THE ALGEBRA OF FACTORIAL POLYNOMIALS

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ABSTRACT. We discuss the formal aspects of the factorial polynomials and of the associated series. We develop the theory using the formalism of quasi-monomials and prove the usefulness of the method for the solutions of nontrivial difference equations.

In this note we consider the so called factorial polynomials [1], [2] by framing the relevant theory within the context of quasi-monomials [3]. The method we propose allows a significant simplification of the study of their properties and of their applications in the theory of multi-loop Feynman integrals [4].

The factorial polynomials (f.p.)

$$\varphi_n(x) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \tag{1}$$

are quasi-monomials in the sense that they behave as ordinary monomials under the action of the operators

$$\hat{M} = x e^{-\partial_x} \qquad \hat{P} = e^{\partial_x} - 1, \qquad (2)$$

namely¹

$$\hat{M}\varphi_n(x) = \varphi_{n+1}(x) \qquad \hat{P}\varphi_n(x) = n\varphi_{n-1}(x).$$
(3)

These operators are called multiplicative and derivative operators, respectively, and satisfy the commutation relation

$$\left[\hat{P},\hat{M}\right] = 1. \tag{4}$$

The polynomials $\varphi_n(x)$ satisfy the "differential" equation

$$\tilde{M} \tilde{P} \varphi_n(x) = n \varphi_n(x), \qquad (5)$$

that, according to eq. (2), is equivalent to the difference equation

$$x \left[\varphi_n(x) - \varphi_n(x-1)\right] = n \,\varphi_n(x) \,. \tag{6}$$

By defining the operator

$$\phi(\hat{M}) = \sum_{n=0}^{\infty} a_n \,\hat{M}^n \,, \tag{7}$$

the identity

$$\hat{M}^n \varphi_0(x) = \hat{M}^n \, 1 = \varphi_n(x) \tag{8}$$

allows us to define a f.p.-based function as follows

$$f(x) = \phi(\hat{M}) \, 1 = \sum_{n=0}^{\infty} a_n \, \varphi_n(x) \,.$$
 (9)

For example, to the operator $\exp(\lambda \hat{M})$ it is associated the f.p.-based function

$$e(x,\lambda) = e^{\lambda \hat{M}} 1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \varphi_n(x) = (1+\lambda)^x$$
(10)

¹Though not in the explicit realization of Eq. (2), the operators \hat{M} and \hat{P} have been introduced in ref. [3].

that plays the same role of exponential function for ordinary monomials since it is an eigenfunction of the operator \hat{P} with eigenvalue λ

$$\hat{P} e(x, \lambda) = \lambda e(x, \lambda).$$
(11)

In terms of it, the f.p.-based cosine and sine functions can be defined as follows

$$c(x) = \frac{1}{2} \left[e(x,i) + e(x,-i) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \hat{M}^{2k} \, 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \, \varphi_{2k}(x)$$
$$s(x) = \frac{1}{2i} \left[e(x,i) - e(x,-i) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \, \hat{M}^{2k+1} \, 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \, \varphi_{2k+1}(x) \,. \tag{12}$$

The f.p.-based function defined in eq. (10) is involved in the solution of the finite difference heat equation

$$\hat{P}_{\tau} W(x,\tau) = \partial_x^2 W(x,\tau) \qquad \qquad W(x,0) = w(x) \tag{13}$$

where

$$\hat{P}_{\tau} W(x,\tau) = W(x,\tau+1) - W(x,\tau).$$
(14)

The solution formally writes²

$$W(x,\tau) = \exp(\hat{M}_{\tau} \,\partial_x^2) \,w(x) \,\mathbf{1}_{\tau} \,, \tag{15}$$

that, taking into account eq. (10), allows us to get the solution of eq. (13) in the form

$$W(x,\tau) = e(\tau,\partial_x^2) w(x).$$
(16)

The expansion in terms of factorial series is

$$W(x,\tau) = \sum_{n=0}^{m} \frac{\varphi_n(\tau_m)}{n!} w^{(2n)}(x) , \qquad (17)$$

where the upper index (2n) denotes the order of the derivative and $\tau_m = m \tau_0$ is the discrete time, corresponding to the definition given in eq. (14) of the time derivative ($\tau_0 = 1$ because we have chosen a unity time interval). The sum run up to m as a consequence of the fact that $\varphi_{m+1}(m) = 0$. The method could be usefully exploited to deal with discrete time Schrödinger equations [5].

In the same way, starting with the operators

$$\beta_n(\hat{M}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (n+k)! \, 2^{n+2k}} \, \hat{M}^{n+2k} \,, \tag{18}$$

the f.p.-based Bessel functions can be defined as $B_n(x) = \beta(\hat{M}) 1$. In analogy with the ordinary cylindrical Bessel functions, the operators (18) satisfy the recurrences

$$2\hat{P}\beta_{n}(\hat{M}) = \left[\beta_{n-1}(\hat{M}) - \beta_{n+1}(\hat{M})\right]$$

$$2n\beta_{n}(\hat{M}) = \hat{M}\left[\beta_{n-1}(\hat{M}) + \beta_{n+1}(\hat{M})\right].$$
 (19)

that, in terms of finite difference, gives

$$B_n(x+1) = B_n(x) + \frac{1}{2} [B_{n-1}(x) - B_{n+1}(x)]$$

$$2 n B_n(x) = x [B_{n-1}(x-1) + B_{n+1}(x-1)].$$
(20)

²The suffix τ indicates that we consider only the space of this variable.

By denoting with \hat{N} a number operator such that $\hat{N} B_n(x) = n B_n(x)$, the combination of eqs. (19) allows the introduction of the following index shifting operators

$$\hat{E}_{+} = -\hat{P} + \hat{M}^{-1} \hat{N}, \qquad \hat{E}_{-} = \hat{P} + \hat{M}^{-1} \hat{N}, \qquad (21)$$

whose action on the operators in eq. (18) is given by

$$\hat{E}_{\pm}\,\beta_n(\hat{M}) = \beta_{n\pm 1}(\hat{M}) \tag{22}$$

from which it is easy to show that

$$\left[(\hat{M}\,\hat{P})^2 + (\hat{M}^2 - n^2) \right] \,\beta_n(\hat{M}) = 0 \,.$$
⁽²³⁾

The corresponding difference equation writes

$$2x(x-1)B_n(x-2) - x(2x-1)B_n(x-1) + (x^2 - n^2)B_n(x) = 0.$$
 (24)

So far a differential equation has been translated into a corresponding difference equation using the realization (2) of the multiplicative and derivative operators. The correspondence between the solutions (i.e. their isospectrality) is ensured by the fact that the operators \hat{P} , \hat{M} , and $\hat{1}$ realize a Weyl algebra. Let us now consider the same problem from a reversed perspective, i.e., given a difference equation we obtain from it a corresponding differential equation in terms of derivative and multiplicative operators.

Let us consider the following difference equation

$$(4x+2)F(x+1) + 4xF(x-1) - (8x-1)F(x) = 0.$$
(25)

By performing the following substitutions

$$e^{\partial_x} \to 1 + \hat{P}, \qquad x \to \hat{M} \left(1 + \hat{P} \right),$$
(26)

we find

$$\left(4\,\hat{M}\,\hat{P}^2 + 2\,\hat{P} + 1\right)\,\Phi(\hat{M}) = 0\tag{27}$$

where $\Phi(\hat{M}) = F(x)$. The solution of the previous differential equation can be found by using the Frobenius method. Hence, we look for solutions of the type

$$\Phi(\hat{M}) = \sum_{k=0}^{\infty} a_k M^{k+c}$$
(28)

and the associated indicial equation is

$$2a_0 c (2c-1) = 0. (29)$$

By using the solution c = 0 of this equation, we get

$$F(x) = \Phi(\hat{M}) \, 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k)!} \, \hat{M}^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k)!} \, \varphi_k(x) \,. \tag{30}$$

The other solution, c = 1/2, implies the definition of the operator $\hat{M}^{1/2}$, whose meaning will be discussed in a forthcoming paper, where the problems associated with the fractional powers of operators of the type (2) will be considered.

The method we have envisaged can also be extended to the non-homogeneous case. For example, the equation

$$a y(x+1) + b y(x) = g(x)$$
 (31)

can be cast in the form

$$\hat{Q}\psi(\hat{M}) = g(\hat{M}(1+\hat{P}))$$
(32)

where $\hat{Q} = a \hat{P} + (a + b)$ and $y(x) = \psi(\hat{M}) 1$. The particular solution of this equation formally writes

$$\psi_p(\hat{M}) = \hat{Q}^{-1} g(\hat{M} (1 + \hat{P}))$$
(33)

and the explicit form can be written as

$$\psi_p(\hat{M}) = \int_0^\infty ds \, e^{-s\hat{Q}} \, g(\hat{M} \, (1+\hat{P})) \tag{34}$$

from which, taking into account that

$$e^{\alpha \hat{P}} g(\hat{M} (1+\hat{P})) = g((\hat{M}+\alpha) (1+\hat{P})), \qquad (35)$$

we obtain

$$\psi_p(\hat{M}) = \int_0^\infty \mathrm{d}s \,\mathrm{e}^{-s\,(a+b)} \,g((\hat{M} - a\,s)\,(1+\hat{P}))\,. \tag{36}$$

As an example, in the case $g(x) = x^2$, by using eqs. (3) and (8), one obtains:

$$y_p(x) = \frac{z}{a} \left\{ (x-z)^2 - z (1-z) \right\} \qquad \left(z = \frac{a}{a+b} \right).$$
(37)

This note has just been aimed at fixing the general rule of the use of factorial series in the theory of difference equations. More specific problems regarding applications like those treated in Ref. [4] will be discussed elsewhere.

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