJ}{}_0$, $\lambda^{IJ}{}_a, \lambda_{0a}{}^{IJ}$ and $\lambda_{ab}{}^{IJ}$ are Lagrange multipliers enforcing the constraints.

For the theory under study, can be identified the next non-vanishing fundamental brackets

$$
\begin{aligned}
\{A_\alpha{}^{IJ}(x), \Pi^\beta{}_{KL}(y)\} &= \frac{1}{2}\delta^\beta{}_\alpha \left(\delta^I{}_K\delta^J{}_L - \delta^I{}_L\delta^J{}_K\right)\delta^3(x-y), \\
\{R_{\mu\nu}{}^{IJ}(x), \Pi^{\alpha\beta}{}_{KL}(y)\} &= \frac{1}{4}\left(\delta^\alpha{}_\mu\delta^\beta{}_\nu - \delta^\alpha{}_\nu\delta^\beta{}_\mu\right)\left(\delta^I{}_K\delta^J{}_L - \delta^I{}_L\delta^J{}_K\right)\delta^3(x-y). \qquad (57)
\end{aligned}
$$

Now, we need to identify if our modified theory presents secondary constraints. For this aim, we compute the 60×60 matrix whose entries are the Poisson brackets among the primary constraints (54)

$$
\begin{aligned}
\{\phi_{IJ}{}^0(x), \phi_{KL}{}^0(y)\} &= 0, \\
\{\phi_{IJ}{}^0(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^0(x), \phi^{0a}{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^0(x), \phi^{ab}{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^a(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^a(x), \phi^{0a}{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^a(x), \phi^{cd}{}_{KL}(y)\} &= -\frac{\alpha}{4}\eta^{acd}\left(\eta_{IK}\eta_{JL} - \eta_{IH}\eta_{JF}\right)\delta^3(x-y), \\
\{\phi_{IJ}{}^{0a}(x), \phi^{0b}{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^{0a}(x), \phi^{cd}{}_{KL}(y)\} &= 0, \\
\{\phi_{IJ}{}^{ab}(x), \phi^{cd}{}_{KL}(y)\} &= 0,
\end{aligned}
\qquad (58)
$$

this matrix has rank= 36 and 24 linearly independent null-vectors. Consistency and the null vectors yields to identify the next 24 secondary constraints

$$
\begin{aligned}
\dot{\phi}^0{}_{IJ} &= \{\phi^0{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi_{IJ} := D_a\Pi^a{}_{IJ} \approx 0. \\
\dot{\phi}^{0a}{}_{IJ} &= \{\phi^{0a}{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi^{0a}{}_{IJ} := \Pi^a{}_{IJ} - \frac{\alpha}{2}\epsilon^{abc}F_{bcIJ} \approx 0, \qquad (59)
\end{aligned}
$$

and the next values for the Lagrange multipliers

$$
\dot{\phi}^a{}_{IJ} = \{\phi^a{}_{IJ}(x), H_P\} \approx 0 \qquad \Rightarrow \qquad \frac{1}{2}\left[\Pi^a{}_{JL}\eta_{KI} - \Pi^a{}_{IL}\eta_{KJ}\right]A_0{}^{KL} - \alpha\eta^{abi}D_iR_{0bIJ}
$$

$$
\begin{aligned}
& - \quad \frac{\alpha}{2}\eta^{acd}\lambda_{cdIJ} \approx 0, \\
\dot{\phi}^{ab}{}_{IJ} = \{\phi^{ab}{}_{IJ}(x), H_P\} \approx 0 \qquad & \Rightarrow \qquad \eta^{abc}\lambda_{cIJ} \approx 0.
\end{aligned} \tag{60}
$$

Consistency requires that the conservation in the time of the constraints vanish as well. For this theory there no, third constraints. At this point, we need to identify from primary and secondary constrains which ones corresponds to first and second class. For this purpose, we need to calculate the rank and the null-vectors of the 84× 84 matrix whose entries will be the Poisson brackets among primary and secondary constraints, this is

$$
\begin{aligned}
\{\phi^0{}_{IJ}(x), \phi^0{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^{0a}{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \phi^{ab}{}_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \psi_{KL}(y)\} &= 0, \\
\{\phi^0{}_{IJ}(x), \psi^{0a}{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^a{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^{0a}{}_{KL}(y)\} &= 0, \\
\{\phi^a{}_{IJ}(x), \phi^{cd}{}_{KL}(y)\} &= -\frac{\alpha}{4}\eta^{acd}\left(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK}\right)\delta^3(x-y), \\
\{\phi^a{}_{IJ}(x), \psi_{KL}(y)\} &= -\frac{1}{2}\left[\Pi^a{}_{JL}\eta_{KI} - \Pi^a{}_{IL}\eta_{KJ} + \Pi^a{}_{KJ}\eta_{LI} - \Pi^a{}_{KI}\eta_{LJ}\right]\delta^3(x-y), \\
\{\phi^a{}_{IJ}(x), \psi^{0b}{}_{KL}(y)\} &= \frac{\alpha}{2}\eta^{abc}\Big\{\partial_c\delta^3(x-y)\left(\eta_{KI}\eta_{LJ} - \eta_{KJ}\eta_{LI}\right) + \left(\omega_{cIL}\eta_{KJ} - \omega_{cJL}\eta_{KI}\right)\delta^3(x-y) \\
&+ \left(\omega_{cKI}\eta_{LJ} - \omega_{cKJ}\eta_{LI}\right)\delta^3(x-y)\Big\}, \\
\{\phi^{0a}{}_{IJ}(x), \phi^{0b}{}_{KL}(y)\} &= 0, \\
\{\phi^{0a}{}_{IJ}(x), \phi^{cd}{}_{KL}(y)\} &= 0, \\
\{\phi^{0a}{}_{IJ}(x), \psi_{KL}(y)\} &= 0, \\
\{\phi^{0a}{}_{IJ}(x), \psi^{0b}{}_{KL}(y)\} &= 0, \\
\{\phi^{ab}{}_{IJ}(x), \phi^{cd}{}_{KL}(y)\} &= 0, \\
\{\phi^{ab}{}_{IJ}(x), \psi_{KL}(y)\} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\{\phi^{ab}{}_{IJ}(x),\psi^{0c}{}_{KL}(y)\} &= 0,\\
\{\psi_{IJ}(x),\psi_{KL}(y)\} &= \frac{1}{2}\left(\psi_{IJ}\eta_{KI}+\psi_{JK}\eta_{LI}+\psi_{IL}\eta_{KJ}+\psi_{KI}\eta_{LJ}\right)\delta^3(x-y)\approx 0,\\
\{\psi_{IJ}(x),\psi^{0a}{}_{KL}(y)\} &= \frac{1}{2}\left(\psi^{0a}{}_{LJ}\eta_{KI}+\psi^{0a}{}_{JK}\eta_{LI}+\psi^{0a}{}_{IL}\eta_{KJ}+\psi^{0a}{}_{KI}\eta_{LJ}\right)\delta^3(x-y)\approx 0,\\
\{\psi^{0a}{}_{IJ}(x),\psi^{0b}{}_{KL}(y)\} &= 0,
\end{aligned}
\tag{61}
$$

this matrix has rank=36 and 48 null-vectors. From the null vectors we can identify the next 48 first class constraints

$$
\begin{aligned}
\gamma^{0}{}_{IJ} &:= \Pi^{0}{}_{IJ}\approx 0,\\
\gamma^{0a}{}_{IJ} &:= \Pi^{0a}{}_{IJ}\approx 0,\\
\gamma_{IJ} &:= D_a\Pi^{a}{}_{IJ}-\left(\Pi^{ab}{}_{I}{}^{F}R_{abFJ}-\Pi^{ab}{}_{J}{}^{F}R_{abFI}\right)\approx 0,\\
\gamma^{0a}{}_{IJ} &:= \Pi^{a}{}_{IJ}-\frac{\alpha}{2}\eta^{abc}F_{bcIJ}+2D_b\Pi^{ab}{}_{IJ}\approx 0,
\end{aligned}
\tag{62}
$$

We can observe, that the third equation of (62) can be identified as the Gauss constraint for this extended Pontryagin theory. On the other hand, the rank of the matrix (61) yields to identify the following t 36 second class constraints

$$
\begin{aligned}
\chi^{a}{}_{IJ} &:= \Pi^{a}{}_{IJ}-\frac{\alpha}{2}\eta^{abc}R_{bcIJ}\approx 0,\\
\chi^{ab}{}_{IJ} &:= \Pi^{ab}{}_{IJ}\approx 0.
\end{aligned}
\tag{63}
$$

The identification of first and second class constraints will allow us to carry out the counting of degrees of freedom; we have 120 canonical variables $(A_\alpha{}^{IJ}, R_{\mu\nu}{}^{IJ}, \Pi^{\alpha}{}_{IJ}, \Pi^{\mu\nu}{}_{IJ})$, 48 first class constraints $(\gamma^{0}{}_{IJ}, \gamma^{0a}{}_{IJ}, \gamma_{IJ}, \gamma^{0a}{}_{IJ})$ and 36 second class constraints $(\chi^{a}{}_{IJ}, \chi^{ab}{}_{IJ})$. However, just as for Pontryagin theory Bianchi's identities $DF=0$ implies 6 reducibility conditions among the first class constraints. We can see that for the modified Pontryagin theory, reducibility conditions has a longer expression than Pontryagin (see (40)): $D_a\gamma^{0a}{}_{IJ}-\gamma_{IJ}-\left(\chi^{ab}{}_{I}{}^{F}R_{abFJ}-\chi^{ab}{}_{J}{}^{F}R_{abFI}\right)-2D_aD_b\chi^{ab}{}_{IJ}=0$. Therefore, we have 42 independent first class constraints. By using this fact, the counting of degrees of freedom yields to conclude that this modified Pontryagin theory is devoid of degrees of freedom and defines a topological field theory too. It is important to remark that we can reproduce the results found for the action (29) by considering the second class constraints (63) as strong identities, thus, the constraints (62) will be reduced to (41).

By following with the method, we need to compute the algebra of constraints, for this fact it is convenient rewrite them as

$$
\begin{aligned}
\phi_1 &:= \gamma^0{}_{IJ}\left[A\right] = \int dx^3 A^{IJ}\left[\Pi^0{}_{IJ}\right], \\
\phi_2 &:= \gamma^{0a}{}_{IJ}\left[B\right] = \int dx^3 B_{0a}{}^{IJ}\left[\Pi^{0a}{}_{IJ}\right], \\
\phi_3 &:= \gamma_{IJ}\left[C\right] = \int dx^3 C_a{}^{IJ}\left[D_a\Pi^a{}_{IJ} - \left(\Pi^{ab}{}_I{}^F R_{FJab} - \Pi^{ab}{}_J{}^F R_{FIab}\right)\right], \\
\phi_4 &:= \gamma^{0a}{}_{IJ}\left[\mathbf{D}\right] = \int dx^3 \mathbf{D}_{0a}{}^{IJ}\left[\Pi^a{}_{IJ} - \frac{\alpha}{2}\epsilon^{abc}F_{bcIJ} + 2D_b\Pi^{ab}{}_{IJ}\right], \\
\phi_5 &:= \chi^a{}_{IJ}\left[F\right] = \int dx^3 F_a{}^{IJ}\left[\Pi^a{}_{IJ} - \frac{\alpha}{2}\eta^{abc}R^{IJ}{}_{bc}\right], \\
\phi_6 &:= \chi^{ab}{}_{IJ}\left[G\right] = \int dx^3 G_{ab}{}^{IJ}\left[\Pi^{ab}{}_{IJ}\right].
\end{aligned} \tag{64}
$$

Thus, the algebra of constraints is given by

$$
\begin{aligned}
\left\{\phi_1\left[A^{IJ}\right], \phi_1\left[A'^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[A^{IJ}\right], \phi_2\left[B_{0a}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[A^{IJ}\right], \phi_3\left[C^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[A^{IJ}\right], \phi_4\left[\mathbf{D}_{0a}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[A^{IJ}\right], \phi_5\left[F_a{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_1\left[A^{IJ}\right], \phi_6\left[G_{ab}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B_{0a}{}^{IJ}\right], \phi_2\left[B'_{0b}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B_{0a}{}^{IJ}\right], \phi_3\left[C^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B_{0a}{}^{IJ}\right], \phi_4\left[\mathbf{D}_{0b}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B_{0a}{}^{IJ}\right], \phi_5\left[F_b{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_2\left[B_{0a}{}^{IJ}\right], \phi_6\left[G_{cd}{}^{KL}\right]\right\} &= 0, \\
\left\{\phi_3\left[C^{IJ}\right], \phi_3\left[C'^{KL}\right]\right\} &= \int dx^3\left[C^{IK}C'_K{}^J - C^{JK}C'_K{}^I\right]\gamma_{IJ} \approx 0, \\
\left\{\phi_3\left[C^{IJ}\right], \phi_4\left[\mathbf{D}_{0a}{}^{KL}\right]\right\} &= \int dx^3\left[C^{IK}\mathbf{D}_{0aK}{}^J - C^{JK}\mathbf{D}_{0aK}{}^I\right]\gamma^{0a}{}_{IJ} \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
\left\{\phi_3\left[C^{IJ}\right],\phi_5\left[F_a{}^{KL}\right]\right\} &= \int dx^3\left[C^{IK}F_{aK}{}^J - C^{JK}F_{aK}{}^I\right]\chi^a{}_{IJ}\approx 0,\\
\left\{\phi_3\left[C^{IJ}\right],\phi_6\left[G_{ab}{}^{KL}\right]\right\} &= 0,\\
\left\{\phi_4\left[\mathbf{D}_{0a}{}^{IJ}\right],\phi_4\left[\mathbf{D}'_{0b}{}^{KL}\right]\right\} &= 0,\\
\left\{\phi_4\left[\mathbf{D}_{0a}{}^{IJ}\right],\phi_5\left[F_b{}^{KL}\right]\right\} &= 0,\\
\left\{\phi_4\left[\mathbf{D}_{0a}{}^{IJ}\right],\phi_6\left[G'_{cd}{}^{KL}\right]\right\} &= 0,\\
\left\{\phi_5\left[F_a{}^{IJ}\right],\phi_5\left[F'_a{}^{KL}\right]\right\} &= 0,\\
\left\{\phi_5\left[F_a{}^{IJ}\right],\phi_6\left[G'_{ab}{}^{KL}\right]\right\} &= -\frac{\alpha}{4}\eta^{aij}\int dx^3\left[F_{aKH}G_{ij}{}^{KH} - F_{iKH}G_{aj}{}^{KH}\right],\\
\left\{\phi_6\left[G_{ab}{}^{IJ}\right],\phi_6\left[G'_{cd}{}^{KL}\right]\right\} &= 0,
\end{aligned}
\tag{65}
$$

where we can see clearly that (62) and (63) form a first and second class constraints set respectively. It is important to observe, that the algebra among the constraints for this modified Pontryagin theory shares a closed relation with the constraint algebra for the Chern-Simons theory (18) (see the Poisson's brackets between ϕ_3, ϕ_4, ϕ_5 of (65) and ϕ_2, ϕ_3 of (18)). In addition, now this modified theory presents second class constraints as well.

With all these results at hand, we can use the Lagrange's multipliers values (60), the first class constraints (62) and the second class constraints (63) to identify the extended action for the theory expressed by

$$
\begin{aligned}
S_E &\left[A_\alpha{}^{IJ},\Pi^\alpha{}_{IJ},R_{\mu\nu}{}^{IJ},\Pi^{\mu\nu}{}_{IJ},u_0{}^{IJ},u_{0a}{}^{IJ},u^{IJ},u_a{}^{IJ},v_a{}^{IJ},v_{ab}{}^{IJ}\right] = \int\Big\{\dot{A}_\alpha{}^{IJ}\Pi^\alpha{}_{IJ} + \dot{R}_{0a}{}^{IJ}\Pi^{0a}{}_{IJ}\\
+ &\ \dot{R}_{ab}{}^{IJ}\Pi^{ab}{}_{IJ} - H - u_0{}^{IJ}\gamma^0{}_{IJ} - u_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} - u^{IJ}\gamma_{IJ} - u_a{}^{IJ}\gamma^{0a}{}_{IJ} - v_a{}^{IJ}\chi^a{}_{IJ} - v_{ab}{}^{IJ}\chi^{ab}{}_{IJ}\Big\}dx^4,
\end{aligned}
\tag{66}
$$

where H is only linear combination of first class constraints

$$
H = \frac{1}{2}A_0{}^{IJ}\left[D_a\Pi^a{}_{IJ} - \left(\Pi^{ab}{}_I{}^F R_{abFJ} - \Pi^{ab}{}_J{}^F R_{abFI}\right)\right] - R_{0a}{}^{IJ}\left[\Pi^a{}_{IJ} - \frac{\alpha}{2}\epsilon^{abc}F_{bcIJ} + 2D_b\Pi^{ab}{}_{IJ}\right],
\tag{67}
$$

and $u_0{}^{IJ}, u_{0a}{}^{IJ}, u^{IJ}, u_a{}^{IJ}, v_a{}^{IJ}, v_{ab}{}^{IJ}$ are the Lagrange multipliers enforcing the first and second class constraints. We can observe, that by considering the second class constraints as strong equations the Hamiltonian (67) is reduced to that Hamiltonian quantized in [12] where was performed the Hamiltonian analysis on a smaller phase space. In this manner, we have here a best description at classical level than that reported in [12].

From the extended action we can identify the extended Hamiltonian which is given by

$$H_E = H - u_0{}^{IJ}\gamma^0{}_{IJ} - u_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} - u^{IJ}\gamma_{IJ} - u_a{}^{IJ}\gamma^{0a}{}_{IJ}. \tag{68}$$

As we Know, the equations of motion obtained by means of the extended Hamiltonian in general are mathematically different with the Euler-Lagrange equations, but the difference is unphysical [7]. We will continue this section computing the equations of motion obtained from the extended action. The equations of motion derived from the extended action are given by

$$\begin{aligned}
\delta A_0{}^{IJ} : \dot{\Pi}^0{}_{IJ} &= \frac{1}{2}\left[D_a\Pi^a{}_{IJ} - \left(\Pi^{ab}{}_I{}^F R_{abFJ} - \Pi^{ab}{}_J{}^F R_{abFI}\right)\right],\\
\delta\Pi^0{}_{IJ} : \dot{A}_0{}^{IJ} &= u_0{}^{IJ},\\
\delta A_a{}^{IJ} : \dot{\Pi}^a{}_{IJ} &= \left[A_{0J}{}^F + u_J{}^F\right]\Pi^a{}_{IF} - \left[A_{0I}{}^F + u_I{}^F\right]\Pi^a{}_{JF} - \alpha\eta^{abc}\left[D_bR_{0cIJ} - D_bu_{cIJ}\right]\\
&+ 2\left[u_{bI}{}^F - R_{0bI}{}^F\right]\Pi^{ab}{}_{JF} - 2\left[u_{bJ}{}^F - R_{0bJ}{}^F\right]\Pi^{ab}{}_{IF},\\
\delta\Pi^a{}_{IJ} : \dot{A}_a{}^{IJ} &= -D_a\left(\frac{1}{2}A_0{}^{IJ} + u^{IJ}\right) + \left(u_a{}^{IJ} - R_{0a}{}^{IJ}\right) + v_a{}^{IJ},\\
\delta R_{0a}{}^{IJ} : \dot{\Pi}^{0a}{}_I &= -\left[\Pi^a{}_{IJ} - \frac{\alpha}{2}\eta^{abc}F_{bcIJ} + 2D_b\Pi^{ab}{}_{IJ}\right],\\
\delta\Pi^{0a}{}_{IJ} : \dot{R}_{0a}{}^{IJ} &= u_{0a}{}^{IJ},\\
\delta R_{ab}{}^{IJ} : \dot{\Pi}^{ab}{}_{IJ} &= \left[\frac{1}{2}A_0{}^F{}_J + u^F{}_J\right]\Pi^{ab}{}_{FI} - \left[\frac{1}{2}A_0{}^F{}_I + u^F{}_I\right]\Pi^{ab}{}_{FJ} + \frac{\alpha}{2}\eta^{abc}v_{cIJ},\\
\delta\Pi^{ab}{}_{IJ} : \dot{R}_{ab}{}^{IJ} &= \left[\frac{1}{2}A_0{}^{JF} + u^{JF}\right]R_{ab}{}^I{}_F - \left[\frac{1}{2}A_0{}^{IF} + u^{IF}\right]R_{ab}{}^J{}_F + D_a\left(u_b{}^{IJ} - R_{0b}{}^{IJ}\right)\\
&- D_b\left(u_a{}^{IJ} - R_{0a}{}^{IJ}\right) + v_{ab}{}^{IJ},\\
\delta u_0{}^{IJ} : \gamma^0{}_{IJ} &= 0,\\
\delta u_{0a}{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0,\\
\delta u^{IJ} : \gamma_{IJ} &= 0,\\
\delta u_a{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0,\\
\delta v_a{}^{IJ} : \chi^a{}_{IJ} &= 0,\\
\delta v_{ab}{}^{IJ} : \chi^{ab}{}_{IJ} &= 0.
\end{aligned} \tag{69}$$

III.I Gauge generator

As we have showed, our modified theory presents a set of first class constraints. Therefore, we need to identify the form of gauge transformations generated for these constraints. For this part, we will find the gauge transformations generated by the first class constraints (62) by using the Castellani's algorithm in essence constructing the follow gauge generator

$$G=\int_{\Sigma}\left[D_0\varepsilon_0{}^{IJ}\gamma^0{}_{IJ}+D_0\varepsilon_{0a}{}^{IJ}\gamma^{0a}{}_{IJ}+\varepsilon^{IJ}\gamma_{IJ}+\varepsilon_a{}^{IJ}\gamma^{0a}{}_{IJ}\right], \tag{70}$$

thus, we find the following gauge transformations on the phase space

$$\begin{aligned}
\delta_0 A_0{}^{IJ} &= D_0\varepsilon_0{}^{IJ},\\
\delta_0 A_a{}^{IJ} &= -D_a\varepsilon^{IJ}+\varepsilon_a{}^{IJ},\\
\delta_0 R_{0a}{}^{IJ} &= D_0\varepsilon_{0a}{}^{IJ},\\
\delta_0 R_{ab}{}^{IJ} &= \left[D_a\varepsilon_b{}^{IJ}-D_b\varepsilon_a{}^{IJ}\right]+\left[\varepsilon^{IF}R_{abF}{}^{J}-\varepsilon^{JF}R_{abF}{}^{I}\right],\\
\delta_0\Pi^0{}_{IJ} &= \varepsilon_{0J}{}^{L}\Pi^0{}_{IL}-\varepsilon_{0I}{}^{L}\Pi^0{}_{JL}+\varepsilon_{0J}{}^{L}\Pi^{0a}{}_{IL}-\varepsilon_{0J}{}^{L}\Pi^{0a}{}_{JL},\\
\delta_0\Pi^a{}_{IJ} &= \left[\Pi^a{}_{IL}\varepsilon_J{}^{L}-\Pi^a{}_{JL}\varepsilon_I{}^{L}\right]+\alpha\eta^{adc}D_d\varepsilon_{cIJ}+2\left[\Pi^{ab}{}_{KI}\varepsilon_b{}^{L}{}_{J}-\Pi^{ab}{}_{KJ}\varepsilon_b{}^{L}{}_{I}\right],\\
\delta_0\Pi^{0a}{}_{I} &= 0,\\
\delta_0\Pi^{ab}{}_{IJ} &= -\left[\Pi^{ab}{}_{IF}\varepsilon^{F}{}_{J}-\Pi^{ab}{}_{JF}\varepsilon^{F}{}_{I}\right].
\end{aligned} \tag{71}$$

It is important to observe, that we obtain relevant results when are considered the second class constraints as strong equations. By taking the second class constraints as strong equations, the gauge transformations obtained above are reduced to (47) which corresponds to Pontryagin theory. On the other hand, with all these results at hand, we can find in particular a close relation among the gauge transformations of this modified Pontirjagin theory and the gauge transformations reported in the case of a BF theory [21]. In fact, the modified version for the Pontryagin theory (equation (49)) has a BF form. However, in this work we find a big difference respect to [21], since in [21] we can observe that there exists only first class constraints, while in this work we have first and second class kind. The reason for that difference is because in [21] the Hamiltonian analysis was performed on a smaller phase space context and a complete analysis was not reported.

Just as in last section, we can introduce the new set of parameters ${\varepsilon_0}^{IJ} = \varepsilon^{IJ} = -\xi^\rho {A_\rho}^{IJ}$, ${\varepsilon_\mu}^{IJ} = -\xi^\rho R^{IJ}_{\rho\mu}$ and taking on to account the equations of motion (51), we find that the gauge transformations take the next form

$$
\begin{aligned}
{A'_\mu}^{IJ} &\rightarrow {A_\mu}^{IJ} + \mathcal{L}_\xi {A_\mu}^{IJ} + \xi^\rho \left({F_{\rho\mu}}^{IJ} - {R_{\rho\mu}}^{IJ} \right), \\
{R'_{\mu\nu}}^{IJ} &\rightarrow {R_{\mu\nu}}^{IJ} + \mathcal{L}_\xi {R_{\mu\nu}}^{IJ} + \xi^\rho \left[D_\nu {R_{\mu\rho}}^{IJ} + D_\mu {R_{\rho\nu}}^{IJ} + D_\rho {R_{\nu\mu}}^{IJ} \right],
\end{aligned} \tag{72}
$$

where we can observe that corresponds to diffeomorphisms. In this manner, diffeomorphisms corresponds to an internal symmetry for the theory. It is important to remark that this result becomes to be important, because we have extended the number of dynamical variables by considering the 1-form conexion A^{IJ} and the two-form R_{IJ} as independent. Nevertheless, we have not lost the symmetries of Pontryagin theory.

We can compare the results reported in this paper with the reported in [13] and [21] where the Hamiltonian analysis for topological theories has been performed on a smaller phase space context. However, in our work we have identified the complete form of the first class and second class constraints, the extended Hamiltonian and the gauge transformations. In this sense, our methodology extends and complete the previous ones, thus we are showing a clear advantage when is applied a pure Dirac method for the theories under study.

VI. Conclusions and prospects

In this paper, we could present a clear and consistent application of a pure Dirac's method for constrained systems. By working with the original phase space we could perform the Hamiltonian dynamics for the Chern-Simons theory and for the Pontryagin invariant. With the present analysis we could identify for both theories the extended action, the extended Hamiltonian and the full constraints program. The correct identification of the constraints as first and second class, allowed us carry out the counting of degrees of freedom, concluding that the theories under study corresponds to topological field theories. We could observe, that Chern-Simons theory and the Pontryagin invariant has a closed relation at Hamiltonian level. From one side, the Chern-Simons theory has presence of first and second class constraints. From other side, Pontryagin theory presented only first class constraints and reducibility conditions among them. Thus, both theories are related by the action (1) and his Hamiltonian study indicates that the theories shares the principal symmetries namely; zero degrees of freedom and diffeomorphisms as gauge transformations.

On the other hand, by extending the original configuration space for the Pontryagin theory we could perform the Hamiltonian analysis for this modified theory. We could observe that this extended theory shares the same symmetries with unmodified Pontryagin. Nevertheless, the price to pay for extending the configuration space is that now we have the presence of second class constraints while for unmodified Pontryagin we do not have it. But, by considering the second class constraints as strong equations, we can reproduce the results found for the Pontryagin invariant.

As final conclusion of this paper, the results presented in this work allowed us to understand at Hamiltonian level the existing relation among Chern-Simons theory and the Pontryagin invariant. In this manner, we expect that these results will be useful to develop the quantum treatment for both theories, and thus, to obtain a best understanding for the quantum theory. In particular, the results of this article presents a best classical description than the results reported in [12] and [13]. Therefore, we can analyze the quantization aspects for the Pontryagin theory by using the context presented in this work, taking on to account the original configuration space. However, this important part will be reported in forthcoming works.

Acknowledgements

This work was supported by CONACyT under grant 95560. Leopoldo Carbajal wants to thank to IFUAP's Theoretical Physics group under grant promep: 169.

References

[1] M. Montesinos and A. Perez Phys.Rev.D77:104020,2008.

[2] G. T. Horowitz, Commun. Math.Phys.125 (1989) 417.

[3] G. T. Horowitz and M. Srednicki, Commun. Math.Phys.130 (1990) 83.

[4] J. F. Plebanski, *On the separation of Einstenian substructures*, J. Math. Phys. 18 (1977) 2511.

[5] Derek K. Wise, *MacDowell Mansouri Gravity and Cartan Geometry*, gr-qc/0611154.

[6] Freidel L and Starodubtsev A *Quantum Gravity in terms of topological observables*, arXiv: hep-th/0501191.

[7] Milutin Blagojević, *Gravitation and gauge symmetries*. Series in high energy Physics, cosmology and gravitation, (2002).

[8] E. Witten, Nucl. Phys. B311, 46 (1988).

[9] R. Jackiw, Topological Investigations in Quantized Gauge Theories, p.258, exercise 3.7, in Current Algebra and Anomalies, edited by S. B. Treiman et al (World Scientific, 1985).

[10] I. Oda, hep-th/0311149.

[11] L. Smolin, hep-th/0209079, L. Freidel and L. Smolin, Class.Quant.Grav.21:3831-3844,2004.

[12] Alberto Escalante, Phys. Lett B: 676, (2009), p. 105 - 111.

[13] R. Cartas-Fuentevilla, J.F. Tlapanco-Limon, Phys.Lett.B623:165-170,2005.

[14] A. Mardones and J. Zanelli, Class. Quantum Grav. 8. 1545 (1991).

[15] T. Kimura, Prog. Theo. Phys. 42, 1191 (1969).

[16] R. Delbourgo and A. Salam, Phys. Lett. 40B, 381 (1972).

[17] T. Eguchi and P. Freund, Phys. Rev. Lett., 1251 (1976).

[18] L. Alvarez-Gaume and E. Witten, Nucl.Phys. B234, 269.

[19] Osvaldo Chandia, and Jorge Zanelli, Phys.Rev.D55:7580-7585,1997.

[20] D. M. Gitman and I.V.Tyutin, Quantization of fields with constraints. (Berlin, Germany: Springer. (Springer series in nuclear and particle physics, (1990)).

A. Hanson, T. Regge and C. Teitelboim. Constrained Hamiltonian Systems (Accademia Nazionale dei Lincei, Roma, (1978)).

[21] M. Mondragon, M. Montesinos, J.Math.Phys.47: 022301, (2006).

[22] Alberto Esclante, *Hamiltonian dynamics for Einstein's action in G—0 limit*, 48: 24862498 DOI 10.1007/s10773-009-0035-9, 2009.

[23] V. Cuesta, M. Montesinos, Phys. Rev. D76: 025025, 2007.

[24] M. Montesinos, Class.Quant.Grav.23:2267-2278,2006.