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GLOBAL COMPACTNESS FOR A CLASS OF QUASI-LINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We prove a global compactness result for Palais-Smale sequences associated with a class of quasi-linear elliptic equations on exterior domains.

1. INTRODUCTION AND MAIN RESULT

Let Ω be a smooth domain of $\mathbb{R}^N$ with a bounded complement and $N > p > m > 1$. The main goal of this paper is to obtain a global compactness result for the Palais-Smale sequences of the energy functional associated with the following quasi-linear elliptic equation

$$-\operatorname{div}(L_\xi(Du)) - \operatorname{div}(M_\xi(u, Du)) + M_s(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega, \tag{1.1}$$

where $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, meant as the completion of the space $\mathcal{D}(\Omega)$ of smooth functions with compact support, with respect to the norm $\|u\|_{W^{1,p}(\Omega)\cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$, having set $\|u\|_p := \|u\|_{W^{1,p}(\Omega)}$ and $\|u\|_m := \|Du\|_{L^m(\Omega)}$. We assume that V is a continuous function on Ω,

$$\lim_{|x|\to\infty} V(x) = V_\infty \quad \text{and} \quad \inf_{x\in\Omega} V(x) = V_0 > 0.$$

As known, lack of compactness may occur due to the lack of compact embeddings for Sobolev spaces on Ω and since the limiting equation on $\mathbb{R}^N$

$$-\operatorname{div}(L_\xi(Du)) - \operatorname{div}(M_\xi(u, Du)) + M_s(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

with $u \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, is invariant by translations. A particular case of (1.1) is

$$-\Delta_p u - \operatorname{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m}a'(u)|Du|^m + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega, \tag{1.3}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, for a suitable function $a \in C^1(\mathbb{R}; \mathbb{R}^+)$, or the even simpler case where a is constant, namely

$$-\Delta_p u - \Delta_m u + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega. \tag{1.4}$$

Since the pioneering work of Benci and Cerami [2] dealing with the case $L(\xi) = |\xi|^2/2$ and $M(s,\xi) \equiv 0$, many papers have been written on this subject, see for instance the bibliography of [12]. Quite recently, in [12], the case $L(\xi) = |\xi|^p/p$ and $M(s,\xi) \equiv 0$ was investigated. The main point in the present contribution is the fact that we allow, under suitable assumptions, a quasi-linear term $M(u, Du)$ depending on the unknown u itself. The typical tools exploited in [2, 12], in addition to the point-wise convergence of the gradients, are some decomposition (splitting) results both for the energy functional and for the equation, along a given bounded Palais-Smale sequence (u_n). To this regard, the explicit dependence on u in the term $M(u, Du)$ requires a rather careful analysis. In particular, we can handle it for

$$\nu|\xi|^m \le M(s,\xi) \le C|\xi|^m, \qquad p - 1 \le m < p - 1 + p/N.$$

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The restriction on m, together with the sign condition (1.9) provides, thanks to the presence of L, the needed a priori regularity on the weak limit of (u_n), see Theorems 3.2 and 3.4.
Besides the aforementioned motivations, which are of mathematical interest, it is worth pointing out that in recent years, some works have been devoted to quasi-linear operators with double homogeneity, which arise from several problems of Mathematical Physics. For instance, the reaction diffusion problem $u_t = -\mathrm{div}(\mathbb{D}(u)Du) + \ell(x,u)$, where $\mathbb{D}(u) = d_p|Du|^{p-2} + d_m|Du|^{m-2}$, $d_p > 0$ and $d_m > 0$, admitting a rather wide range of applications in biophysics [10], plasma physics [16] and in the study of chemical reactions [1]. In this framework, u typically describes a concentration and $\mathrm{div}(\mathbb{D}(u)Du)$ corresponds to the diffusion with a coefficient $\mathbb{D}(u)$, whereas $\ell(x,u)$ plays the rǒle of reaction and relates to source and loss processes. We refer the interested reader to [5] and to the reference therein. Furthermore, a model for elementary particles proposed by Derrick [9] yields to the study of standing wave solutions $\psi(x,t) = u(x)e^{\mathrm{i}\omega t}$ of the following nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \Delta_2\psi - b(x)\psi + \Delta_p\psi - V(x)|\psi|^{p-2}\psi + |\psi|^{\sigma-2}\psi = 0 \quad \text{in } \mathbb{R}^N,$$

for which we refer the reader e.g. to [3].

In order to state the first main result, assume $N > p > m \geq 2$ and

$$p - 1 \leq m < p - 1 + p/N, \qquad p < \sigma < p^*, \tag{1.5}$$

and consider the C^2 functions $L : \mathbb{R}^N \to \mathbb{R}$ and $M : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that both the functions $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s,\xi)$ are strictly convex and

$$\nu|\xi|^p \leq |L(\xi)| \leq C|\xi|^p, \quad |L_\xi(\xi)| \leq C|\xi|^{p-1}, \quad |L_{\xi\xi}(\xi)| \leq C|\xi|^{p-2}, \tag{1.6}$$

for all $\xi \in \mathbb{R}^N$. Furthermore, we assume

$$\nu|\xi|^m \leq M(s,\xi)| \leq C|\xi|^m, \qquad |M_s(s,\xi)| \leq C|\xi|^m, \qquad |M_\xi(s,\xi)| \leq C|\xi|^{m-1}, \tag{1.7}$$

$$|M_{ss}(s,\xi)| \leq C|\xi|^m, \qquad |M_{s\xi}(s,\xi)| \leq C|\xi|^{m-1}, \qquad |M_{\xi\xi}(s,\xi)| \leq C|\xi|^{m-2}, \tag{1.8}$$

for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and that the sign condition (cf. [14])

$$M_s(s,\xi)s \geq 0, \tag{1.9}$$

holds for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, $G : \mathbb{R} \to \mathbb{R}$ is a C^2 function with $G'(s) := g(s)$ and

$$|G'(s)| \leq C|s|^{\sigma-1}, \quad |G''(s)| \leq C|s|^{\sigma-2}, \tag{1.10}$$

for all $s \in \mathbb{R}$. We define

$$j(s,\xi) := L(\xi) + M(s,\xi) - G(s), \tag{1.11}$$

and on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with $\|u\|_{W^{1,p}(\Omega)\cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$ the functional

$$\phi(u) := \int_\Omega j(u,Du) + \int_\Omega V(x)\frac{|u|^p}{p}.$$

Finally, on $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)} = \|u\|_p + \|u\|_m$ we define

$$\phi_\infty(u) := \int_{\mathbb{R}^N} j(u,Du) + \int_{\mathbb{R}^N} V_\infty\frac{|u|^p}{p}.$$

See Section 2 for some properties of the functionals ϕ and ϕ_∞.
The first main global compactness type result is the following

Theorem 1.1. *Assume that (1.5)-(1.11) hold and let* $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *be a bounded sequence such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$$

Then, up to a subsequence, there exists a weak solution $v_0 \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *of*

$$-\mathrm{div}(L_\xi(Du)) - \mathrm{div}(M_\xi(u, Du)) + M_s(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega,$$

a finite sequence $\{v_1, ..., v_k\} \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ *of weak solutions of*

$$-\mathrm{div}(L_\xi(Du)) - \mathrm{div}(M_\xi(u, Du)) + M_s(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N$$

and k *sequences* $(y_n^i) \subset \mathbb{R}^N$ *satisfying*

$$|y_n^i| \to \infty, \quad |y_n^i - y_n^j| \to \infty, \quad i \neq j, \quad \text{as } n \to \infty,$$

$$\|u_n - v_0 - \sum_{i=1}^k v_i((\cdot - y_n^i)\|_{W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)} \to 0, \qquad \text{as } n \to \infty,$$

$$\|u_n\|_p^p \to \sum_{i=0}^k \|v_i\|_p^p, \qquad \|u_n\|_m^m \to \sum_{i=0}^k \|v_i\|_m^m, \qquad \text{as } n \to \infty,$$

as well as

$$\phi(v_0) + \sum_{i=1}^k \phi_\infty(v_i) = c.$$

Let us now come to a statement for the cases $1 < m \leq 2$ or $1 < p \leq 2$. Let us define

$$\mathfrak{L}(\xi, h) := \frac{|L_\xi(\xi + h) - L_\xi(\xi)|}{|h|^{p-1}}, \qquad \text{if } 1 < p < 2,$$

$$\mathfrak{G}(s, t) := \frac{|G'(s + t) - G'(s)|}{|t|^{\sigma-1}}, \qquad \text{if } 1 < \sigma < 2,$$

$$\mathfrak{M}(s, \xi, h) := \frac{|M_\xi(s, \xi + h) - M_\xi(s, \xi)|}{|h|^{m-1}}, \qquad \text{if } 1 < m < 2.$$

If either $p < 2$, $\sigma < 2$ or $m < 2$, we shall weaken the twice differentiability assumptions, by requiring $L_\xi \in C^1(\mathbb{R}^N \setminus \{0\})$, $G' \in C^1(\mathbb{R} \setminus \{0\})$, $M_\xi \in C^1(\mathbb{R} \times (\mathbb{R}^N \setminus \{0\}))$, $M_{s\xi} \in C^0(\mathbb{R} \times \mathbb{R}^N)$ and $M_{ss} \in C^0(\mathbb{R} \times \mathbb{R}^N)$. Moreover we assume the same growth conditions for L, M, G and their derivatives, replacing only the growth assumptions for $L_{\xi\xi}, M_{\xi\xi}, G''$ by the following hypotheses:

$$\sup_{h\neq 0,\, \xi\in\mathbb{R}^N} \mathfrak{L}(\xi, h) < \infty, \tag{1.12}$$

$$\sup_{t\neq 0,\, s\in\mathbb{R}} \mathfrak{G}(s, t) < \infty, \tag{1.13}$$

$$\sup_{h\neq 0,\, (s,\xi)\in\mathbb{R}\times\mathbb{R}^N} \mathfrak{M}(s, \xi, h) < \infty. \tag{1.14}$$

Conditions (1.12)-(1.13), in some more concrete situations, follow immediately by homogeneity of L_ξ and G' (see, for instance, [12, Lemma 3.1]). Similarly, (1.14) is satisfied for instance when M is of the form $M(s, \xi) = a(s)\mu(\xi)$, being $a : \mathbb{R} \to \mathbb{R}^+$ a bounded function and $\mu : \mathbb{R}^N \to \mathbb{R}^+$ a C^1 strictly convex function such that μ_ξ is homogeneous of degree $m - 1$.

Theorem 1.2. *Under the additional assumptions (1.12)-(1.14) in the sub-quadratic cases, the assertion of Theorem 1.1 holds true.*

As a consequence of the above results we have the following compactness criterion.

Corollary 1.3. *Assume* (2.1) *below for some* $\delta > 0$ *and* $\mu > p$. *Under the hypotheses of Theorem 1.1 or 1.2, if* $c < c^*$, *then* (u_n) *is relatively compact in* $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *where*

$$c^* := \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p}V_\infty\right\}\left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma-p}},$$

and $S_{p,\sigma}$ *and* C_g *are constants such that* $S_{p,\sigma}\|u\|_p^\sigma \geq \|u\|^\sigma_{L^\sigma(\mathbb{R}^N)}$ *and* $|g(s)| \leq C_g|s|^{\sigma-1}$.

Remark 1.4. It would be interesting to get a global compactness result in the case $L = 0$ and $p = m$, namely for the model case

$$-\operatorname{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m}a'(u)|Du|^m + V(x)|u|^{m-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega. \tag{1.15}$$

Notice that, even assuming a' bounded, $a'(u)|Du|^m$ is merely in $L^1(\Omega)$ for $W_0^{1,m}(\Omega)$ distributional solutions. In general, in this setting, the splitting properties of the equation are hard to formulate in a reasonable fashion.

Remark 1.5. The restriction of between m and p in assumption (1.5) is no longer needed in the case where M is independent of the first variable s, namely $M_s \equiv 0$.

Remark 1.6. We prove the above theorems under the a-priori boundedness assumption of (u_n). This occurs in a quite large class of problems, as Proposition 2.2 shows.

Remark 1.7. With no additional effort, we could cover the case where an additional term $W(x)|u|^{m-2}u$ appears in (1.1) and the functional framework turns into $W_0^{1,p}(\Omega) \cap W_0^{1,m}(\Omega)$.

In the spirit of [11], we also get the following

Corollary 1.8. *Let* $N > p \geq m > 1$ *and assume that* $\xi \mapsto L(\xi)$ *is* p*-homogeneous,* $\xi \mapsto M(\xi)$ *is* m*-homogeneous,* $L(\xi) \geq |\xi|^p$, $M(\xi) \geq |\xi|^m$ *(we put* $\nu = 1$*) and set*

$$\mathbb{S}_\Omega := \inf_{\|u\|_{L^\sigma(\Omega)}=1} \int_\Omega L(Du) + M(Du) + V(x)|u|^p, \tag{1.16}$$

$$\mathbb{S}_{\mathbb{R}^N} := \inf_{\|u\|_{L^\sigma(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} |Du|^p + |u|^p,$$

with $V(x) \to 1$ *as* $|x| \to \infty$. *Assume furthermore that*

$$\mathbb{S}_\Omega < \Big(\frac{\sigma - p}{\sigma - m}\frac{m}{p}\Big)^{\frac{\sigma-p}{\sigma}}\mathbb{S}_{\mathbb{R}^N}. \tag{1.17}$$

Then (1.16) *admits a minimizer.*

Remark 1.9. We point out that, some conditions guaranteeing the nonexistence of nontrivial solutions in the star-shaped case $\Omega = \mathbb{R}^N$ can be provided. For the sake of simplicity, assume that L is p-homogeneous and that $\xi \mapsto M(s,\xi)$ is m-homogeneous. Then, in view of [13, Theorem 3], that holds for C^1 solutions by virtue of the results of [8], we have that (1.1) admits no nontrivial C^1 solution well behaved at infinity, namely satisfying condition (19) of [13], provided that there exists a number $a \in \mathbb{R}^+$ such that a.e. in $\mathbb{R}^N$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$\begin{aligned}(N - p(a+1))L(\xi) + (N - m(a+1))M(s,\xi) + (asg(s) - NG(s))\\ + \frac{(N-ap)V(x) + x\cdot DV(x)}{p}|s|^p - aM_s(s,\xi)s \geq 0,\end{aligned}$$

holding, for instance, if there exists $0 \leq a \leq \frac{N-p}{p}$ such that

$$asg(s) - NG(s) \geq 0, \qquad (N-ap)V(x) + x\cdot DV(x) \geq 0, \qquad M_s(s,\xi)s \leq 0,$$

for a.e. $x \in \mathbb{R}^N$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Also, in the more particular case where $g(s) = |s|^{\sigma-2}s$ and $V(x) = V_\infty > 0$, then the above conditions simply rephrase into

$$\sigma \geq p^*, \qquad M_s(s,\xi)s \leq 0,$$

for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. In fact, in (1.9), we consider the opposite assumption on M_s.

2. Some preliminary facts

It is a standard fact that, under condition (1.6) and (1.10), the functionals

$$u \mapsto \int_\Omega L(Du), \qquad u \mapsto \int_\Omega V(x)|u|^p, \qquad u \mapsto \int_\Omega G(u)$$

are C^1 on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Analogously, although M depends explicitly on s, the functional

$$\mathbb{M} : W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \to \mathbb{R}, \quad \mathbb{M}(u) = \int_\Omega M(u, Du),$$

admits, thanks to condition (1.5), directional derivatives along any $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ and

$$\mathbb{M}'(u)(v) = \int_\Omega M_\xi(u,Du)\cdot Dv + \int_\Omega M_s(u,Du)v,$$

as it can be easily verified observing that $p \leq \frac{p}{p-m} \leq p^*$ and that, for $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, by Young's inequality, for some constant C it holds

$$\begin{aligned} |M_\xi(u,Du)\cdot Dv| &\leq C|Du|^m + C|Dv|^m \in L^1(\Omega), \\ |M_s(u,Du)v| &\leq C|Du|^p + C|v|^{\frac{p}{p-m}} \in L^1(\Omega). \end{aligned}$$

Furthermore, if $u_k \to u$ in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ as $k \to \infty$ then $\mathbb{M}'(u_k) \to \mathbb{M}'(u)$ in the dual space $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, as $k \to \infty$. Indeed, for $\|v\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \leq 1$, we have

$$\begin{aligned} &|\mathbb{M}'(u_k)(v) - \mathbb{M}'(u)(v)| \\ &\leq \int_\Omega |M_\xi(u_k,Du_k) - M_\xi(u,Du)||Dv| + \int_\Omega |M_s(u_k,Du_k) - M_s(u,Du)|\,|v| \\ &\leq \|M_\xi(u_k,Du_k) - M_\xi(u,Du)\|_{L^{m'}}\|Dv\|_{L^m} + \|M_s(u_k,Du_k) - M_s(u,Du)\|_{L^{p/m}}\|v\|_{L^{p/(p-m)}} \\ &\leq \|M_\xi(u_k,Du_k) - M_\xi(u,Du)\|_{L^{m'}} + \|M_s(u_k,Du_k) - M_s(u,Du)\|_{L^{p/m}}. \end{aligned}$$

This yields the desired convergence, using (1.7) and the Dominated Convergence Theorem. Notice that the same argument carried out before applies either to integrals defined on Ω or on $\mathbb{R}^N$. Hence the following proposition is proved.

Proposition 2.1. *In the hypotheses of Theorems 1.1 and 1.2, the functionals ϕ and ϕ_∞ are C^1.*

In addition to the assumptions on L, M and g, G set in the introduction, assume now that there exist positive numbers $\delta > 0$ and $\mu > p$ such that

$$\mu M(s,\xi) - M_s(s,\xi)s - M_\xi(s,\xi)\cdot\xi \geq \delta|\xi|^m, \quad \mu L(\xi) - L_\xi(\xi)\cdot\xi \geq \delta|\xi|^p, \quad sg(s) - \mu G(s) \geq 0, \tag{2.1}$$

for any $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$. This hypothesis is rather well established in the framework of quasi-linear problems (cf. [14]) and it allows an arbitrary Palais-Smale sequence (u_n) to be bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, as shown in the following

Proposition 2.2. *Let j be as in (1.11) and assume that (1.5) holds. Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ be a sequence such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$$

Then, if condition (2.1) holds, (u_n) is bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$.

Proof. Let $(w_n) \subset (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ with $w_n \to 0$ as $n \to \infty$ be such that $\phi'(u_n)(v) = \langle w_n, v\rangle$, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Whence, by choosing $v = u_n$, it follows

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Du_n + \int_\Omega j_s(u_n, Du_n)u_n + \int_\Omega V(x)|u_n|^p = \langle w_n, u_n\rangle.$$

Combining this equation with $\mu\phi(u_n) = \mu c + o(1)$ as $n \to \infty$, namely

$$\int_\Omega \mu j(u_n, Du_n) + \frac{\mu}{p}\int_\Omega V(x)|u_n|^p = \mu c + o(1),$$

recalling the definition of j, and using condition (2.1), yields

$$\frac{\mu - p}{p}\int_\Omega V(x)|u_n|^p + \delta\int_\Omega |Du_n|^p + \delta\int_\Omega |Du_n|^m \le \mu c + \|w_n\|\|u_n\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} + o(1),$$

as $n \to \infty$, implying, due to $V \ge V_0$ that

$$\|u_n\|^p_{W^{1,p}(\Omega)} + \|u_n\|^m_{D^{1,m}(\Omega)} \le C + C\|u_n\|_{W^{1,p}(\Omega)} + C\|u_n\|_{D^{1,m}(\Omega)} + o(1),$$

as $n \to \infty$. The assertion then follows immediately. □

From now on we shall always assume to handle *bounded* Palais-Smale sequences, keeping in mind that condition (2.1) can guarantee the boundedness of such sequences.

Proposition 2.3. *Let j be as in (1.11) and assume that $1 < m < p < N$ and $p < \sigma < p^*$. Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ bounded be such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*.$$

Then, up to a subsequence, (u_n) converges weakly to some u in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, $u_n(x) \to u(x)$ and $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$.

Proof. It is sufficient to justify that $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$. Given an arbitrary bounded subdomain $\omega \subset \overline{\omega} \subset \Omega$ of Ω, from the fact that $\phi'(u_n) \to 0$ in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, we can write

$$\int_\omega a(u_n, Du_n) \cdot Dv = \langle w_n, v\rangle + \langle f_n, v\rangle + \int_\omega v\, d\mu_n, \quad \text{for all } v \in \mathcal{D}(\omega),$$

where $(w_n) \subset (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ is vanishing, and hence in particular $w_n \in W^{-1,p'}(\omega)$, with $w_n \to 0$ in $W^{-1,p'}(\omega)$ as $n \to \infty$ and we have set

$$a_n(x, s, \xi) := L_\xi(\xi) + M_\xi(s, \xi), \qquad \text{for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$
$$f_n := -V(x)|u_n|^{p-2}u_n + g(u_n) \in W^{-1,p'}(\omega), \qquad n \in \mathbb{N},$$
$$\mu_n := -M_s(u_n, Du_n) \in L^1(\omega), \qquad n \in \mathbb{N}.$$

Due to the strict convexity assumptions on the maps $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s, \xi)$ and the growth conditions on L_ξ, M_ξ, M_s and g, all the assumptions of [6, Theorem 1] are fulfilled. Precisely,

$$|a_n(x, s, \xi)| \le |L_\xi(\xi)| + |M_\xi(s, \xi)| \le C|\xi|^{p-1} + C|\xi|^{m-1} \le C + C|\xi|^{p-1},$$

for a.e. $x \in \omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, and

$$f_n \to f, \quad f := -V(x)|u|^{p-2}u + g(u), \quad \text{strongly in } W^{-1,p'}(\omega),$$
$$\mu_n \rightharpoonup \mu, \quad \text{weakly* in } \mathcal{M}(\omega), \quad \text{since } \sup_{n\in\mathbb{N}} \|M_s(u_n, Du_n)\|_{L^1(\omega)} < +\infty.$$

Then, it follows that $Du_n(x) \to Du(x)$ for a.e. $x \in \omega$. Finally, a simple Cantor diagonal argument allows to recover the convergence over the whole domain Ω. □

Next we prove a regularity result for the solutions of equation (1.1).

Proposition 2.4. *Let j be as in (1.11) and assume (1.5) and (1.9). Let $u \in W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$ be a solution of (1.1). Then*

$$u \in \bigcap_{q\ge p} L^q(\Omega), \quad u \in L^\infty(\Omega) \text{ and } \lim_{|x|\to\infty} u(x) = 0.$$

Proof. Let $k, i \in \mathbb{N}$. Then, setting $v_{k,i}(x) := (u_k(x))^i$ with $u_k(x) := \min\{u^+(x), k\}$, it follows that $v_{k,i} \in W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$ can be used as a test function in (1.1), yielding

$$\int_\Omega L_\xi(Du)\cdot Dv_{k,i} + \int_\Omega M_\xi(u,Du)\cdot Dv_{k,i} + \int_\Omega M_s(u,Du)v_{k,i} + \int_\Omega V(x)|u|^{p-2}uv_{k,i} = \int_\Omega g(u)v_{k,i}.$$

Taking into account that $Dv_{k,i}$ is equal to $iu^{i-1}Du\chi_{\{0<u<k\}}$, by convexity and positivity of the map $\xi \mapsto M(s,\xi)$ we deduce that $M_\xi(u,Du)\cdot Dv_{k,i} \ge 0$. Moreover, by the sign condition (1.9) it follows $M_s(u,Du)v_{k,i} \ge 0$ a.e. in Ω. Then, we reach

$$\int_\Omega i(u_k)^{i-1}L_\xi(Du_k)\cdot Du_k + \int_\Omega V(x)|u|^{p-2}u(u_k(x))^i \le \int_\Omega g(u)(u_k(x))^i,$$

yielding in turn, by (1.10), that for all $k, i \ge 1$

$$\nu i \int_\Omega (u_k)^{i-1}|Du_k|^p \le C\int_\Omega (u^+(x))^{\sigma-1+i}. \tag{2.2}$$

If $\hat u_k := \min\{u^-(x), k\}$, a similar inequality

$$\nu i \int_\Omega (\hat u_k)^{i-1}|D\hat u_k|^p \le C\int_\Omega (u^-(x))^{\sigma-1+i}, \tag{2.3}$$

can be obtained by using $\hat v_{k,i} := -(\hat u_k)^i$ as a test function in (1.1), observing that by (1.9),

$$M_s(u,Du)\hat v_{k,i} = -M_s(u,Du)\chi_{\{-k<u<0\}}(-u)^i \ge 0,$$
$$M_\xi(u,Du)\cdot Dv_{k,i} = i(-u)^{i-1}\chi_{\{-k<u<0\}}M_\xi(u,Du)\cdot Du \ge 0.$$

Once (2.2)-(2.3) are reached, the assertion follows exactly as in [15, Lemma 2, (a) and (b)]. □

We now recall the following version of [7, Lemma 4.2] which turns out to be a rather useful tool in order to establish convergences in our setting. Roughly speaking, one needs some kind of sub-criticality in the growth conditions.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^N$ and $h : \Omega\times\mathbb{R}\times\mathbb{R}^N$ be a Carathéodory function, $p, m > 1$, $\mu \ge 1$, $p \le \sigma \le p^*$ and assume that, for every $\varepsilon > 0$ there exist $a_\varepsilon \in L^\mu(\Omega)$ such that*

$$|h(x,s,\xi)| \le a_\varepsilon(x) + \varepsilon|s|^{\sigma/\mu} + \varepsilon|\xi|^{p/\mu} + \varepsilon|\xi|^{m/\mu}, \tag{2.4}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$. Assume that $u_n \to u$ a.e. in Ω, $Du_n \to Du$ a.e. in Ω and

$$(u_n) \text{ is bounded in } W^{1,p}_0(\Omega), \quad (u_n) \text{ is bounded in } D^{1,m}_0(\Omega).$$

Then $h(x,u_n,Du_n)$ converges to $h(x,u,Du)$ in $L^\mu(\Omega)$.

Proof. The proof follows as in [7, Lemma 4.2] and we shall sketch it here for self-containedness. By Fatou's Lemma, it immediately holds that $u\in W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$. Furthermore, there exists a positive constant C such that

$$|h(x,s_1,\xi_1)-h(x,s_2,\xi_2)|^\mu \le C(a_\varepsilon(x))^\mu + C\varepsilon^\mu|s_1|^\sigma + C\varepsilon^\mu|s_2|^\sigma + C\varepsilon^\mu|\xi_1|^m + C\varepsilon^\mu|\xi_2|^m + C\varepsilon^\mu|\xi_1|^p + C\varepsilon^\mu|\xi_2|^p,$$

a.e. in Ω and for all $(s_1,\xi_1)\in\mathbb{R}\times\mathbb{R}^N$ and $(s_2,\xi_2)\in\mathbb{R}\times\mathbb{R}^N$. Then, taking into account the boundedness of (Du_n) in $L^p(\Omega)\cap L^m(\Omega)$ and of (u_n) in $L^\sigma(\Omega)$ by interpolation being $p\le\sigma\le p^*$, the assertion follows by applying Fatou's Lemma to the sequence of functions $\psi_n:\Omega\to[0,+\infty]$

$$\psi_n(x) := -\,|h(x,u_n,Du_n)-h(x,u,Du)|^\mu + C(a_\varepsilon(x))^\mu + C\varepsilon^\mu|u_n|^\sigma + C\varepsilon^\mu|u|^\sigma + C\varepsilon^\mu|Du_n|^m + C\varepsilon^\mu|Du|^m + C\varepsilon^\mu|Du_n|^p + C\varepsilon^\mu|Du|^p,$$

and, finally, exploiting the arbitrariness of ε. □

3. Proof of the result

3.1. **Energy splitting.** The next result allows to perform an energy splitting for the functional

$$J(u)=\int_\Omega j(u,Du),\quad u\in W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega),$$

along a bounded Palais-Smale sequence $(u_n)\subset W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$. The result is in the spirit of the classical Brezis-Lieb Lemma [4].

Lemma 3.1. *Let the integrand j be as in (1.11) and*

$$p-1\le m<p-1+p/N,\qquad p\le\sigma\le p^*.$$

Let $(u_n)\subset W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$ with $u_n\rightharpoonup u$, $u_n\to u$ a.e. in Ω and $Du_n\to Du$ a.e. in Ω. Then

$$\lim_{n\to\infty}\int_\Omega j(u_n-u,Du_n-Du)-j(u_n,Du_n)+j(u,Du)=0. \tag{3.1}$$

Proof. We shall apply Lemma 2.5 to the function

$$h(x,s,\xi):=j(s-u(x),\xi-Du(x))-j(s,\xi),\qquad \text{for a.e. } x\in\Omega \text{ and all } (s,\xi)\in\mathbb{R}\times\mathbb{R}^N.$$

Given $x\in\Omega$, $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$, consider the C^1 map $\varphi:[0,1]\to\mathbb{R}$ defined by setting

$$\varphi(t):=j(s-tu(x),\xi-tDu(x)),\quad \text{for all } t\in[0,1].$$

Then, for some $\tau\in[0,1]$ depending upon $x\in\Omega$, $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$, it holds

$$\begin{aligned}
h(x,s,\xi)&=\varphi(1)-\varphi(0)=\varphi'(\tau)\\
&=-j_s(s-\tau u(x),\xi-\tau Du(x))u(x)-j_\xi(s-\tau u(x),\xi-\tau Du(x))\cdot Du(x)\\
&=-L_\xi(\xi-\tau Du(x))\cdot Du(x)\\
&\quad -M_s(s-\tau u(x),\xi-\tau Du(x))u(x)\\
&\quad -M_\xi(s-\tau u(x),\xi-\tau Du(x))\cdot Du(x)+G'(s-\tau u(x))u(x).
\end{aligned}$$

Hence, for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, it follows that

$$\begin{aligned}
|h(x,s,\xi)| &\le |L_\xi(\xi - \tau Du(x))||Du(x)| + |M_s(s - \tau u(x), \xi - \tau Du(x))||u(x)| \\
&\quad + |M_\xi(s - \tau u(x), \xi - \tau Du(x))||Du(x)| + |G'(s - \tau u(x))||u(x)| \\
&\le C(|\xi|^{p-1} + |Du(x)|^{p-1})|Du(x)| + C(|\xi|^m + |Du(x)|^m)|u(x)| \\
&\quad + C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)| + C(|s|^{\sigma-1} + |u(x)|^{\sigma-1})|u(x)| \\
&\le \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + C_\varepsilon|u(x)|^{p/(p-m)} \\
&\quad + \varepsilon|\xi|^m + C_\varepsilon|Du(x)|^m + \varepsilon|s|^\sigma + C_\varepsilon|u(x)|^\sigma \\
&= a_\varepsilon(x) + \varepsilon|s|^\sigma + \varepsilon|\xi|^p + \varepsilon|\xi|^m,
\end{aligned}$$

where $a_\varepsilon : \Omega \to \mathbb{R}$ is defined a.e. by

$$a_\varepsilon(x) := C_\varepsilon|Du(x)|^p + C_\varepsilon|Du(x)|^m + C_\varepsilon|u(x)|^{p/(p-m)} + C_\varepsilon|u(x)|^\sigma.$$

Notice that, as $p - 1 \le m < p - 1 + p/N$ it holds $p \le p/(p-m) \le p^*$, yielding $u \in L^{p/(p-m)}(\Omega)$ and in turn, $a_\varepsilon \in L^1(\Omega)$. The assertion follows directly by Lemma 2.5 with $\mu = 1$. □

We have the following splitting result

Theorem 3.2. *Let the integrand j be as in (1.11) and*

$$p - 1 \le m \le p - 1 + p/N, \qquad p < \sigma < p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ is a bounded Palais-Smale sequence for ϕ at the level $c \in \mathbb{R}$ weakly convergent to some $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Then

$$\lim_{n\to\infty} \Big(\int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \Big) = c - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p},$$

namely

$$\lim_{n\to\infty} \phi_\infty(u_n - u) = c - \phi(u),$$

being u_n and u regarded as elements of $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ after extension to zero out of Ω.

Proof. In light of Proposition 2.3, up to a subsequence, (u_n) converges weakly to some function u in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, $u_n(x) \to u(x)$ and $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$. Also, recalling that by assumption $V(x) \to V_\infty$ as $|x| \to \infty$, we have [4, 17]

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n - u|^p - V_\infty|u_n - u|^p = 0, \tag{3.2}$$

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n - u|^p - V(x)|u_n|^p + V(x)|u|^p = 0. \tag{3.3}$$

Therefore, by virtue of Lemma 3.1, we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \phi_\infty(u_n - u) &= \lim_{n\to\infty} \Big(\int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \Big) \\
&= \lim_{n\to\infty} \Big(\int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V(x) \frac{|u_n - u|^p}{p} \Big) \\
&= \lim_{n\to\infty} \Big(\int_\Omega j(u_n, Du_n) + \int_\Omega V(x) \frac{|u_n|^p}{p} \Big) - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p} \\
&= \lim_{n\to\infty} \phi(u_n) - \phi(u) = c - \phi(u),
\end{aligned}$$

concluding the proof. □

Remark 3.3. In order to shed some light on the restriction (1.5) of m, it is readily seen that it is a sufficient condition for the following local compactness property to hold. Assume that ω is a smooth domain of $\mathbb{R}^n$ with finite measure. Then, if (u_h) is a bounded sequence in $W_0^{1,p}(\omega)$, there exists a subsequence (u_{h_k}) such that

$$\Upsilon(x, u_{h_k}, Du_{h_k}) \text{ converges strongly to some } \Upsilon_0 \text{ in } W^{-1,p'}(\omega),$$

where $\Upsilon(x,s,\xi) = g(s) - M_s(s,\xi) - V(x)|s|^{p-2}s$. In fact, taking into account the growth condition on g and M_s, this can be proved observing that, for every $\varepsilon > 0$, there exists C_ε such that

$$|\Upsilon(x,s,\xi)| \leq C_\varepsilon + \varepsilon|s|^{\frac{N(p-1)+p}{N-p}} + \varepsilon|\xi|^{p-1+p/N},$$

for a.e. $x \in \omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

3.2. **Equation splitting I (super-quadratic case).** We shall assume that $m, p \geq 2$ and that conditions (1.7)-(1.8) hold. The following Theorem 3.4 and the forthcoming Theorem 3.5 (see next subsection) are in the spirit of the Brezis-Lieb Lemma [4], in a dual framework. For the particular case

$$M(s,\xi) = 0 \quad \text{and} \quad L(\xi) = \frac{|\xi|^p}{p},$$

we refer the reader to [12].

Theorem 3.4. *Assume that* (1.5)-(1.11) *hold and that*

$$p - 1 \leq m < p - 1 + p/N, \qquad p < \sigma < p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *is such that* $u_n \rightharpoonup u$, $u_n \to u$ *a.e. in* Ω, $Du_n \to Du$ *a.e. in* Ω *and there is* (w_n) *in the dual space* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ *such that* $w_n \to 0$ *as* $n \to \infty$ *and, for all* $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$,

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v = \langle w_n, v\rangle. \tag{3.4}$$

Then $\phi'(u) = 0$. *Moreover, there exists a sequence* (ξ_n) *that goes to zero in* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, *such that*

$$\begin{aligned}\langle \xi_n, v\rangle := &\int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv \\ &- \int_\Omega j_s(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u, Du)v + \int_\Omega j_\xi(u, Du) \cdot Dv,\end{aligned} \tag{3.5}$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$.
Furthermore, there exists a sequence (ζ_n) *in* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ *such that*

$$\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v = \langle \zeta_n, v\rangle$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *and* $\zeta_n \to 0$ *as* $n \to \infty$, *namely* $\phi'_\infty(u_n - u) \to 0$ *as* $n \to \infty$.

Proof. Fixed some $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, let us define for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\begin{aligned}f_v(x,s,\xi) := &\, j_s(s - u(x), \xi - Du(x))v(x) \\ &+ j_\xi(s - u(x), \xi - Du(x)) \cdot Dv(x) - j_s(s,\xi)v(x) - j_\xi(s,\xi) \cdot Dv(x).\end{aligned}$$

In order to prove 3.5 we are going to show that

$$\lim_{n\to\infty} \sup_{\|v\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \leq 1} \Big| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du) \Big| = 0. \tag{3.6}$$

As it can be easily checked, there holds

$$
\begin{aligned}
-f_v(x,s,\xi) = &\int_0^1 j_{ss}(s-\tau u(x),\xi-\tau Du(x))u(x)v(x)d\tau \\
&+\int_0^1 j_{s\xi}(s-\tau u(x),\xi-\tau Du(x))\cdot[Du(x)v(x)+Dv(x)u(x)]d\tau \\
&+\int_0^1 [j_{\xi\xi}(s-\tau u(x),\xi-\tau Du(x))\,Du(x)]\cdot Dv(x)d\tau.
\end{aligned}
$$

Hence, by plugging the particular form of j in the above equation yields

$$-f_v(x,s,\xi) = a(x,s,\xi)v(x)+b(x,s)v(x)+c_1(x,s,\xi)\cdot Dv(x)+c_2(x,s,\xi)\cdot Dv(x)+d(x,\xi)\cdot Dv(x)$$

where

$$
\begin{aligned}
a(x,s,\xi) &:= \int_0^1 [M_{ss}(s-\tau u(x),\xi-\tau Du(x))u(x)+M_{s\xi}(s-\tau u(x),\xi-\tau Du(x))\cdot Du(x)]d\tau, \\
b(x,s) &:= -\int_0^1 G''(s-\tau u(x))u(x)d\tau, \\
c_1(x,s,\xi) &:= \int_0^1 M_{\xi s}(s-\tau u(x),\xi-\tau Du(x))u(x)d\tau, \\
c_2(x,s,\xi) &:= \int_0^1 M_{\xi\xi}(s-\tau u(x),\xi-\tau Du(x))\,Du(x)d\tau, \\
d(x,\xi) &:= \int_0^1 L_{\xi\xi}(\xi-\tau Du(x))\,Du(x)d\tau.
\end{aligned}
$$

We claim that, as $n\to\infty$, it holds

$$
\begin{aligned}
a(\cdot,u_n,Du_n) &\to a(\cdot,u,Du) && \text{in } L^{(p^*)'}(\Omega), \\
b(\cdot,u_n) &\to b(\cdot,u) && \text{in } L^{\sigma'}(\Omega), \\
c_1(\cdot,u_n,Du_n) &\to c_1(\cdot,u,Du) && \text{in } L^{p'}(\Omega), \\
c_2(\cdot,u_n,Du_n) &\to c_2(\cdot,u,Du) && \text{in } L^{m'}(\Omega), \\
d(\cdot,Du_n) &\to d(\cdot,Du) && \text{in } L^{p'}(\Omega).
\end{aligned}
$$

Then, using Hölder's inequality and the embeddings of $W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)$ into $L^\sigma(\Omega)$ and $L^{p^*}(\Omega)$ we obtain

$$
\begin{aligned}
\sup_{\|v\|_{W^{1,p}_0(\Omega)\cap D^{1,m}_0(\Omega)}\le 1}&\Big|\int_\Omega f_v(x,u_n,Du_n)-f_v(x,u,Du)\Big| \\
&\le C\|a(\cdot,u_n,Du_n)-a(\cdot,u,Du)\|_{L^{(p^*)'}(\Omega)} \\
&+C\|b(\cdot,u_n)-b(\cdot,u)\|_{L^{\sigma'}(\Omega)}, \\
&+C\|c_1(\cdot,u_n,Du_n)-c_1(\cdot,u,Du)\|_{L^{p'}(\Omega)}, \\
&+C\|c_2(\cdot,u_n,Du_n)-c_2(\cdot,u,Du)\|_{L^{m'}(\Omega)}, \\
&+C\|d(\cdot,Du_n)-d(\cdot,Du)\|_{L^{p'}(\Omega)},
\end{aligned}
$$

yielding the desired conclusion (3.6). It remains to prove the convergences we claimed above. For each term, we shall exploit Lemma 2.5. Since $m<p-1+p/N$, we can set

$$\alpha := \frac{m}{p^*-1}, \qquad \beta := \frac{pN}{pN-N+p-mN}$$

it follows $\beta > 0$ and $m < m + \alpha < p$. Young's inequality yields in turn

$$y^{(m+\alpha)/(p^*)'} \le C y^{m/(p^*)'} + C y^{p/(p^*)'}, \quad \text{for all } y \ge 0.$$

Since $\beta/(p^*)' > 1$ and $(m+\alpha)/(p^*)' > 1$, by the growths of M_{ss} and $M_{s\xi}$, we have

$$\begin{aligned}
|a(x,s,\xi)| &\le C(|\xi|^m + |Du(x)|^m)|u(x)| + C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)| \\
&\le \varepsilon|\xi|^{p/(p^*)'} + C_\varepsilon|u(x)|^{\beta/(p^*)'} + C_\varepsilon|Du(x)|^{p/(p^*)'} + \varepsilon|\xi|^{(m+\alpha)/(p^*)'} + C_\varepsilon|Du(x)|^{(m+\alpha)/(p^*)'} \\
&\le \varepsilon|\xi|^{p/(p^*)'} + \varepsilon|\xi|^{m/(p^*)'} + C_\varepsilon|u(x)|^{\beta/(p^*)'} + C_\varepsilon|Du(x)|^{p/(p^*)'} + C_\varepsilon|Du(x)|^{m/(p^*)'}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|b(x,s)| &\le C(|s|^{\sigma-2} + |u(x)|^{\sigma-2})|u(x)| \le \varepsilon|s|^{\sigma/\sigma'} + C_\varepsilon|u|^{\sigma/\sigma'}, \\
|c_1(x,s,\xi)| &\le C(|\xi|^{m-1} + |Du(x)|^{m-1})|u(x)| \\
&\le \varepsilon|\xi|^{p/p'} + C_\varepsilon|u(x)|^{p/((p-m)p')} + C_\varepsilon|Du(x)|^{p/p'}, \\
|c_2(x,s,\xi)| &\le C(|\xi|^{m-2} + |Du(x)|^{m-2})|Du(x)| \\
&\le \varepsilon|\xi|^{m/m'} + C_\varepsilon|Du(x)|^{m/m'}, \\
|d(x,\xi)| &\le C(|\xi|^{p-2} + |Du(x)|^{p-2})|Du(x)| \le \varepsilon|\xi|^{p/p'} + C_\varepsilon|Du(x)|^{p/p'}.
\end{aligned}$$

From the point-wise convergence of the gradients and the growth estimates of j_ξ, j_s and g that u is a week solutions to the problem, namely for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$

$$\int_\Omega L_\xi(Du)\cdot Dv + \int_\Omega M_\xi(u,Du)\cdot Dv + \int_\Omega M_s(u,Du)v + \int_\Omega V(x)|u|^{p-2}uv = \int_\Omega g(u)v. \tag{3.7}$$

To get this, recall that $v \in L^{(p/m)'}(\Omega)$ and the sequence $(M_s(u_n, Du_n))$ is bounded in $L^{p/m}(\Omega)$ and hence it converges weakly to $M_s(u, Du)$ in $L^{p/m}(\Omega)$. Thanks to Proposition 2.4 (recall that $\beta \ge p$ if and only if $m \ge p-2+p/N$ and this is the case since $m \ge p-1$), we have $L^\beta(\Omega)$. Hence,

$$u \in L^\sigma(\Omega) \cap L^{\frac{p}{p-m}}(\Omega) \cap L^\beta(\Omega),$$

being $p \le p/(p-m) < p^*$ and $p < \sigma < p^*$. By the previous inequalities the claim follows by Lemma 2.5 with the choice $\mu = (p^*)', \sigma', p', m'$ and p' respectively. Let us now recall a dual version of properties (3.2)-(3.3) (cf. [17]), namely there exist two sequences (μ_n) and (ν_n) in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ which converge to zero as $n \to \infty$ and such that

$$\begin{aligned}
\int_\Omega V_\infty|u_n - u|^{p-2}(u_n-u)v &= \int_\Omega V(x)|u_n-u|^{p-2}(u_n-u)v + \langle \nu_n, v\rangle, \\
\int_\Omega V(x)|u_n - u|^{p-2}(u_n-u)v &= \int_\Omega V(x)|u_n|^{p-2}u_n v - \int_\Omega V(x)|u|^{p-2}uv + \langle \mu_n, v\rangle,
\end{aligned}$$

for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Whence, by collecting (3.4), (3.5), (3.6), (3.7), we get

$$\begin{aligned}
&\int_\Omega j_\xi(u_n-u, Du_n-Du)\cdot Dv + \int_\Omega j_s(u_n-u, Du_n-Du)v + \int_\Omega V_\infty|u_n-u|^{p-2}(u_n-u)v \\
&= \int_\Omega j_\xi(u_n, Du_n)\cdot Dv + \int_\Omega j_s(u_n,Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v \\
&- \int_\Omega j_\xi(u,Du)\cdot Dv - \int_\Omega j_s(u,Du)v - \int_\Omega V(x)|u|^{p-2}uv + \langle \xi_n + \mu_n + \nu_n, v\rangle = \langle \zeta_n, v\rangle,
\end{aligned}$$

where $\langle \zeta_n, v\rangle := \langle w_n + \xi_n + \mu_n + \nu_n, v\rangle$ and $\zeta_n \to 0$ as $n \to \infty$. This concludes the proof. □

3.3. **Equation splitting II (sub-quadratic case).** We assume that (1.12)-(1.14) hold.

Theorem 3.5. *Assume (1.9), let the integrand j be as in (1.11) and $p \le 2$ or $m \le 2$ or $\sigma \le 2$,*

$$p-1 \le m < p-1+p/N, \qquad p<\sigma<p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ is such that $u_n \rightharpoonup u$, $u_n \to u$ a.e. in Ω, $Du_n \to Du$ a.e. in Ω and there exists (w_n) in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ such that $w_n \to 0$ as $n\to\infty$ and, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$,

$$\int_\Omega j_\xi(u_n,Du_n)\cdot Dv + \int_\Omega j_s(u_n,Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v = \langle w_n, v\rangle.$$

Then $\phi'(u)=0$. Moreover, there exists a sequence $(\hat{\xi}_n)$ that goes to zero in $(W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega))^*$, such that

$$\begin{aligned}(3.8)\qquad \langle \hat{\xi}_n, v\rangle &:= \int_\Omega j_s(u_n-u,Du_n-Du)v + \int_\Omega j_\xi(u_n-u,Du_n-Du)\cdot Dv \\ &- \int_\Omega j_s(u_n,Du_n)v - \int_\Omega j_\xi(u_n,Du_n)\cdot Dv + \int_\Omega j_s(u,Du)v + \int_\Omega j_\xi(u,Du)\cdot Dv,\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$.
Furthermore, there exists a sequence $(\hat{\zeta}_n)$ in $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ with

$$\int_\Omega j_\xi(u_n-u,Du_n-Du)\cdot Dv + \int_\Omega j_s(u_n-u,Du_n-Du)v + \int_\Omega V_\infty|u_n-u|^{p-2}(u_n-u)v = \langle \hat{\zeta}_n, v\rangle$$

for all $v \in W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ and $\hat{\zeta}_n \to 0$ as $n\to\infty$, namely $\phi'_\infty(u_n-u)\to 0$ as $n\to\infty$.

Proof. Keeping in mind the argument in proof of Theorem 3.4, here we shall be more sketchy. For every $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ we plug L, M, G into the equation

$$\begin{aligned}f_v(x,s,\xi) &= j_s(s-u(x),\xi-Du(x))v(x) \\ &\quad + j_\xi(s-u(x),\xi-Du(x))\cdot Dv(x) - j_s(s,\xi)v(x) - j_\xi(s,\xi)\cdot Dv(x),\end{aligned}$$

thus obtaining

$$\begin{aligned}f_v(x,s,\xi) &= (M_s(s-u(x),\xi-Du(x)) - M_s(s,\xi))v(x) - (G'(s-u(x)) - G'(s))v(x) \\ &\quad + (M_\xi(s-u(x),\xi-Du(x)) - M_\xi(s,\xi))\cdot Dv(x) + (L_\xi(\xi-Du(x)) - L_\xi(\xi))\cdot Dv(x) \\ &= a'v(x) + b'v(x) + c'\cdot Dv(x) + d'\cdot Dv(x).\end{aligned}$$

We write the term $M_\xi(s-u(x),\xi-Du(x)) - M_\xi(s,\xi)$ in a more suitable form, namely

$$\begin{aligned}c' &= M_\xi(s-u(x),\xi-Du(x)) - M_\xi(s,\xi) \\ &= \underbrace{M_\xi(s-u(x),\xi-Du(x)) - M_\xi(s,\xi-Du(x))}_{c_1'(x,s,\xi)} + \underbrace{M_\xi(s,\xi-Du(x)) - M_\xi(s,\xi)}_{c_2'(x,s,\xi)},\end{aligned}$$

so that

$$f_v(x,s,\xi) = a'(x,s,\xi)v(x) + b'(x,s)v(x) + (c_1'(x,s,\xi)+c_2'(x,s,\xi))\cdot Dv(x) + d'(x,\xi)\cdot Dv(x).$$

The term a' admits the same growth condition of a, cf. the proof of Theorem 3.4. Also, since

$$c_1'(x,s,\xi) = -\int_0^1 M_{\xi s}(s-\tau u(x),\xi-Du(x))u(x)d\tau,$$

as for the term c_1 in the proof of Theorem 3.4 we obtain

$$|c_1'(x,s,\xi)| \le \varepsilon|\xi|^{p/p'} + C_\varepsilon|u(x)|^{p/((p-m)p')} + C_\varepsilon|Du(x)|^{p/p'}.$$

On the other hand, directly from assumptions (1.12)-(1.14) we get

$$|b'(x,s)| \leq C|u(x)|^{\sigma/\sigma'}, \quad |c_2'(x,s,\xi)| \leq C|Du(x)|^{m/m'}, \quad |d'(x,\xi)| \leq C|Du(x)|^{p/p'}.$$

The conclusion follows then by the same argument carried out in Theorem 3.4. □

In the spirit of [17, Lemma 8.3], we have the following

Lemma 3.6. *Under the hypotheses of Theorem 1.1 or 1.2, let $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to \infty$,*

$$\begin{aligned} &u_n(\cdot + y_n) \rightharpoonup u \qquad \text{in } W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N), \\ &u_n(\cdot + y_n) \to u \qquad \text{a.e. in } \mathbb{R}^N, \\ &Du_n(\cdot + y_n) \to Du \qquad \text{a.e. in } \mathbb{R}^N, \\ &\phi_\infty(u_n) \to c, \\ &\phi'_\infty(u_n) \to 0 \qquad \text{in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*. \end{aligned}$$

Then $\phi'_\infty(u) = 0$ and, setting $v_n := u_n - u(\cdot - y_n)$, we have

$$\phi_\infty(v_n) \to c - \phi_\infty(u) \tag{3.9}$$

$$\phi'_\infty(v_n) \to 0 \quad \text{in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*, \tag{3.10}$$

and $\|v_n\|_p^p = \|u_n\|_p^p - \|u\|_p^p + o(1)$ and $\|v_n\|_m^m = \|u_n\|_m^m - \|u\|_m^m + o(1)$ as $n \to \infty$.

Proof. The energy splitting (3.9) follows by Theorem 3.2 applied with $\Omega = \mathbb{R}^N$ and the sequence (u_n) replaced by $(u_n(\cdot + y_n))$. Take now $\varphi \in \mathcal{D}(\Omega)$ with $\|\varphi\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \leq 1$ and define $\varphi_n := \varphi(\cdot + y_n)$. Then $\varphi_n \in \mathcal{D}(\Omega_n)$, where $\Omega_n = \Omega - \{y_n\} \subset \Omega$ for n large. For any $n \in \mathbb{N}$, we get

$$\langle \phi'_\infty(v_n), \varphi\rangle = \langle \phi'_\infty(u_n(\cdot + y_n) - u), \varphi_n\rangle.$$

By the splitting argument in the proof of Theorem 3.4, it follows that

$$\langle \phi'_\infty(u_n(\cdot + y_n) - u), \varphi_n\rangle = \langle \phi'_\infty(u_n(\cdot + y_n)), \varphi_n\rangle - \langle \phi'_\infty(u), \varphi_n\rangle + \langle \zeta_n, \varphi_n\rangle,$$

where $\zeta_n \to 0$ in the dual of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. If we prove that u is critical for ϕ_∞, then the right-hand side reads as $\langle \phi'_\infty(u_n), \varphi\rangle + \langle \zeta_n, \varphi_n\rangle$, and also the second limit (3.10) follows. To prove that $\phi'_\infty(u) = 0$ we observe that, for all φ in $\mathcal{D}(\mathbb{R}^N)$,

$$\langle \phi'_\infty(u_n(\cdot + y_n)), \varphi\rangle \to \langle \phi'_\infty(u), \varphi\rangle, \quad |\langle \phi'_\infty(u_n(\cdot + y_n)), \varphi\rangle| \leq \|\phi'_\infty(u_n)\|_* \|\varphi\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \to 0.$$

Indeed, defining $\hat\varphi_n := \varphi(\cdot - y_n)$, since $|y_n| \to \infty$ as $n \to \infty$, we have $\operatorname{supp} \hat\varphi_n \subset \Omega$, for n large enough and $\|\hat\varphi_n\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} = \|\varphi\|_{W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)}$. The last assertion follows by using Brezis-Lieb Lemma [4]. □

We can finally come to the proof of the main results.

4. Proof of Theorems 1.1 and 1.2 completed

We follow the scheme of the proof given in [17, p.121]. Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ be a bounded Palais-Smale sequence for ϕ at the level $c \in \mathbb{R}$. Hence, there exists a sequence (w_n) in the dual of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ such that $w_n \to 0$ and $\phi(u_n) \to c$ as $n \to \infty$ and, for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, we have

$$\int_\Omega L_\xi(Du_n) \cdot Dv + \int_\Omega M_\xi(u_n, Du_n) \cdot Dv + \int_\Omega M_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v = \int_\Omega g(u_n)v + \langle w_n, v\rangle.$$

Since (u_n) is bounded in $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$, up to a subsequence, it converges weakly to some function $v_0\in W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ and, by virtue of Proposition 2.3, (u_n) and (Du_n) converge to v_0 and Dv_0 a.e. in Ω, respectively. In turn (see also the proof of Theorem 3.4) it follows

$$\int_\Omega L_\xi(Dv_0)\cdot Dv+\int_\Omega M_\xi(v_0,Dv_0)\cdot Dv+\int_\Omega M_s(v_0,Dv_0)v+\int_\Omega V(x)|v_0|^{p-2}v_0v=\int_\Omega g(v_0)v,$$

for any $v\in W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$. By combining Theorem 3.2 and Theorem 3.4, setting $u_n^1:=u_n-v_0$ and thinking the functions on $\mathbb{R}^N$ after extension to zero out of Ω, get

$$\phi_\infty(u_n^1)\to c-\phi(v_0),\quad n\to\infty, \tag{4.1}$$

$$\begin{aligned}\int_{\mathbb{R}^N} L_\xi(Du_n^1)\cdot Dv+\int_{\mathbb{R}^N} M_\xi(u_n^1,Du_n^1)\cdot Dv+\int_{\mathbb{R}^N} M_s(u_n^1,Du_n^1)v\\ +\int_{\mathbb{R}^N} V_\infty|u_n^1|^{p-2}u_n^1v=\int_{\mathbb{R}^N} g(u_n^1)v+\langle w_n^1,v\rangle.\end{aligned} \tag{4.2}$$

where (w_n^1) is a sequence in the dual of $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ with $w_n^1\to 0$ as $n\to\infty$. In turn, it follows that (u_n^1) is Palais-Smale sequence for ϕ_∞ at the energy level $c-\phi(v_0)$. In addition,

$$\|u_n^1\|_p^p=\|u_n\|_p^p-\|v_0\|_p^p+o(1),\qquad \|u_n^1\|_m^m=\|u_n\|_m^m-\|v_0\|_m^m+o(1),\quad \text{as } n\to\infty,$$

by the Brezis-Lieb Lemma [4]. Let us now define

$$\varpi:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|u_n^1|^p.$$

If it is the case that $\varpi=0$, then, according to [11, Lemma I.1], (u_n^1) converges to zero in $L^r(\mathbb{R}^N)$ for every $r\in(p,p^*)$. Then, one obtains that

$$\lim_{n\to\infty}\int_\Omega g(u_n^1)u_n^1=0,\qquad \int_\Omega M_s(u_n^1,Du_n^1)u_n^1\geq 0,$$

where the inequality follows by the sign condition (1.9). In turn, testing equation (4.2) with $v=u_n^1$, by the coercivity and convexity of $\xi\mapsto L(\xi),M(s,\xi)$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\Big[\nu\int_{\mathbb{R}^N}|Du_n^1|^p+\nu\int_{\mathbb{R}^N}|Du_n^1|^m+V_\infty\int_{\mathbb{R}^N}|u_n^1|^p\Big]\\ &\leq\limsup_{n\to\infty}\Big[\int_{\mathbb{R}^N}L_\xi(Du_n^1)\cdot Du_n^1+\int_{\mathbb{R}^N}M_\xi(u_n^1,Du_n^1)\cdot Du_n^1+\int_{\mathbb{R}^N}V_\infty|u_n^1|^p\Big]\leq 0,\end{aligned}$$

yielding that (u_n^1) strongly converges to zero in $W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)$, concluding the proof in this case. If, on the contrary, it holds $\varpi>0$, then, there exists an unbounded sequence $(y_n^1)\subset\mathbb{R}^N$ with $\int_{B(y_n^1,1)}|u_n^1|^p>\varpi/2$. Whence, let us consider $v_n^1:=u_n^1(\cdot+y_n^1)$, which, up to a subsequence, converges weakly and pointwise to some $v_1\in W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)$, which is nontrivial, due to the inequality $\int_{B(0,1)}|v_1|^p\geq\varpi/2$. Notice that, of course,

$$\lim_{n\to\infty}\phi_\infty(v_n^1)=\lim_{n\to\infty}\phi_\infty(u_n^1)=c-\phi(v_0).$$

Moreover, since $|y_n^1| \to \infty$ and Ω is an exterior domain, for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we have $\varphi(\cdot - y_n^1) \in \mathcal{D}(\Omega)$ for $n \in \mathbb{N}$ large enough. Whence, in light of equation (4.2), for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we get

$$\begin{aligned}&\int_{\mathbb{R}^N} L_\xi(Dv_n^1)\cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1)\cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi \\ &+ \int_{\mathbb{R}^N} V_\infty |v_n^1|^{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \int_{\mathbb{R}^N} L_\xi(Du_n^1)\cdot D\varphi(\cdot - y_n^1) \\ &+ \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1)\cdot D\varphi(\cdot - y_n^1) + \int_{\mathbb{R}^N} M_s(u_n^1, Du_n^1)\varphi(\cdot - y_n^1) + \int_{\mathbb{R}^N} V_\infty |u_n^1|^{p-2}(u_n^1)\varphi(\cdot - y_n^1) \\ &- \int_{\mathbb{R}^N} g(u_n^1)\varphi(\cdot - y_n^1) = \langle w_n^1, \varphi(\cdot + y_n^1)\rangle.\end{aligned}$$

Defining the form $\langle \hat{w}_n^1, \varphi\rangle := \langle w_n^1, \varphi(\cdot - y_n^1)\rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} L_\xi(Dv_n^1)\cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1)\cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi \\ + \int_{\mathbb{R}^N} V_\infty |v_n^1|^{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \langle \hat{w}_n^1, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).\end{aligned}$$

Since $(\hat{w}_n^1)$ converges to zero in the dual of $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, it follows by Proposition 2.3 (with $V = V_\infty$ and $\Omega = \mathbb{R}^N$) that the gradients Dv_n^1 converge point-wise to Dv_1, namely

$$Dv_n^1(x) \to Dv_1(x), \qquad \text{a.e. in } \mathbb{R}^N. \tag{4.3}$$

Setting $u_n^2 := u_n^1 - v_1(\cdot - y_n^1)$, in light of (4.1)-(4.2) and (4.3), we can apply Lemma 3.6 to the sequence (v_n^1), getting

$$\lim_{n\to\infty} \phi_\infty(u_n^2) = c - \phi(v_0) - \phi_\infty(v_1),$$

as well as $\phi_\infty(v_1) = 0$ and, furthermore, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} L_\xi(Du_n^2)\cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^2, Du_n^2)\cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^2, Du_n^2)v \\ &+ \int_{\mathbb{R}^N} V_\infty |u_n^2|^{p-2}u_n^2 v - \int_{\mathbb{R}^N} g(u_n^2)v = \langle \zeta_n^2, v\rangle,\end{aligned}$$

where (ζ_n^2) goes to zero in the dual of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. In turn, $(u_n^2) \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ is a Palais-Smale sequence for ϕ_∞ at the energy level $c - \phi(v_0) - \phi(v_1)$. Arguing on (u_n^2) as it was done for (u_n^1), either u_n^2 goes to zero strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ or we can generate a new (u_n^3). By iterating the above procedure, one obtains diverging sequences (y_n^i), $i = 1, \dots, k-1$, solutions v_i on $\mathbb{R}^N$ to the limiting problem, $i = 1, \dots, k-1$ and a sequence

$$u_n^k = u_n - v_0 - v_1(\cdot - y_n^1) - v_2(\cdot - y_n^2) - \cdots - v_{k-1}(\cdot - y_n^{k-1}),$$

such that (recall again Lemma 3.6) as $n \to \infty$

$$\begin{aligned}\|u_n^k\|_p^p &= \|u_n\|_p^p - \|v_0\|_p^p - \|v_1\|_p^p - \cdots - \|v_{k-1}\|_p^p + o(1), \\ \|u_n^k\|_m^m &= \|u_n\|_m^m - \|v_0\|_m^m - \|v_1\|_m^m - \cdots - \|v_{k-1}\|_m^m + o(1),\end{aligned} \tag{4.4}$$

as well as $\phi'_\infty(u_n^k) \to 0$ in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ and

$$\phi_\infty(u_n^k) \to c - \phi(v_0) - \sum_{j=1}^{k-1} \phi_\infty(v_j).$$

Notice that the iteration is forced to end up after a finite number $k \geq 1$ of steps. Indeed, for every nontrivial critical point $v \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ of ϕ_∞ we have,

$$\int_{\mathbb{R}^N} L_\xi(Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(v, Dv) \cdot Dv + \int_{\mathbb{R}^N} M_s(v, Dv)v + \int_{\mathbb{R}^N} V_\infty |v|^p = \int_{\mathbb{R}^N} g(v)v,$$

yielding by the sign condition, the coercivity-convexity conditions and the growth of g,

$$\min\{\nu, V_\infty\}\|v\|_p^p + \|Dv\|^m_{L^m(\mathbb{R}^N)} \leq C_g \|v\|^\sigma_{L^\sigma(\mathbb{R}^N)} \leq C_g S_{p,\sigma} \|v\|_p^\sigma, \tag{4.5}$$

so that, due to $\sigma > p$, it holds

$$\|v\|_p^p \geq \left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}} =: \Gamma_\infty > 0, \tag{4.6}$$

thus yielding from (4.4)

$$\|u_n^k\|_p^p \leq \|u_n\|_p^p - \|v_0\|_p^p - (k-1)\Gamma_\infty + o(1).$$

By boundedness of (u_n), k has to be finite. Hence $u_n^k \to 0$ strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ at some finite index $k \in \mathbb{N}$. This concludes the proof. □

5. Proof of Corollary 1.3

As a byproduct of the proof of the Theorems 1.1 and 1.2, since the p norm is bounded away from zero on the set of nontrivial critical points of ϕ_∞, cf. (4.5),we can estimate ϕ_∞ from below on that set. In order to do so, we use condition (2.1). For any nontrivial critical point of the functional ϕ_∞, we have (see the proof of Proposition 2.2)

$$\mu\phi_\infty(v) \geq \delta \int_\Omega |Dv|^p + \frac{\mu - p}{p} V_\infty \int_{\mathbb{R}^N} |v|^p \geq \min\left\{\delta, \frac{\mu - p}{p} V_\infty\right\} \|v\|_p^p.$$

An analogous argument applies to ϕ, yielding for any nontrivial critical point

$$\mu\phi(u) \geq \delta \int_\Omega |Du|^p + \frac{\mu - p}{p} V_0 \int_\Omega |u|^p \geq \min\left\{\delta, \frac{\mu - p}{p} V_0\right\} \|u\|_p^p.$$

Now notice that, recalling (4.6) and a similar variant for the norm of the critical points of ϕ in place of ϕ_∞, setting also

$$e_\infty := \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_\infty\right\}\Gamma_\infty, \quad e_0 := \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_0\right\}\Gamma_0, \quad \Gamma_0 := \left[\frac{\min\{\nu, V_0\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}} > 0,$$

from Theorems 1.1 or 1.2 we have $c \geq \ell e_0 + k e_\infty$ for some $\ell \in \{0, 1\}$ and non-negative integer k. Condition $c < c^* := e_\infty$ implies necessarily $k < 1$, namely $k = 0$. This provides the desired compactness result, using Theorems 1.1 or 1.2. □

6. Proof of Corollary 1.8

Defining the functionals $J, M : W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{p}\int_\Omega L(Du) + \frac{1}{m}\int_\Omega M(Du) + \frac{1}{p}\int_\Omega V(x)|u|^p, \qquad Q(u) := \frac{\mathbb{S}_\Omega}{\sigma}\int_\Omega |u|^\sigma,$$

and given a minimization sequence (u_n) for problem (1.16), by Ekeland's variational principle, without loss of generality we can replace it by a new minimization sequence, still denoted by (u_n) for which there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$

$$J'(u_n)(v) - \lambda_n Q'(u_n)(v) = \langle w_n, v\rangle, \quad \text{with } w_n \to 0 \text{ in the dual of } W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega).$$

Taking into account the homogeneity of L and M, choosing $v = u_n$ this means

$$\int_\Omega L(Du_n) + \int_\Omega M(Du_n) + \int_\Omega V(x)|u_n|^p - \mathbb{S}_\Omega \lambda_n \int_\Omega |u_n|^\sigma = \langle w_n, u_n \rangle.$$

Since $\|u_n\|_{L^\sigma(\Omega)=1}$ for all n and $\int_\Omega L(Du_n) + M(Du_n) \to \mathbb{S}_\Omega$ as $n \to \infty$, this means that (u_n) is a Palais-Smale sequence for the functional $I(u) := J(u) - Q(u)$ at an energy level

$$c \leq \frac{\sigma - m}{\sigma m} \mathbb{S}_\Omega, \tag{6.1}$$

since it holds (recall that $p \geq m$), as $n \to \infty$,

$$\begin{aligned} I(u_n) &= \frac{1}{p}\int_\Omega L(Du_n) + \frac{1}{m}\int_\Omega M(Du_n) + \frac{1}{p}\int_\Omega V(x)|u_n|^p - \frac{\mathbb{S}_\Omega}{\sigma} \\ &\leq \frac{1}{m}\int_\Omega L(Du_n) + \frac{1}{m}\int_\Omega M(Du_n) + \frac{1}{m}\int_\Omega V(x)|u_n|^p - \frac{\mathbb{S}_\Omega}{\sigma} = \Big(\frac{1}{m} - \frac{1}{\sigma}\Big)\mathbb{S}_\Omega + o(1). \end{aligned}$$

From Corollary 1.3 (applied with $L(Du)$ replaced by $L(Du)/p$, $M(u, Du)$ replaced by $M(Du)/m$ and $G \equiv 0$), the compactness of (u_n) holds provided that (in the notations of Corollary 1.3)

$$c < \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_\infty\right\} \left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}}.$$

In our case, we can take $\mu = \sigma$, $\delta = \frac{\sigma - p}{p}$, $C_g = \mathbb{S}_\Omega$, $V_\infty = 1$, $\nu = 1$, $S_{p,\sigma} = \mathbb{S}_{\mathbb{R}^N}^{-\sigma/p}$, yielding

$$c < \frac{\sigma - p}{\sigma p} \mathbb{S}_{\mathbb{R}^N}^{\frac{\sigma}{\sigma - p}} / \mathbb{S}_\Omega^{\frac{p}{\sigma - p}}.$$

Hence, finally, by combining this conclusion with (6.1) the compactness (and in turn the solvability of the minimization problem) holds if (1.17) holds, concluding the proof.

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