INTEGRALITY PROPERTIES OF THE CM-VALUES OF CERTAIN WEAK MAASS FORMS

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Abstract. In a recent paper, Bruinier and Ono prove that the coefficients of certain weight $-1/2$ harmonic Maass forms are traces of singular moduli for weak Maass forms. In particular, for the partition function $p(n)$, they prove that

$$p(n) = \frac{1}{24n-1} \cdot \sum P(\alpha_Q),$$

where P is a weak Maass form and α_Q ranges over a finite set of discriminant $-24n+1$ CM points. Moreover, they show that $6 \cdot (24n-1) \cdot P(\alpha_Q)$ is always an algebraic integer, and they conjecture that $(24n-1) \cdot P(\alpha_Q)$ is always an algebraic integer. Here we prove a general theorem which implies this conjecture as a corollary.

1. Introduction and Statement of Results

A *partition* of a positive integer n is any nonincreasing sequence of positive integers which sum to n. The partition function $p(n)$, which counts the number of partitions of n, is an important function in number theory whose study has a long history. One of the celebrated results of Hardy and Ramanujan on this function, giving rise to the "circle" method, quantifies the growth rate:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

This asymptotic and its method of proof were later refined by Rademacher, yielding an "exact" formula in terms of a modified Bessel function of the first kind $I_{\frac{3}{2}}(\cdot)$ and a Kloosterman sum $A_k(n)$:

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

One can compute values of $p(n)$ from this formula by using sufficiently accurate truncations. Bounding the resulting error is a well-known difficult problem; the best-known bounds are due to Folsom and Masri [4].

In recent work [2], Bruinier and Ono prove a new formula for $p(n)$ as a finite sum of algebraic numbers. These numbers are *singular moduli* for a *weak Maass form* which they describe in terms of Dedekind's eta function and the quasimodular Eisenstein series E_2, which are defined in terms of $q := e^{2\pi iz}$ as

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad \text{and} \quad E_2(z) := 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n.$$

They then define the $\Gamma_0(6)$ weight -2 meromorphic modular form:

$$F_p(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q - \cdots \tag{1}$$

Using the convention that $z := x + iy$, with $x, y \in \mathbb{R}$, they define the weak Maass form:

$$P_p(z) := -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F_p(z) = \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y} q\right) + \cdots \tag{2}$$

Bruinier and Ono give a formula for $p(n)$ in terms of discriminant $-24n+1 = b^2 - 4ac$ positive definite integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ satisfying the condition $6 \mid a$. The group $\Gamma_0(6)$ acts on such forms, and we let $\mathcal{Q}_n$ be any set of representatives of those equivalence classes with $a > 0$ and $b \equiv 1 \pmod{12}$. To each such Q, we associate the CM point α_Q defined to be the root of $Q(x,1) = 0$ lying in the upper half of the complex plane. Then the formula of Bruinier and Ono states:

$$p(n) = \frac{1}{24n - 1} \cdot \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q). \tag{3}$$

They further prove that each $6 \cdot (24n - 1) \cdot P_p(\alpha_Q)$ is an algebraic integer. They also show that the numbers $P(\alpha_Q)$, as Q varies over $\mathcal{Q}_n$, form a multiset which is a union of Galois orbits for the discriminant $-24n + 1$ ring class field. Based on numerics, they made the following conjecture:

Conjecture (Bruinier and Ono [2])**.** *For the Maass form $P(z)$ above and for the α_Q in the formula for $p(n)$, we have that $(24n - 1) \cdot P_p(\alpha_Q)$ is an algebraic integer.*

We prove that this is indeed the case. In fact, we prove that this is the true for all the CM points of discriminant $-24n + 1$ of a wider class of Maass forms. Namely, we have the following:

Theorem 1.1. *Suppose F is a weakly holomorphic, weight -2 modular form on a congruence subgroup such that the Fourier expansions of*

$$F \quad \textit{and} \quad q\frac{dF}{dq} + F \cdot \frac{E_2E_4 - E_6}{6E_4}$$

at all cusps have coefficients that are algebraic integers. Let α_Q be the CM point in $\mathbb{H}$ corresponding to a quadratic form $Q(x,y)$ of discriminant $-24n+1$, and let $P(z)$ be the weak Maass form

$$P(z) = -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z).$$

Then $(24n-1) \cdot P(\alpha_Q)$ is an algebraic integer.

Remark. We recall that a meromorphic modular form is said to be *weakly holomorphic* if its poles are supported on the cusps.

The form $F_p(z)$ studied by Bruinier and Ono satisfies these conditions. One can see this because $F_p(z)$ has level 6, so the group of Atkin-Lehner involutions acts transitively on the cusps. Since $F_p(z)$ is an eigenform for all of the Atkin-Lehner involutions and has an integral Fourier expansion at infinity, it follows that the Fourier expansions of F_p at all cusps is integral. Moreover, since the Atkin-Lehner involutions commute with the Maass raising operator

$$R_{-2} = -4\pi q \frac{d}{dq} - \frac{2}{y},$$

the Fourier expansion of

$$q\frac{dF}{dq} + F \cdot \frac{E_2E_4 - E_6}{6E_4} = F \cdot \frac{\left(E_2 - \frac{3}{\pi \operatorname{Im} z}\right) E_4 - E_6}{6E_4} - \frac{1}{4\pi} R_{-2} F$$

at all cusps is integral as well. Therefore, Theorem 1.1 implies the following:

Corollary 1.2. *The conjecture of Bruinier and Ono is true.*

Remark. Corollary 1.2 is sharp for small (and possibly all) n. For example, we have

$$\prod_{m=1}^{3} (x - P_p(\alpha_{Q_m})) = x^3 - 23x^2 + \frac{3592}{23}x - 419,$$

where Q_m ranges over any choice of representatives of $\mathcal{Q}_1$.

Returning to the general case of $P(z)$ as in Theorem 1.1, the work of Bruinier and Ono (Theorem 4.5 of [2]) implies $6 \cdot (24n-1) \cdot P(\alpha_Q)$ is an algebraic integer. Although this theorem is stated for squarefree level and when F is an eigenfunction of the Atkin-Lehner involutions, an inspection of the proof shows that the assumptions in the statement of Theorem 1.1 are also sufficient. Thus it suffices to show that $P(\alpha_Q)$ is integral at primes lying over 6. We will henceforth refer to this property as 6-*integrality.* For this purpose, it is convenient to decompose P as

$$P = A + B \cdot C, \tag{4}$$

where

$$A = -q\frac{dF}{dq} - \frac{1}{6}FE_2 + \frac{FE_6(7j - 6912)}{6E_4(j - 1728)}, \tag{5}$$

$$B = \frac{FE_6 j}{E_4}, \tag{6}$$

$$C = \frac{E_4}{6E_6 j}\left(E_2 - \frac{3}{\pi \operatorname{Im} z}\right) - \frac{7j - 6912}{6j(j - 1728)}. \tag{7}$$

To establish the 6-integrality of $P(\alpha_Q)$, it suffices to establish the 6-integrality of each of $A(\alpha_Q)$, $B(\alpha_Q)$ and $C(\alpha_Q)$. In Section 2, we will use methods similar to those of [2] to show that $A(\alpha_Q)$ and $B(\alpha_Q)$ are 6-integral. Then in Section 3, we show that $C(\alpha_Q)$ is 6-integral using a description of C in terms of classical modular polynomials due to Masser.

Remark. For the remainder of this paper, we fix $D \equiv 1 \bmod 24$ with $D < 0$, and we let α_Q denote any CM point of discriminant D.

2. Proof of 6-Integrality of A and B

In this section, we prove the 6-integrality of A and B at the CM-points α_Q. We begin by showing that $j(\alpha_Q)$ is a unit at 2 and 3.

Lemma 2.1. *Let $p \in \{2, 3\}$ and E be an elliptic curve defined over a number field K having complex multiplication by an order in a quadratic field F. If E has good ordinary reduction at all primes lying over p, then $j(E)$ is coprime to p.*

Proof. Assume to the contrary that $j(E)$ was not coprime to p; write $\mathfrak{p}$ for a prime ideal lying over p containing $j(E)$, and write k for the residue field $\mathcal{O}_K/\mathfrak{p}$.

When $p = 2$, the elliptic curve $E^2 = \mathbb{C}/\mathbb{Z}[\omega]$ (where ω is a primitive cube root of unity) has good supersingular reduction at $\mathfrak{p}$. But $j(\omega) = 0$, so $E^2_{/k} \simeq E_{/k}$, so $E_{/k}$ is supersingular, which is a contradiction.

Similarly, when $p = 3$, the elliptic curve $E^3 = \mathbb{C}/\mathbb{Z}[i]$ has good supersingular reduction at $\mathfrak{p}$. But $j(\omega) = 1728$, so $E^3_{/k} \simeq E_{/k}$, so $E_{/k}$ is supersingular, which is a contradiction. $\square$

By this lemma, it suffices to show that both B and $A' := A \cdot j \cdot (j - 1728)$ assume integral values at all CM-points.

Lemma 2.2. *The modular functions A' and B are weakly holomorphic and have integral Fourier expansions at all cusps.*

Proof. By definition, we have

$$B = F \cdot E_6 \cdot \frac{j}{E_4},$$

and by direct examination, all three of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps. Similarly, by definition, we have

$$A' = F \cdot E_6(j-864) \cdot \frac{j}{E_4} - (j-1728) \cdot \left[j \cdot \left(q\frac{dF}{dq} + F \cdot \frac{E_2E_4 - E_6}{6E_4} \right) \right]$$

and all of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps. □

Lemma 2.3. *A weakly holomorphic modular function g for a congruence subgroup Γ_g that has integral Fourier expansions at all cusps is integral at any CM-point.*

Proof. (The following argument is due to Bruinier and Ono; see Lemma 4.3 of [2].) We consider the polynomial

$$\Psi_g(X, z) = \prod_{\gamma \in \Gamma_g \backslash \Gamma(1)} (X - g(\gamma z)).$$

This is a monic polynomial in X of degree $[\Gamma(1) : \Gamma_g]$ whose coefficients are weakly holomorphic modular functions in z for the group $\Gamma(1)$, so $\Psi_g(X, z) \in \mathbb{C}[j(z), X]$.

Our assumption that g has integral Fourier expansion at all cusps implies that for any $\gamma \in \Gamma(1)$, the modular function $g \mid \gamma$ has a Fourier expansion at infinity whose coefficients are algebraic integers. Thus, the coefficients of $\Psi_g(X, z)$ are polynomials in $j(z)$ whose coefficients are algebraic integers.

Since j is integral at any CM-point α, the value $g(\alpha)$ satisfies a monic polynomial whose coefficients are algebraic integers, and is therefore an algebraic integer. □

Remark. In Appendix A, we give values of the polynomials $\Psi_{A'}$ and Ψ_B for the form F_p considered by Bruinier and Ono, thus providing a direct proof of the integrality of $A'(\alpha_Q)$ and $B(\alpha_Q)$ in this case.

Lemma 2.4. *If α_Q is an CM-point with discriminant $D \equiv 1 \bmod 24$, then $A(\alpha_Q)$ and $B(\alpha_Q)$ are 6-integral.*

Proof. This follows from combining Lemmas 2.1, 2.2, and 2.3. □

3. Proof of 6-Integrality for $C(\alpha_Q)$

In this section, we finish the proof of Theorem 1.1 by showing that $C(\alpha_Q)$ is 6-integral at the required CM points α_Q. To do this, we study the classical modular polynomials Φ_{-D}, in a fashion similar to Appendix 1 of [3]. We begin by reviewing the definition of Φ_{-D}.

Definition 1. We say that two matrices B_1 and B_2 are *equivalent* if $B_1 = X \cdot B_2$ for some $X \in \mathrm{SL}_2(\mathbb{Z})$.

It is well-known that there are only finitely many equivalence classes of primitive integer matrices of determinant $-D$. Write $M_1, M_2, \ldots, M_n$ for these equivalence classes and suppose M_1 is such that $\alpha_Q = M_1\alpha_Q$.

Definition 2. We write $\Phi_{-D}(X, Y)$ for the *classical modular polynomial*, i.e. the polynomial such that

$$\Phi_{-D}(j(z), Y) = \prod_{i=1}^{n}(Y - j(M_i z)).$$

By [1], Theorem 1 of Section 3.4, the polynomial $\Phi_{-D}(X, Y)$ is symmetric in X and Y and has coefficients that are rational integers. In particular, we can expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

$$\Phi(X, Y) = \sum_{\mu,\nu} \beta_{\mu,\nu}(X - j(\alpha_Q))^{\mu}(Y - j(\alpha_Q))^{\nu},$$

where $\beta_{\mu,\nu} = \beta_{\nu,\mu}$. We write $\beta = \beta_{0,1} = \beta_{1,0}$.

We define Q to be *special* is there is more than one equivalence class of matrices M such that $M\alpha_Q = \alpha_Q$. This can only happen if $D = 3d^2$ for some integer d (see [3], Appendix 1), so in particular forms of discriminant $-24n + 1$ are not special.

Lemma 3.1 (Masser)**.** *If Q is not special, we have $\beta \neq 0$ and*

$$C(\alpha_Q) = \frac{\beta_{0,2} - \beta_{1,1} + \beta_{2,0}}{\beta}.$$

Proof. See [3], Appendix 1 (in particular, equations (100) and (106), and the definition of γ on page 118). □

By definition, the $\beta_{\mu,\nu}$ are algebraic integers. Thus, to prove that $C(\alpha_Q)$ is integral at primes lying over 6, it suffices to show that β is a unit at primes lying over 6. From the definition of β, we have

$$\beta = \prod_{i=2}^{n}(j(\alpha_Q) - j(M_i\alpha_Q)).$$

Thus, it suffices to show that for any prime $\mathfrak{p}$ lying over 6, we have $j(\alpha_Q) \not\equiv j(M_i\alpha_Q)$ mod $\mathfrak{p}$. To show this, it is enough to establish the following lemma:

Lemma 3.2. *Suppose $\mathfrak{p}$ is a prime ideal of a number field K. Suppose E and E' are two elliptic curves over K with complex multiplication by the same order R in a quadratic field F. Suppose the index $[\mathcal{O}_F : R]$ is coprime to the residue characteristic of $\mathfrak{p}$. If both curves have good ordinary reduction at $\mathfrak{p}$ and the reduced curves are isomorphic, then E and E' are also isomorphic.*

Proof. Write k for the residue field $\mathcal{O}_K/\mathfrak{p}$ and $p = \mathrm{char}(k)$. As the index of R in $\mathcal{O}_F$ is coprime to p, there is an isogeny $f\colon E \to E'$ whose degree is coprime to p. Since E has ordinary reduction at $\mathfrak{p}$, its endomorphisms over k are a rank-2 submodule S of $\mathcal{O}_F$ which contains R. As the index of R in $\mathcal{O}_F$ is coprime to p, the index d of R in S is also coprime to p. Choose an isomorphism between the reductions $E_{/k}$ and $E'_{/k}$. Composing this with the isogeny f gives an endomorphism of $E_{/k}$, and multiplying this endomorphism by d gives an endomorphism which lifts to an endomorphism g of E whose degree is coprime to p. Now the specializations of the kernels of $f \circ d$ and g coincide by construction, and both kernels are subgroups whose order is coprime to p. Thus, $\ker f \circ d = \ker g$, and therefore $E \cong E'$. □

This completes the proof of the 6-integrality of $C(\alpha_Q)$, as the assumption $D \equiv 1 \mod 24$ shows that the conditions of the above lemma are satisfied. By the discussion in Section 1, this establishes Theorem 1.1.

Appendix A. The Polynomials $\Psi_{A'}$ and Ψ_B for $F = F_p$

Here, we give the explicit values of the polynomials $\Psi_{A'}$ and Ψ_B when $F = F_p$ is the form considered by Bruinier and Ono in [2]. Namely, we have

$$\Psi_{A'} = X^{12} + \sum_{i=0}^{11} a_i X^i \quad \text{and} \quad \Psi_B = X^{12} + \sum_{i=0}^{11} b_i X^i,$$

where the a_i and b_i are the polynomials in j with integer coefficients given below.

$$\begin{aligned}
a_{11} &= -2 \cdot (j - 2^6 \cdot 3^3) \cdot (j - 2^5 \cdot 3^3) \cdot j \\
a_{10} &= -(j - 2^6 \cdot 3^3) \cdot j^2 \cdot (7 \cdot 67 \cdot j^2 - 2^6 \cdot 3^2 \cdot 2053 \cdot j + 2^{11} \cdot 3^5 \cdot 31 \cdot 53) \\
a_9 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \cdot (3^2 \cdot j^4 - 2^3 \cdot 6379 \cdot j^3 + 2^6 \cdot 3^2 \cdot 162713 \cdot j^2 \\
&\quad - 2^{12} \cdot 3^5 \cdot 72797 \cdot j + 2^{25} \cdot 3^{12}) \\
a_8 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \cdot (2 \cdot 7 \cdot 13^2 \cdot j^5 - 3^2 \cdot 409 \cdot 3373 \cdot j^4 \\
&\quad + 2^7 \cdot 3^4 \cdot 1237 \cdot 1973 \cdot j^3 - 2^{14} \cdot 3^7 \cdot 5 \cdot 311 \cdot 443 \cdot j^2 \\
&\quad + 2^{21} \cdot 3^{10} \cdot 31 \cdot 2897 \cdot j - 2^{31} \cdot 3^{14} \cdot 163) \\
a_7 &= 2^2 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (11 \cdot 61 \cdot 193 \cdot j^5 - 2^3 \cdot 3 \cdot 27510443 \cdot j^4 \\
&\quad + 2^9 \cdot 3^3 \cdot 97550587 \cdot j^3 - 2^{16} \cdot 3^6 \cdot 11 \cdot 2599451 \cdot j^2 \\
&\quad + 2^{23} \cdot 3^9 \cdot 5 \cdot 739 \cdot 1109 \cdot j - 2^{34} \cdot 3^{13} \cdot 4691)
\end{aligned}$$

$$
\begin{aligned}
a_6 = {} & 2^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (2^4 \cdot 3^2 \cdot j^8 + 7 \cdot 199 \cdot 1373 \cdot j^7 \\
& - 2^2 \cdot 29 \cdot 37 \cdot 281 \cdot 13913 \cdot j^6 + 2^{13} \cdot 3^3 \cdot 7 \cdot 233 \cdot 143281 \cdot j^5 \\
& - 2^{15} \cdot 3^7 \cdot 5 \cdot 11 \cdot 21117827 \cdot j^4 + 2^{23} \cdot 3^9 \cdot 3943 \cdot 117577 \cdot j^3 \\
& - 2^{31} \cdot 3^{12} \cdot 769 \cdot 45317 \cdot j^2 + 2^{41} \cdot 3^{16} \cdot 7 \cdot 15923 \cdot j - 2^{50} \cdot 3^{20} \cdot 269) \\
a_5 = {} & 2^4 \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5 \cdot (2^6 \cdot 3^4 \cdot 5 \cdot j^8 - 7 \cdot 5051 \cdot 5939 \cdot j^7 \\
& + 2^3 \cdot 3^2 \cdot 5 \cdot 61 \cdot 101 \cdot 330037 \cdot j^6 - 2^9 \cdot 3^5 \cdot 96289 \cdot 119173 \cdot j^5 \\
& + 2^{16} \cdot 3^9 \cdot 17 \cdot 77252741 \cdot j^4 - 2^{22} \cdot 3^{11} \cdot 11 \cdot 71 \cdot 523 \cdot 4091 \cdot j^3 \\
& + 2^{35} \cdot 3^{14} \cdot 5 \cdot 673 \cdot 977 \cdot j^2 - 2^{41} \cdot 3^{18} \cdot 79 \cdot 1831 \cdot j + 2^{55} \cdot 3^{24}) \\
a_4 = {} & (j - 2^6 \cdot 3^3)^4 \cdot j^6 \cdot (2^8 \cdot 3^3 \cdot 5 \cdot 2003 \cdot j^9 - 409 \cdot 39157 \cdot 44483 \cdot j^8 \\
& + 2^9 \cdot 3 \cdot 2092618568983 \cdot j^7 - 2^{20} \cdot 3^4 \cdot 98512996093 \cdot j^6 \\
& + 2^{20} \cdot 3^7 \cdot 41 \cdot 242261 \cdot 608831 \cdot j^5 - 2^{28} \cdot 3^{10} \cdot 5 \cdot 1231 \cdot 155631757 \cdot j^4 \\
& + 2^{32} \cdot 3^{13} \cdot 521 \cdot 3077579657 \cdot j^3 - 2^{42} \cdot 3^{16} \cdot 997 \cdot 1607 \cdot 16657 \cdot j^2 \\
& + 2^{52} \cdot 3^{20} \cdot 23 \cdot 541 \cdot 6863 \cdot j - 2^{63} \cdot 3^{24} \cdot 5 \cdot 11987) \\
a_3 = {} & 2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^6 \cdot (3^2 \cdot 377732207 \cdot j^{10} - 2^6 \cdot 5^2 \cdot 7 \cdot 101 \cdot 28520381 \cdot j^9 \\
& + 2^{11} \cdot 11 \cdot 337 \cdot 17990477821 \cdot j^8 - 2^{20} \cdot 3^3 \cdot 179 \cdot 389 \cdot 171956657 \cdot j^7 \\
& + 2^{23} \cdot 3^6 \cdot 5 \cdot 479 \cdot 37193046587 \cdot j^6 - 2^{30} \cdot 3^9 \cdot 1283 \cdot 28703 \cdot 758137 \cdot j^5 \\
& + 2^{36} \cdot 3^{12} \cdot 7 \cdot 31 \cdot 54791988203 \cdot j^4 - 2^{45} \cdot 3^{15} \cdot 19^2 \cdot 151 \cdot 7738067 \cdot j^3 \\
& + 2^{55} \cdot 3^{20} \cdot 41 \cdot 12810583 \cdot j^2 - 2^{65} \cdot 3^{24} \cdot 1103107 \cdot j + 2^{76} \cdot 3^{27} \cdot 1447) \\
a_2 = {} & 2^2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^7 \cdot (42967 \cdot 2406947 \cdot j^{11} - 2^3 \cdot 557 \cdot 1783 \cdot 140768209 \cdot j^{10} \\
& + 2^9 \cdot 3^4 \cdot 6205891 \cdot 21226039 \cdot j^9 - 2^{19} \cdot 3^7 \cdot 5 \cdot 11 \cdot 251872948013 \cdot j^8 \\
& + 2^{24} \cdot 3^9 \cdot 5 \cdot 13 \cdot 23 \cdot 37 \cdot 521 \cdot 3203149 \cdot j^7 - 2^{29} \cdot 3^{13} \cdot 47242981376477 \cdot j^6 \\
& + 2^{35} \cdot 3^{16} \cdot 227 \cdot 112292655271 \cdot j^5 - 2^{41} \cdot 3^{18} \cdot 107 \cdot 269749728667 \cdot j^4 \\
& + 2^{54} \cdot 3^{22} \cdot 43 \cdot 449215127 \cdot j^3 - 2^{61} \cdot 3^{27} \cdot 5 \cdot 653 \cdot 54193 \cdot j^2 \\
& + 2^{72} \cdot 3^{30} \cdot 139 \cdot 3719 \cdot j - 2^{82} \cdot 3^{35} \cdot 139)
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= 2^3 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (1847032397279 \cdot j^{11} - 2^6 \cdot 47 \cdot 157 \cdot 3691 \cdot 11660843 \cdot j^{10} \\
&\quad + 2^{14} \cdot 3^4 \cdot 383 \cdot 25679 \cdot 7797631 \cdot j^9 - 2^{20} \cdot 3^6 \cdot 400129001343469 \cdot j^8 \\
&\quad + 2^{24} \cdot 3^9 \cdot 5 \cdot 41 \cdot 503 \cdot 67307 \cdot 267271 \cdot j^7 \\
&\quad - 2^{30} \cdot 3^{12} \cdot 19 \cdot 509 \cdot 13597 \cdot 11431571 \cdot j^6 \\
&\quad + 2^{37} \cdot 3^{15} \cdot 31 \cdot 3038701 \cdot 4610147 \cdot j^5 - 2^{43} \cdot 3^{20} \cdot 7^2 \cdot 41 \cdot 73 \cdot 2381 \cdot 56891 \cdot j^4 \\
&\quad + 2^{52} \cdot 3^{21} \cdot 5 \cdot 139 \cdot 9239401667 \cdot j^3 - 2^{62} \cdot 3^{25} \cdot 5 \cdot 1381 \cdot 3698087 \cdot j^2 \\
&\quad + 2^{73} \cdot 3^{29} \cdot 11 \cdot 47 \cdot 58693 \cdot j - 2^{85} \cdot 3^{33} \cdot 8161) \\
a_0 &= -2^4 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (2^3 \cdot 3^2 \cdot 7^6 \cdot j^{14} - 5 \cdot 13 \cdot 3109 \cdot 76441597 \cdot j^{13} \\
&\quad + 2^4 \cdot 3449 \cdot 4363 \cdot 873750089 \cdot j^{12} - 2^{11} \cdot 3^4 \cdot 7 \cdot 2087 \cdot 57859 \cdot 9420337 \cdot j^{11} \\
&\quad + 2^{16} \cdot 3^8 \cdot 11^2 \cdot 73 \cdot 125183 \cdot 10636957 \cdot j^{10} - 2^{26} \cdot 3^9 \cdot 691 \cdot 14434308694753 \cdot j^9 \\
&\quad + 2^{31} \cdot 3^{13} \cdot 101 \cdot 283 \cdot 252059913139 \cdot j^8 \\
&\quad - 2^{37} \cdot 3^{16} \cdot 11 \cdot 13 \cdot 17 \cdot 647 \cdot 863 \cdot 4253233 \cdot j^7 \\
&\quad + 2^{43} \cdot 3^{18} \cdot 631819 \cdot 16451871913 \cdot j^6 - 2^{48} \cdot 3^{23} \cdot 149 \cdot 233 \cdot 90533 \cdot 330413 \cdot j^5 \\
&\quad + 2^{59} \cdot 3^{25} \cdot 23 \cdot 1408302006413 \cdot j^4 - 2^{70} \cdot 3^{27} \cdot 726838208711 \cdot j^3 \\
&\quad + 2^{80} \cdot 3^{32} \cdot 7 \cdot 263 \cdot 337 \cdot 1327 \cdot j^2 - 2^{90} \cdot 3^{37} \cdot 569731 \cdot j + 2^{100} \cdot 3^{39} \cdot 17^3) \\
b_{11} &= -(j - 2^6 \cdot 3^3) \cdot j \\
b_{10} &= -2 \cdot 13 \cdot 3^2 \cdot (j - 2^6 \cdot 3^3) \cdot j^2 \\
b_9 &= 2^2 \cdot (j - 2^3 \cdot 3^6) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \\
b_8 &= 3^4 \cdot (13 \cdot j - 2^5 \cdot 3 \cdot 163) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \\
b_7 &= 5 \cdot 2^5 \cdot 3^6 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \\
b_6 &= 2^2 \cdot 3^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (j^2 + 2^4 \cdot 3^5 \cdot 13 \cdot j - 2^9 \cdot 3^5 \cdot 269) \\
b_5 &= 2^5 \cdot 3^5 \cdot (5 \cdot j - 2^6 \cdot 3^4) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5 \\
b_4 &= 2^5 \cdot 3^8 \cdot (31 \cdot j - 2^3 \cdot 3^2 \cdot 1471) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^6 \\
b_3 &= 2^8 \cdot 3^8 \cdot (383 \cdot j - 2^6 \cdot 3 \cdot 1447) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^6 \\
b_2 &= 2^9 \cdot 3^9 \cdot (3923 \cdot j - 2^6 \cdot 3^5 \cdot 139) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^7 \\
b_1 &= 13 \cdot 19 \cdot 3^{11} \cdot 2^{15} \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \\
b_0 &= -2^8 \cdot 3^9 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (j^2 - 2^7 \cdot 3^3 \cdot 1399 \cdot j + 2^{12} \cdot 3^6 \cdot 17^3)
\end{aligned}
$$

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