ON SOME NEW HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some Hadamard-type inequalities based on co-ordinated quasi-convexity. Also we define a new mapping associated to co-ordinated convexity and we prove some properties of this mapping.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality. We recall some definitions;

Definition 1. *(See* [4]*) A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. In [5], Dragomir defined convex functions on the co-ordinates as following:

Definition 2. *Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)$$

for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

In [5], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *(see [5], Theorem 1) Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (1.2) \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\ &\quad \left.\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\ &\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \end{aligned}$$

The above inequalities are sharp.

Similar results for co-ordinated $m-$convex and $(\alpha,m)-$convex functions can be found in [2]. In [**?**], Dragomir considered a mapping which closely connected with above inequalities and established main properties of this mapping as following:

Now, for a mapping $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ, we can define the mapping $H:[0,1]^2\to\mathbb{R}$,

$$H(t,s):=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dxdy$$

Theorem 2. *Suppose that $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is convex on the co-ordinates on $\Delta=[a,b]\times[c,d]$. Then:*

(i) The mapping H is convex on the co-ordinates on $[0,1]^2$.

(ii) We have the bounds

$$\sup_{(t,s)\in[0,1]^2}H(t,s)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy=H(1,1)$$

$$\inf_{(t,s)\in[0,1]^2}H(t,s)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=H(0,0)$$

(iii) The mapping H is monotonic nondecreasing on the co-ordinates.

Definition 3. *Consider a function $f:V\to\mathbb{R}$ defined on a subset V of $\mathbb{R}_n$, $n\in\mathbb{N}$. Let $L=(L_1,L_2,...,L_n)$ where $L_i\geq 0$, $i=1,2,...,n$. We say that f is $L-$Lipschitzian function if*

$$|f(x)-f(y)|\leq\sum_{i=1}^n L\,|x_i-y_i|$$

for all $x,y\in V$.

In [3], Özdemir *et al.* defined quasi-convex function on the co-ordinates as following:

Definition 4. *A function* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *is said quasi-convex function on the co-ordinates on* Δ *if the following inequality*

$$f\left(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w\right) \leq \max\left\{f\left(x, y\right), f\left(z, w\right)\right\}$$

holds for all $(x, y)\,,\ (z, w) \in \Delta$ *and* $\lambda \in [0, 1]\,.$

Let consider a bidimensional interval $\Delta := [a, b] \times [c, d]\,,\ f : \Delta \to \mathbb{R}$ will be called co-ordinated quasi-convex on the co-ordinates if the partial mappings

$$f_y : [a, b] \to \mathbb{R}, \quad f_y\left(u\right) = f\left(u, y\right)$$

and

$$f_x : [c, d] \to \mathbb{R}, \quad f_x\left(v\right) = f\left(x, v\right)$$

are convex where defined for all $y \in [c, d]$ and $x \in [a, b]\,.$ We denote by $QC(\Delta)$ the class of quasi-convex functions on the co-ordinates on Δ.

In [1], Sarıkaya *et al.* proved following Lemma and established some inequalities for co-ordinated convex functions.

Lemma 1. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$*. If* $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$*, then the following equality holds:*

$$\begin{aligned}
& \frac{f\left(a, c\right) + f\left(a, d\right) + f\left(b, c\right) + f\left(b, d\right)}{4} + \frac{1}{\left(b-a\right)\left(d-c\right)} \int_a^b \int_c^d f\left(x, y\right) dy dx \\
& -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f\left(x, c\right) + f\left(x, d\right)\right] dx + \frac{1}{d-c}\int_c^d \left[f\left(a, y\right) + f\left(b, y\right)\right] dy\right] \\
= & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2t)(1-2s)\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)b, sc + (1-s)d\right) dt ds.
\end{aligned}$$

The main purpose of this paper is to obtain some inequalities for co-ordinated quasi-convex functions by using Lemma 1 and elemantery analysis.

2. MAIN RESULTS

Theorem 3. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$*. If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is quasi-convex on the co-ordinates on* Δ*, then one has the inequality:*

$$\begin{aligned}
& \left|\frac{f\left(a, c\right) + f\left(a, d\right) + f\left(b, c\right) + f\left(b, d\right)}{4} + \frac{1}{\left(b-a\right)\left(d-c\right)} \int_a^b \int_c^d f\left(x, y\right) dy dx - A\right| \\
\leq & \frac{(b-a)(d-c)}{16}\max\left\{\left|\frac{\partial^2 f}{\partial t \partial s}(a, b)\right|, \left|\frac{\partial^2 f}{\partial t \partial s}(c, d)\right|\right\}
\end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f\left(x, c\right) + f\left(x, d\right)\right] dx + \frac{1}{d-c}\int_c^d \left[f\left(a, y\right) + f\left(b, y\right)\right] dy\right].$$

Proof. From Lemma 1, we can write

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right. \\ & \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx - A \right| \\ \leq & \frac{(b-a)(d-c)}{4} \\ & \times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right| dtds. \end{aligned}$$

Since $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is quasi-convex on the co-ordinates on Δ, we have

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right. \\ & \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx - A \right| \\ \leq & \frac{(b-a)(d-c)}{4} \\ & \times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \max\left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a,b) \right|, \left| \frac{\partial^2 f}{\partial t \partial s}(c,d) \right| \right\} dtds. \end{aligned}$$

On the other hand, we have

$$\int_0^1 \int_0^1 |(1-2t)(1-2s)|\,dtds = \frac{(b-a)(d-c)}{16}.$$

The proof is complete. □

Theorem 4. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$, $q > 1$, is quasi-convex function on the co-ordinates on Δ, then one has the inequality:*

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx - A \right| \\ \leq & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\max\left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a,b) \right|^q, \left| \frac{\partial^2 f}{\partial t \partial s}(c,d) \right|^q \right\} \right)^{\frac{1}{q}} \end{aligned}$$

where

$$A = \frac{1}{2} \left[\frac{1}{b-a} \int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c} \int_c^d [f(a,y)+f(b,y)]\,dy \right]$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and using Hölder inequality, we get

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
\leq\;& \frac{(b-a)(d-c)}{4} \\
&\times \int_0^1\int_0^1 |(1-2t)(1-2s)| \left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right| dtds \\
\leq\;& \frac{(b-a)(d-c)}{4} \\
&\times \left(\int_0^1\int_0^1 |(1-2t)(1-2s)|^p\, dtds\right)^{\frac{1}{p}} \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is quasi-convex on the co-ordinates on Δ, we have

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
\leq\;& \frac{(b-a)(d-c)}{4} \\
&\times \left(\int_0^1\int_0^1 |(1-2t)(1-2s)|^p\, dtds\right)^{\frac{1}{p}} \left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}} \\
=\;& \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is complete. □

Corollary 1. *Since $\frac{1}{4} < \frac{1}{(p+1)^{\frac{2}{p}}} < 1$, for $p > 1$, we have the following inequality;*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
\leq\;& \frac{(b-a)(d-c)}{4} \left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 5. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q \geq 1$, is quasi-convex function* *on the co-ordinates on* Δ*, then one has the inequality:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right].$$

Proof. From Lemma 1 and using Power Mean inequality, we can write

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}$$
$$\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dtds$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|\,dtds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is quasi-convex on the co-ordinates on Δ, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1|(1-2t)(1-2s)|\,dtds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|\,dtds\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)(d-c)}{16}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(c,d)\right|^q\right\}\right)^{\frac{1}{q}}$$

which completes the proof. □

Remark 1. *Since $\frac{1}{4} < \frac{1}{(p+1)^{\frac{2}{p}}} < 1$, for $p > 1$, the estimation in Theorem 4 is better than Theorem 3.*

Now, for a mapping $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ, we can define the mapping $G : [0,1]^2 \to \mathbb{R}$,

$$\begin{aligned} G(t,s) \quad : \quad &= \frac{1}{4}\left[f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \right. \\ &+ f\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\ &+ f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \\ &\left. + f\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \right] \end{aligned}$$

We will give following theorem which contains some properties of this mapping.

Theorem 6. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a,b] \times [c,d]$. Then:*

(i) The mapping G is convex on the co-ordinates on $[0,1]^2$.

(ii) We have the bounds

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) = G(0,0)$$

$$\sup_{(t,s)\in[0,1]^2} G(t,s) = \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} = G(1,1)$$

(iii) If f is satisfy Lipschitzian conditions, then the mapping G is $L-$Lipschitzian on $[0,1] \times [0,1]$.

(iv) Following inequality holds;

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ \leq \quad &\frac{1}{4}\left[\frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4} \right. \\ &\left. + \frac{f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)}{2} + f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right]. \end{aligned}$$

Proof. (i) Let $s \in [0,1]$. For all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, then we have

$$\begin{aligned}
& G(\alpha t_1 + \beta t_2, s) \\
= & \frac{1}{4}\Bigg[f\left((\alpha t_1 + \beta t_2)\, a + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left((\alpha t_1 + \beta t_2)\, b + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left((\alpha t_1 + \beta t_2)\, a + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \\
& + f\left((\alpha t_1 + \beta t_2)\, b + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \Bigg] \\
= & \frac{1}{4}\Bigg[f\left(\alpha\left(t_1 a + (1-t_1)\frac{a+b}{2} \right) + \beta\left(t_2 a + (1-t_2)\frac{a+b}{2} \right), sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(\alpha\left(t_1 b + (1-t_1)\frac{a+b}{2} \right) + \beta\left(t_2 b + (1-t_2)\frac{a+b}{2} \right), sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(\alpha\left(t_1 a + (1-t_1)\frac{a+b}{2} \right) + \beta\left(t_2 a + (1-t_2)\frac{a+b}{2} \right), sd + (1-s)\frac{c+d}{2} \right) \\
& + f\left(\alpha\left(t_1 b + (1-t_1)\frac{a+b}{2} \right) + \beta\left(t_2 b + (1-t_2)\frac{a+b}{2} \right), sd + (1-s)\frac{c+d}{2} \right) \Bigg].
\end{aligned}$$

Using the convexity of f, we obtain

$$\begin{aligned}
G(\alpha t_1 + \beta t_2, s) \leq & \frac{1}{4}\Bigg[\alpha\Bigg(f\left(t_1 a + (1-t_1)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_1 b + (1-t_1)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_1 a + (1-t_1)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_1 b + (1-t_1)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \Bigg) \\
& + \beta\Bigg(f\left(t_2 a + (1-t_2)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_2 b + (1-t_2)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_2 a + (1-t_2)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t_2 b + (1-t_2)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \Bigg) \Bigg] \\
= & \ \alpha G(t_1, s) + \beta G(t_2, s).
\end{aligned}$$

If $s \in [0,1]$. For all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, then we also have;

$$G(t, \alpha s_1 + \beta s_2) \leq \alpha G(t, s_1) + \beta G(t, s_2)$$

and the statement is proved.

(ii) It is easy to see that by taking $t=s=0$ and $t=s=1$, respectively, in G, we have the bounds

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = f\left(\frac{a+b}{2},\frac{c+d}{2}\right) = G(0,0)$$

$$\sup_{(t,s)\in[0,1]^2} G(t,s) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} = G(1,1).$$

(iii) Let $t_1,t_2,s_1,s_2\in[0,1]$, then we have

$$\begin{aligned}
&|G(t_2,s_2)-G(t_1,s_1)|\\
=\;& \frac{1}{4}\left|f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)\right|\\
&+f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)\\
&-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)\\
&-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)\Bigg|.
\end{aligned}$$

By using the triangle inequality, we get

$$\begin{aligned}
&|G(t_2,s_2)-G(t_1,s_1)|\\
\leq\;& \frac{1}{4}\left|f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)\right|.
\end{aligned}$$

By using the f is satisfy Lipschitzian conditions, then we obtain

$$\begin{aligned}
&\frac{1}{4}\left|f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2c+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2a+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1a+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)\right|\\
&\left|+f\left(t_2b+(1-t_2)\frac{a+b}{2}, s_2d+(1-s_2)\frac{c+d}{2}\right)-f\left(t_1b+(1-t_1)\frac{a+b}{2}, s_1d+(1-s_1)\frac{c+d}{2}\right)\right|\\
\leq\;& \frac{1}{4}\left[L_1(b-a)|t_2-t_1|+L_2(d-c)|s_2-s_1|+L_3(b-a)|t_2-t_1|+L_4(d-c)|s_2-s_1|\right.\\
&\left.+L_5(b-a)|t_2-t_1|+L_6(d-c)|s_2-s_1|+L_7(b-a)|t_2-t_1|+L_8(d-c)|s_2-s_1|\right]\\
=\;& \frac{1}{4}\left[(L_1+L_2+L_3+L_4)(b-a)|t_2-t_1|+(L_5+L_6+L_7+L_8)(d-c)|s_2-s_1|\right]
\end{aligned}$$

this imply that the mapping G is $L-$Lipschitzian on $[0,1]\times[0,1]$.

(iv) By using the convexity of G on $[0,1]\times[0,1]$, we have

$$\begin{aligned}
& f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) \\
& +f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\Bigg] \\
\leq\ & tsf(a,c)+t(1-s)f\left(a,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},c\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& +tsf(b,c)+t(1-s)f\left(b,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},c\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& +tsf(a,d)+t(1-s)f\left(a,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},d\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& +tsf(b,d)+t(1-s)f\left(b,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},d\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

By integrating both sides of the above inequality and by taking into account the change of the variables, we obtain

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
\leq\ & \frac{1}{4}\left[\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\right. \\
& \left.+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].
\end{aligned}$$

Which completes the proof. □

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