

NULL-CONTROL AND MEASURABLE SETS

J. APRAIZ AND L. ESCAURIAZA

ABSTRACT. We prove the interior and boundary null-controllability of some parabolic evolutions with controls acting over measurable sets.

1. INTRODUCTION

The control for evolution equations aims to drive the solution to a prescribed state starting from a certain initial condition. One acts on the equation through a source term, a so-called distributed control, or through a boundary condition. To achieve general results one wishes for the control to only act in part of the domain or its boundary and to have as much latitude as possible in the choice of the control region: location, size, shape.

Here, we focus on the heat equation in a smooth and bounded domain Ω in \mathbb{R}^n for a time interval $(0, T)$, $T > 0$ and for a distributed control f we consider

$$(1.1) \quad \begin{cases} \Delta u - \partial_t u = f(x, t)\chi_\omega(x), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0, & \text{in } \Omega. \end{cases}$$

Here, $\omega \subset \Omega$ is an interior control region. The null controllability of this equation, i.e., the existence for any u_0 in $L^2(\Omega)$ of a control f in $L^2(\omega \times (0, T))$ with

$$(1.2) \quad \|f\|_{L^2(\omega \times (0, T))} \leq N \|u_0\|_{L^2(\Omega)},$$

such that $u(T) = 0$, was proved in [14] by means of local Carleman estimates for the elliptic operator $\Delta + \partial_y^2$ over $\Omega \times \mathbb{R}$. A second approach based on global Carleman estimates for the backward parabolic operator $\Delta + \partial_t$ [9], also led to the null controllability of the heat equation. The first approach has been used for the treatment of time-independent parabolic operators associated to self-adjoint elliptic operators, while the second allows to address time-dependent non-selfadjoint parabolic operators and semi-linear evolutions.

The method introduced in [14] was further extended to study thermoelasticity [15], thermoelastic plates [4] and semigroups generated by fractional orders of elliptic operators [18]. It has also been used to prove null controllability in the case of non smooth coefficients [5, 23]. The method of [14] has also been extended to treat some non-selfadjoint cases, e.g. non symmetric systems [13] and all 1-dimensional time-independent parabolic equations [2].

In the above works, the control region ω is always assumed to contain an open ball. Also, the cost of controllability (the smallest constant N found for the inequality (1.2)) depends on this fact. The reason for these is that the main technique used

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in the arguments, Carleman inequalities, requires to construct suitable Carleman weights: a role for functions which requires smoothness (at least C^2) and to have the extreme values in proper regions associated to the control region ω , the larger body Ω and possibly the value of $T > 0$. The construction of such functions seems to be not possible, when ω does not contain a ball.

Motivated by these facts J.P. Puel and E. Zuazua raised the question whether the null controllability of the heat equation is possible when the control region is a measurable set. A positive partial answer to this question was explained by the second author at the June 2008 meeting *Control of Physical Systems and Partial Differential Equations* held at the Institute Henri Poincaré. Here, we give a formal account of the results.

Theorem 1. *Let $n \geq 2$. Then, $\Delta - \partial_t$ is null-controllable at all positive times, with distributed controls acting over a measurable set $\omega \subset \Omega$ with positive Lebesgue measure, when*

$$\Delta = \nabla \cdot (\mathbf{A}(x)\nabla \cdot) + V(x),$$

is a self-adjoint elliptic operator, the coefficients matrix \mathbf{A} is smooth in $\bar{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of ω . The same holds when $n = 1$,

$$\Delta = \frac{1}{\rho(x)} [\partial_x (a(x)\partial_x) + b(x)\partial_x + c(x)]$$

and a, b, c and ρ are measurable functions in $\Omega = (0, 1)$.

In regard to boundary null controllability, i.e., the existence for any u_0 in $L^2(\Omega)$ of a control h in $L^2(\gamma \times (0, T))$ with

$$(1.3) \quad \|h\|_{L^2(\gamma \times (0, T))} \leq N \|u_0\|_{L^2(\Omega)},$$

such that the solution to

$$(1.4) \quad \begin{cases} \Delta u - \partial_t u = 0, & \text{in } \Omega \times (0, T), \\ u = h(x, t)\chi_\gamma(x), & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

verifies $u(T) \equiv 0$, we have the following result.

Theorem 2. *Let $n \geq 2$. Then, $\Delta - \partial_t$ is null-controllable at all times $T > 0$ with boundary controls acting over a measurable set $\gamma \subset \partial\Omega$ with positive surface measure when*

$$\Delta = \nabla \cdot (\mathbf{A}(x)\nabla \cdot) + V(x)$$

is a self-adjoint elliptic operator, the coefficients matrix \mathbf{A} is smooth in $\bar{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of γ in $\bar{\Omega}$.

The results in Theorems 1 and 2 follow from a straightforward application of the linear construction of the control function for the systems (1.1) and (1.4) developed in [14] and the following observability inequality or propagation of smallness estimate established in [26] (See also [21] and [22]).

Theorem 3. *Assume that $f : B_{2R} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-analytic function verifying*

$$(1.5) \quad |\partial^\alpha f(x)| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \text{ when } x \in B_{2R}, \alpha \in \mathbb{N}^n,$$

for some $M > 0$ and $0 < \rho \leq 1$ and $E \subset B_{\frac{R}{2}}$ is a measurable set with positive measure. Then, there are positive constants $N = N(\rho, |E|/|B_R|)$ and $\theta = \theta(\rho, |E|/|B_R|)$ such that

$$\|f\|_{L^\infty(B_R)} \leq N \left(\int_E |f| dx \right)^\theta M^{1-\theta}.$$

The experts will realize that the word *smooth* describing the regularity (away from the measurable set) of $\partial\Omega$, A and V in Theorems 1 and 2 can be replaced by $\partial\Omega$ is C^2 , A is Lipschitz in $\overline{\Omega}$ and V is bounded (See [9, 14, 15, 24]). In fact, the C^2 regularity of $\partial\Omega$ can be relaxed to require that either $\partial\Omega$ is C^1 or there is $\alpha \in (0, 1]$ such that

$$(P - Q) \cdot \nu(P) \geq -|P - Q|^{1+\alpha}, \text{ for all } P, Q \in \partial\Omega,$$

where $\nu(P)$ is the exterior unit normal vector to $\partial\Omega$. The later holds when Ω is either a convex domain, a polyhedron in \mathbb{R}^n or when $\partial\Omega$ can be locally written as the graphs of Lipschitz functions which are the sum of a convex and a $C^{1,\alpha}$ function over \mathbb{R}^{n-1} .

To simplify the exposition and to show the strength of Theorem 3, we give the proof of Theorems 1 and 2 under the assumptions that $\partial\Omega$, A and V are globally real analytic. We do it because it makes clear how the construction algorithm of the control function in [14] and Theorem 3 can also be applied to prove the *interior* null-controllability (Theorem 1) for other parabolic evolutions whose corresponding observability or spectral inequalities (suitable Carleman inequalities) are otherwise unknown. Examples of these parabolic evolutions are the ones associated to self-adjoint elliptic systems with unknowns $\mathbf{u} = (u^1, \dots, u^m)$,

$$L_\alpha \mathbf{u} = \partial_i (a_{ij}^{\alpha\beta}(x) \partial_j u^\beta), \quad \alpha = 1, \dots, m$$

with $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$, for $\alpha, \beta = 1, \dots, m$, $i, j = 1, \dots, n$, and with coefficients matrices verifying for some $\delta > 0$ the strong ellipticity condition,

$$\sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x) \xi_i^\alpha \xi_j^\beta \geq \delta \sum_{i,\alpha} |\xi_i^\alpha|^2, \text{ when } \boldsymbol{\xi} \in \mathbb{R}^{nm}, x \in \mathbb{R}^n,$$

or the more general Legendre-Hadamard condition

$$(1.6) \quad \sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x) \xi_i \xi_j \eta^\alpha \eta^\beta \geq \delta |\boldsymbol{\xi}|^2 |\boldsymbol{\eta}|^2, \text{ when } \boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{\eta} \in \mathbb{R}^m, x \in \mathbb{R}^n.$$

We recall that the Lamé system of elasticity

$$\nabla \cdot (\mu(x) (\nabla \mathbf{u} + \nabla \mathbf{u}^t)) + \nabla (\lambda(x) \nabla \cdot \mathbf{u}),$$

with $\mu \geq \delta$, $\mu + \lambda \geq 0$ in \mathbb{R}^n , $m = n$ and $a_{ij}^{\alpha\beta} = \mu(\delta_{\alpha\beta} \delta_{ij} + \delta_{i\beta} \delta_{j\alpha}) + \lambda \delta_{j\beta} \delta_{\alpha i}$, are examples of systems verifying (1.6). Here, $a_{ij}^{\alpha\beta}$, μ and λ can either be constants or real analytic functions on $\overline{\Omega}$.

It also makes clear that under such hypothesis one may replace the Carleman inequalities used in the literature, to prove the observability or propagation of smallness inequalities necessary in the process of applying the construction algorithm in [14, 15], by the simpler application of Theorem 3. Of course, it has the drawback that it requires more smoothness on the operators and the boundary of Ω but on the contrary, one can handle with Theorem 3 and the construction methods in [14, 15]

the null-controllability of other parabolic evolutions with internal controls: like the second order evolutions explained above or for higher order evolutions as

$$\partial_t u + (-1)^m \Delta^m u, \quad m = 2, \dots,$$

with Dirichlet boundary conditions on $\partial\Omega$, $u = \nabla u = \dots = \nabla^{m-1} u = 0$.

In section 2 we give the proofs of Theorems 1 and 2. We explain how to extend Theorem 1 to the evolutions (2.19) and (2.21) in Remark 2, while the problems we have to extend Theorem 2 to these evolutions are explained in Remark 3, for the simpler case of parabolic systems. For the sake of completeness, we include a proof of Theorem 3 in section 3. It is built with ideas taken from [16], [21] and [26].

2. PROOF OF THEOREMS 1 AND 2

We begin by setting up the formal hypothesis: first we assume there is $0 < \delta \leq 1$ such that

$$\delta|\xi|^2 \leq \mathbf{A}(x)\xi \cdot \xi \leq \delta^{-1}|\xi|^2, \quad \text{for all } x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

Ω is a bounded open set in \mathbb{R}^n , $n \geq 2$, with a real analytic boundary and \mathbf{A} , V are real analytic in $\overline{\Omega}$, i.e., there are $r > 0$ and $0 < \delta \leq 1$ such that

$$|\partial^\alpha \mathbf{A}(x)| + |\partial^\alpha V(x)| \leq |\alpha|! \delta^{-|\alpha|-1}, \quad \text{when } x \in \overline{\Omega}, \quad \alpha \in \mathbb{N}^n,$$

and for each $x \in \partial\Omega$, there are a new coordinate system (where $x = 0$) and a real analytic function $\varphi : B'_r \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ verifying

$$(2.1) \quad \begin{aligned} \varphi(0') &= 0, \quad |\partial^\alpha \varphi(x')| \leq |\alpha|! \delta^{-|\alpha|-1}, \quad \text{when } x' \in B'_r, \quad \alpha \in \mathbb{N}^{n-1}, \\ B_r \cap \Omega &= B_r \cap \{(x', x_n) : x' \in B'_r, \quad x_n > \varphi(x')\}, \\ B_r \cap \partial\Omega &= B_r \cap \{(x', x_n) : x' \in B'_r, \quad x_n = \varphi(x')\}. \end{aligned}$$

When E is a measurable set, $|E|$ will denote its Lebesgue or surface measure.

Proof of Theorem 1. We may assume that the eigenvalues with zero Dirichlet condition for $\Delta = \nabla \cdot (\mathbf{A}(x)\nabla \cdot) + V(x)$ on Ω are all positive, $0 < \omega_1^2 < \omega_2^2 \leq \omega_3^2 \leq \dots \leq \omega_j^2 \leq \dots$ and $\{e_j\}$ denotes the sequence of $L^2(\Omega)$ -normalized eigenfunctions,

$$\begin{cases} \Delta e_j + \omega_j^2 e_j = 0, & \text{in } \Omega, \\ e_j = 0, & \text{in } \partial\Omega. \end{cases}$$

When $\omega \subset \Omega$ is measurable with positive Lebesgue measure, the method in [14] shows that one find and $L^2(\omega \times (0, T))$ control function f verifying (1.2) for the system (1.1), provided there is $N = N(|\omega|, \Omega, r, \delta)$, such that the inequality

$$(2.2) \quad \sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \leq e^{N\mu} \int \int_{\omega \times [\frac{1}{4}, \frac{3}{4}]} \left| \sum_{\omega_j \leq \mu} (a_j e^{\omega_j y} + b_j e^{-\omega_j y}) e_j \right|^2 dx dy,$$

holds for $\mu \geq \omega_1$ and all sequences a_1, a_2, \dots and b_1, b_2, \dots . Let then,

$$(2.3) \quad u(x, y) = \sum_{\omega_j \leq \mu} (a_j e^{\omega_j y} + b_j e^{-\omega_j y}) e_j,$$

it satisfies, $\Delta u + \partial_y^2 u = 0$, in $\Omega \times \mathbb{R}$, $u = 0$ on the lateral boundary of $\Omega \times \mathbb{R}$ and u is real analytic in $\overline{\Omega} \times \mathbb{R}$. Moreover, given (x_0, y_0) in $\overline{\Omega} \times \mathbb{R}$ and $R \leq 1$, there are

$N = N(r, \delta)$ and $\rho = \rho(r, \delta)$ such that

$$(2.4) \quad \|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_R(x_0, y_0) \cap \Omega \times \mathbb{R})} \leq \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha| + \beta}} \left(\int_{B_{2R}(x_0, y_0) \cap \Omega \times \mathbb{R}} |u|^2 dx dy \right)^{\frac{1}{2}},$$

when $\alpha \in \mathbb{N}^n$ and $\beta \geq 1$. For the later see [20, Chapter 5], [12, Chapter 3].

The orthonormality of $\{e_j\}$ in Ω and (2.4) with $R = 1$ imply

$$(2.5) \quad \|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(\Omega \times [-5, 5])} \leq e^{N\mu} (|\alpha| + \beta)! \rho^{-|\alpha| - \beta} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}, \text{ for } \alpha \in \mathbb{N}^n, \beta \geq 0,$$

and there is $C > 0$ such that replacing the constants N and ρ in (2.5) by CN and ρ/C respectively, u has a real analytic extension to $\Omega_\rho \times [-4, 4]$,

$$\Omega_\rho = \{x \in \mathbb{R}^n : d(x, \Omega) \leq \rho\},$$

with

$$(2.6) \quad \|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(\Omega_\rho \times [-4, 4])} \leq M (|\alpha| + \beta)! (2\rho)^{-|\alpha| - \beta}, \text{ for } \alpha \in \mathbb{N}^n, \beta \geq 0,$$

and

$$M = e^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}.$$

For (x_0, y_0) in $\Omega \times [0, 1]$ with $d(x_0, \partial\Omega) = \rho$, we have $B_{2\rho}(x_0, y_0) \subset \Omega_\rho \times [-4, 4]$, and if we apply Theorem 3 to the real analytic extension of u in $B_{2\rho}(x_0, y_0)$ with $E = B_{\frac{\rho}{4}}(x_0, y_0)$, (2.6) implies there is $0 < \theta_1 < 1$ such that

$$\|u\|_{L^\infty(B_\rho(x_0, y_0))} \leq N \|u\|_{L^\infty(B_{\frac{\rho}{4}}(x_0, y_0))}^{\theta_1} M^{1-\theta_1},$$

From this and a suitable covering argument we get

$$(2.7) \quad \|u\|_{L^\infty(\Omega \times [0, 1])} \leq N \|u\|_{L^\infty(\Omega^\rho \times [-1, 2])}^{\theta_1} M^{1-\theta_1},$$

with

$$\Omega^\rho = \{x \in \Omega : d(x, \partial\Omega) \geq \frac{3\rho}{4}\}.$$

Thus, from Theorem 3 and without Carleman inequalities it is possible to bound except for the factor M all the information related to u over $\Omega \times [0, 1]$ by the information on u over $\Omega^\rho \times [-1, 2]$, a region *located inside* $\Omega \times \mathbb{R}$.

We may also assume that $E = \omega \times (\frac{1-\rho}{4}, \frac{1+\rho}{4}) \subset B_{\frac{\rho}{2}}(0, \frac{1}{2})$ has positive measure inside $B_{\frac{\rho}{2}}(0, \frac{1}{2})$, $B_{8\rho}(0, \frac{1}{2}) \subset \Omega^\rho \times [0, 1]$ and a second application of Theorem 3 gives

$$(2.8) \quad \|u\|_{L^\infty(B_\rho(0, \frac{1}{2}))} \leq N \|u\|_{L^2(\omega \times [\frac{1}{4}, \frac{3}{4}])}^{\theta_2} M^{1-\theta_2}.$$

Proceeding as in [14], we use a covering of $\Omega^\rho \times [-1, 2]$ and successive applications of Theorem 3, with E being a centered-moving ball with fixed radius $R/2$ depending on ρ and the geometry of Ω , with the ball of the same center and radius $2R$ contained $\Omega_\rho \times [-4, 4]$, and where in the last applications of Theorem 3, $E = B_\rho(0, \frac{1}{2})$. Thus, Theorem 3 and (2.6) imply there is $0 < \theta_3 < 1$ with

$$(2.9) \quad \|u\|_{L^\infty(\Omega^\rho \times [-1, 2])} \leq N \|u\|_{L^\infty(B_\rho(0, \frac{1}{2}))}^{\theta_3} M^{1-\theta_3}.$$

The orthonormality of $\{e_j\}$ in Ω and the inequality

$$e^{-\mu} \left(\frac{\sinh \omega_1}{\omega_1} - 1 \right) (a^2 + b^2) \leq \int_0^1 (ae^{\omega y} + be^{-\omega y})^2 dy, \text{ when } \omega_1 \leq \omega \leq \mu, a, b \in \mathbb{R},$$

give

$$(2.10) \quad \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}} \leq e^{N\mu} \|u\|_{L^2(\Omega \times [0,1])} \leq e^{N\mu} |\Omega|^{\frac{1}{2}} \|u\|_{L^\infty(\Omega \times [0,1])}$$

and (2.2) follows from (2.10), (2.7), (2.9) and (2.8).

In [2] it is shown that the null-controllability of the system (1.1) over $\Omega = (0, 1)$ with

$$\Delta = \frac{1}{\rho(x)} [\partial_x (a(x)\partial_x) + b(x)\partial_x + c(x)],$$

$$(2.11) \quad \delta \leq a(x), \rho(x) \leq \delta^{-1}, \quad |b(x)| + |c(x)| \leq \delta^{-1}, \text{ a.e. in } [0, 1],$$

is equivalent to the null-controllability of the system

$$(2.12) \quad \begin{cases} \partial_x^2 z - \rho(x)\partial_t z = f\chi_\omega, & 0 < x < 1, \quad 0 < t < T, \\ z(0, t) = z(1, t) = 0, & 0 \leq t \leq T, \\ z(x, 0) = z_0, & 0 \leq x \leq 1, \end{cases}$$

where ρ is a new function verifying (2.11) and ω a new measurable set in $[0, 1]$ with positive measure. The later follows from the bilipschitz change of variables used in [2]. Let then, $0 < \omega_1^2 < \omega_2^2 \leq \omega_3^2 \leq \dots \leq \omega_j^2 \leq \dots$ and $\{e_j\}$ denote the sequences of eigenvalues and $L^2(\Omega)$ -normalized eigenfunctions verifying

$$\begin{cases} e_j' + \rho(x)\omega_j^2 e_j = 0, & 0 < x < 1, \\ e_j(0) = e_j(1) = 0. \end{cases}$$

From [14], it suffices to show that (2.2) holds in order to find an interior null-control f for (2.12) verifying (1.2). Extend then e_j and ρ to $[-1, 1]$ by odd and even reflections respectively, and to all \mathbb{R} as periodic functions of period 2. The extended e_j is in $C^{1,1}(\mathbb{R})$ and verifies $e_j' + \rho(x)\omega_j^2 e_j = 0$ in \mathbb{R} , $j = 1, 2, \dots$. As before, let u be defined by (2.3), it verifies

$$\partial_x^2 u + \partial_y (\rho(x)\partial_y u) = 0, \text{ in } \mathbb{R}^2.$$

By Chebyshev's inequality, defining

$$(\omega \times [\frac{1}{4}, \frac{3}{4}]) \setminus E = \{(x, y) \in \omega \times [\frac{1}{4}, \frac{3}{4}] : |u(x, y)|/2 > \int_{\omega \times [\frac{1}{4}, \frac{3}{4}]} |u| dx dy\},$$

we have

$$(2.13) \quad |E| \geq \frac{1}{2} |\omega \times [\frac{1}{4}, \frac{3}{4}]| \text{ and } \|u\|_{L^\infty(E)} \leq 2 \int_{\omega \times [\frac{1}{4}, \frac{3}{4}]} |u| dx dy.$$

Set $u_\epsilon(x, y) = u(\frac{x}{\epsilon}, \frac{y}{\epsilon})$, it verifies

$$\partial_x^2 u_\epsilon + \partial_y (\rho(x/\epsilon)\partial_y u_\epsilon) = 0, \text{ in } \mathbb{R}^2,$$

and let v_ϵ be the *stream* function of u_ϵ , i.e., the solution to

$$\begin{cases} \partial_x v_\epsilon = -\rho(\frac{x}{\epsilon})\partial_y u_\epsilon, \\ \partial_y v_\epsilon = \partial_x u_\epsilon, \\ v_\epsilon(0) = 0, \end{cases}$$

Then, $f = u_\epsilon + iv_\epsilon$ is $(1/\delta)$ -quasiregular, i.e.,

$$f \in W_{\text{loc}}^{1,2}(\mathbb{R}^2), \quad \partial_{\bar{z}} f = \nu(z)\partial_z f, \quad |\nu(z)| \leq \frac{1-\delta}{1+\delta}, \quad z \in \mathbb{C},$$

and by the Ahlfors-Bers Representation Theorem [1] (See also [7] or [6]), any $(1/\delta)$ -quasiregular mapping f in B_4 can be written as

$$f = F \circ \Psi,$$

where $F = U + iV$ is holomorphic in B_4 and $\Psi : B_4 \rightarrow B_4$ is a $(1/\delta)$ -quasiconformal mapping, i.e. a $(1/\delta)$ -quasiregular homeomorphism from B_4 onto B_4 verifying

$$(2.14) \quad \begin{aligned} \partial_{\bar{z}} \Psi &= \nu(z)\partial_z \Psi, \quad \Psi(0) = 0, \Psi(4) = 4, \\ N^{-1}|z_1 - z_2|^{\frac{1}{\alpha}} &\leq |\Psi(z_1) - \Psi(z_2)| \leq N|z_1 - z_2|^\alpha, \quad \text{when } z_1, z_2 \in B_4, \end{aligned}$$

for some $0 < \alpha < 1$ and $N \geq 1$ depending on δ . Now, $\epsilon E \subset B_{2\epsilon}$, and from (2.14), $\Psi(\epsilon E) \subset B_{C(2\epsilon)^\alpha}$. Choose then ϵ so that $N(2\epsilon)^\alpha = \frac{1}{2}$. Thus, $\Psi(\epsilon E) \subset B_{\frac{1}{2}}$, $u_\epsilon = U \circ \Psi$,

$$\|U\|_{L^\infty(B_4)} = \|u\|_{L^\infty(B_{\frac{1}{2}})}$$

while the L^∞ interior estimates for subsolutions of elliptic equations [11, §8.6], the periodicity and orthogonality of the eigenfunctions e_j in $L^2([0, 1], \rho dx)$, gives

$$\|u\|_{L^\infty(B_{\frac{1}{2}})} \lesssim \|u\|_{L^2(B_{\frac{1}{2}})} \lesssim e^{6\mu/\epsilon} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}.$$

Thus, U is harmonic in B_4 ,

$$\|U\|_{L^\infty(B_4)} \leq e^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}$$

and from (2.13)

$$(2.15) \quad \|U\|_{L^\infty(\Psi(\epsilon E))} \leq 2 \int_{\omega \times [\frac{1}{4}, \frac{3}{4}]} |u| dx dy.$$

All together, U verifies the conditions in Theorem 3 in B_2 with $R = 1$, with the universal constant $0 < \rho \leq 1$ associated to the quantitative analyticity over B_2 of bounded harmonic functions in B_4 . From (2.15), (2.2) holds provided we can find a lower bound for the Lebesgue measure of $\Psi(\epsilon E) \subset B_{\frac{1}{2}}$. The lower bound follows from (2.13) and the following rescaled version of [3, Theorem 1]:

Let $\Psi : B_4 \rightarrow B_4$ be a $(1/\delta)$ -quasiconformal mapping with $\Psi(0) = 0$ and $E \subset B_4$ be a measurable set. Then, there is $N = N(\delta)$ such that

$$|E|^{\frac{1}{\delta}}/N \leq |\Psi(E)| \leq N|E|^\delta.$$

□

Remark 1. Theorem 3 also implies the version of (2.2) appearing in [15]. For if Ω , \mathbf{A} and V are as above and

$$u(x, y) = \sum_{\omega_j \leq \mu} a_j e^{\omega_j y} e_j(x),$$

u verifies (2.4) and

$$\|\partial_x^\alpha u(\cdot, 0)\|_{L^\infty(\Omega)} \leq M |\alpha|! (2\rho)^{-|\alpha|}, \text{ for } \alpha \in \mathbb{N}^n, \text{ with } M = e^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 \right)^{\frac{1}{2}}$$

and ρ as above. Thus, $u(\cdot, 0)$ has an analytic extension to a ρ -neighborhood of Ω , and after a finite number of applications of Theorem 3 and a covering argument,

$$\|u(\cdot, 0)\|_{L^2(\Omega)} \leq N \|u(\cdot, 0)\|_{L^2(\omega)}^\theta M^{1-\theta}.$$

In particular,

$$\sum_{\omega_j \leq \mu} a_j^2 \leq e^{N\mu} \int_{\omega} \left| \sum_{\omega_j \leq \mu} a_j e_j \right|^2 dx, \text{ with } N = N(|\omega|, \Omega, r, \delta).$$

Proof of Theorem 2. Let u be defined by (2.3). From [14], one can find a boundary control h verifying (1.3), provided there is $N = N(|\gamma|, \Omega, r, \delta)$ such that the inequality

$$(2.16) \quad \sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \leq e^{N\mu} \int \int_{\gamma \times [\frac{1}{4}, \frac{3}{4}]} \left| \sum_{\omega_j \leq \mu} (a_j e^{\omega_j y} + b_j e^{-\omega_j y}) \frac{\partial e_j}{\partial n} \right|^2 d\sigma dy,$$

holds for $\mu \geq \omega_1$ and all sequences a_1, a_2, \dots and b_1, b_2, \dots . Here, ν , σ and $\frac{\partial}{\partial n}$ denote respectively the exterior unit normal vector to Ω , the surface measure on $\partial\Omega$ and the conormal derivative for $\partial_y^2 + \Delta$ on $\partial\Omega \times \mathbb{R}$, $\frac{\partial e}{\partial n} = A \nabla_x e \cdot \nu$. We may also assume that $0 \in \partial\Omega$, $\gamma \subset B_{\frac{\rho}{2}} \cap \partial\Omega$, where $B_{2\rho} \cap \partial\Omega$ is the region above the graph of a real analytic function $\varphi : B'_\rho \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, as in (2.1). From [14, §3 (2)], there is N such that

$$(2.17) \quad \sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \leq e^{N\mu} \left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(\partial\Omega \times [-1, 2])},$$

with

$$\frac{\partial u}{\partial n} = \sum_{\omega_j \leq \mu} (a_j e^{\omega_j y} + b_j e^{-\omega_j y}) \frac{\partial e_j}{\partial n}.$$

From (2.5) and (2.1), there are $N = N(r, \delta)$ and $\rho = \rho(r, \delta)$ such that $h(x', y) = \frac{\partial u}{\partial n}(x', \varphi(x'), y)$ verifies

$$\|\partial_{x'}^\alpha \partial_y^\beta h\|_{L^\infty(B'_{2\rho} \times [-4, 4])} \leq M (|\alpha| + \beta)! (2\rho)^{-|\alpha| - \beta}, \text{ for } \alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N},$$

$$M = e^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}},$$

when $B_{2\rho} \cap \partial\Omega$ is a coordinate chart of $\partial\Omega$ as in (2.1). This fact, a suitable covering argument of $\partial\Omega$ and the *three-spheres inequalities* associated to the *obvious*

extension of Theorem 3 for real analytic functions over a compact analytic surface in \mathbb{R}^{n+1} , imply there are $N = N(|\gamma|, r, \delta)$ and $\theta = \theta(|\gamma|, r, \delta)$ such that

$$(2.18) \quad \left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(\partial\Omega \times [-1, 2])} \leq N \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\gamma \times [\frac{1}{4}, \frac{3}{4}])}^\theta M^{1-\theta}.$$

Finally, (2.16) follows from (2.17) and (2.18). \square

Remark 2. The extension of Theorem 1 to the parabolic system

$$(2.19) \quad \begin{cases} \partial_i(a_{ij}^{\alpha\beta} \partial_j e_k^\beta) - \partial_t u^\alpha = f^\alpha(x, t) \chi_\omega(x), & \text{in } \Omega_T, \alpha = 1, \dots, m, \\ \mathbf{u} = 0, & \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega. \end{cases}$$

with $\partial\Omega$ as in (2.1), $a_{ij}^{\alpha\beta}$ verifying (1.6) and

$$(2.20) \quad |\partial^\gamma a_{ij}^{\alpha\beta}(x)| \leq |\alpha|! \delta^{-|\alpha|-1}, \text{ when } x \in \overline{\Omega}, \gamma \in \mathbb{N}^n,$$

for some $0 < \delta \leq 1$ is now obvious: the symmetry, coerciveness and compactness of the operator $L^2(\Omega)^m \rightarrow W_0^{1,2}(\Omega)^m$, mapping $\mathbf{f} = (f^1, \dots, f^m)$ into the unique solution $\mathbf{u} = (u^1, \dots, u^m)$ to

$$\begin{cases} \partial_i(a_{ij}^{\alpha\beta} \partial_j u^\beta) - \Lambda u^\alpha = f^\alpha, & \text{in } \Omega, \alpha = 1, \dots, m, \\ \mathbf{u} = 0, & \text{in } \partial\Omega \end{cases}$$

where $\Lambda > 0$ is sufficiently large [10, Prop. 2.1], gives the existence of a complete system $\{\mathbf{e}_k\}$ in $L^2(\Omega)^m$, $\mathbf{e}_k = (e_k^1, \dots, e_k^m)$, of eigenfunctions verifying

$$\begin{cases} \partial_i(a_{ij}^{\alpha\beta} \partial_j e_k^\beta) + \omega_k^2 e_k^\alpha = 0, & \text{in } \Omega, \alpha = 1, \dots, m, \\ \mathbf{e}_k = 0, & \text{in } \partial\Omega \end{cases}$$

with eigenvalues $0 \leq \omega_1 \leq \dots \leq \omega_k \leq \dots$ and $\lim_{k \rightarrow +\infty} \omega_k = +\infty$. By separation of variables, the Green's matrix for the system (2.19) over $\Omega \times \mathbb{R}$ is the $m \times m$ matrix

$$\mathbf{\Gamma}(x, y, t - s) = \sum_{k=1}^{+\infty} e^{-\omega_k^2(t-s)} \mathbf{e}_k(x) \otimes \mathbf{e}_k(y).$$

Moreover, the interior and boundary regularity for the elliptic system $\partial_y^2 + \partial_i(a_{ij}^{\alpha\beta} \partial_j)$ in $\Omega \times \mathbb{R}$, shows that (2.4) holds for \mathbf{u} as in (2.3) but with \mathbf{e}_k replacing e_k ([19], [10, Chapter II]). These and [14] suffice to find a control function \mathbf{f} for the system (2.19) verifying (1.2). Furthermore, if you wish to get bounds on the regularity of \mathbf{f} , [14] shows it suffices to know that, $\|e_k\|_{H^s(\Omega)} \leq C_s(1 + \omega_k)^s$, for $s \geq 0$, and that the number of $0 \leq \omega_k \leq \mu$, is bounded by $N\mu^n$, when $\mu \geq 1$. The first holds from elliptic regularity and (2.20), while the second follows from the Gaussian estimates verified by $\mathbf{\Gamma}$, i.e., there are N and κ [8, Corollary 4.14] such that

$$|\mathbf{\Gamma}(x, y, t)| \leq N(1 \wedge t)^{-\frac{n}{2}} e^{\Lambda t - \kappa|x-y|^2/t}, \text{ for } x, y \in \mathbb{R}^n, t > 0.$$

It implies,

$$\int_{\Omega} \int_{\Omega} |\mathbf{\Gamma}(x, y, t)|^2 dx dy = \sum_{k \geq 1} e^{-2\omega_k^2 t} \leq N e^{2\Lambda t} |\Omega| t^{-\frac{n}{2}},$$

and suffices to choose $t = \frac{1}{\mu^2}$. In particular, $\mathbf{f} \in C_0^\infty((0, T), C^\infty(\overline{\Omega}))$.

The null-controllability of the system

$$(2.21) \quad \begin{cases} \partial_t u + (-1)^m \Delta^m u = f(x, t) \chi \omega, & \text{in } \Omega \times (0, T], \\ u = \nabla u = \dots = \nabla^{m-1} u = 0, & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

$m \geq 2$, is better managed with the approach in [15]. If $\{e_j\}$ and $0 \leq \omega_1^{2m} \leq \dots \leq \omega_k^{2m} \leq \dots$ are the eigenvectors and eigenvalues for Δ^m in $W_0^{2,m}(\Omega)$,

$$\begin{cases} (-1)^m \Delta^m e_j - \omega_j^{2m} e_j = 0, & \text{in } \Omega, \\ e_j = \nabla e_j = \dots = \nabla^{m-1} e_j = 0, & \text{in } \partial\Omega, \end{cases}$$

$$u(x, y) = \sum_{\omega_j^m \leq \mu} a_j X_j(y) e_j(x),$$

verifies, $\partial_y^2 u + \Delta^m u = 0$ in $\Omega \times \mathbb{R}$, $u = \nabla u = \dots = \nabla^{m-1} u = 0$ in $\partial\Omega \times \mathbb{R}$, when

$$X_j(y) = \begin{cases} e^{\omega_j y}, & \text{for } m \text{ odd}, \\ e^{\omega_j e^{\frac{\pi i}{2m} y}}, & \text{for } m \text{ even}. \end{cases}$$

Again, u verifies (2.4) [19] and from Theorem 3 applied to $u(\cdot, 0)$ as in Remark 1,

$$\sum_{\omega_j^m \leq \mu} a_j^2 \leq e^{N\mu^{\frac{1}{m}}} \int_{\omega} \left| \sum_{\omega_j^m \leq \mu} a_j e_j \right|^2 dx, \quad \text{with } N = N(|\omega|, \Omega, r, \delta, m).$$

From [15], the last inequality suffices to find a control f in $L^2(0, T, L^2(\Omega))$ for (2.21).

Remark 3. The proof of Theorem 2 for the scalar case is based on the bound (2.17). In the literature it is obtained from (2.10) and the interpolation inequality below, proved via Carleman inequalities: there are N and θ such that

$$\|u\|_{L^2(\Omega \times [0,1])} \leq N \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\partial\Omega \times [-1,2])}^{\theta} \|u\|_{L^2(\Omega \times [-3,3])}^{1-\theta},$$

holds whenever u verifies $\Delta u + \partial_y^2 u = 0$ in $\Omega \times \mathbb{R}$ and $u = 0$ on $\partial\Omega \times \mathbb{R}$. However, the authors are not aware whether the corresponding interpolation inequality for solutions \mathbf{u} of the elliptic system

$$\begin{cases} \partial_i (a_{ij}^{\alpha\beta} \partial_j u^\beta) + \partial_y^2 u^\alpha = 0, & \text{in } \Omega \times \mathbb{R}, \quad \alpha = 1, \dots, m, \\ \mathbf{u} = 0, & \text{in } \partial\Omega \times \mathbb{R}, \end{cases}$$

$$\|\mathbf{u}\|_{L^2(\Omega \times [0,1])} \leq N \left\| \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \right\|_{L^2(\partial\Omega \times [-1,2])}^{\theta} \|\mathbf{u}\|_{L^2(\Omega \times [-3,3])}^{1-\theta},$$

holds. Here, $(\frac{\partial \mathbf{u}}{\partial \mathbf{n}})^\alpha = a_{ij}^{\alpha\beta} \partial_j u^\beta \nu_i$, $\alpha = 1, \dots, m$, and for this reason we can not extend Theorem 2 to parabolic systems. On the other hand, there is $\rho > 0$ such that the mapping $\partial\Omega \times (0, \rho) \rightarrow U_\rho$, $(Q, t) \rightsquigarrow Q + t\nu(Q)$, is an analytic diffeomorphism onto $U_\rho = \{x \in \mathbb{R}^n \setminus \bar{\Omega} : 0 < d(x, \partial\Omega) < \rho\}$. Because the null-control of the parabolic system over $\bar{\Omega} \cup U_\rho$ and with controls acting over $\omega = U_{\frac{3\rho}{4}} \setminus U_{\frac{\rho}{2}}$ is possible, standard arguments show that the system

$$\begin{cases} \partial_i (a_{ij}^{\alpha\beta} \partial_j e_k^\beta) - \partial_t u^\alpha = 0, & \text{in } \Omega_T, \quad \alpha = 1, \dots, m, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega. \end{cases}$$

can be null-controlled with controls \mathbf{g} acting over the full lateral boundary of Ω .

3. PROOF OF THEOREM 3

First we recall Hadamard's three-circle theorem [17] and prove two Lemmas before the proof of Theorem 3.

Theorem 4. *Let F be a holomorphic function of a complex variable in the ball B_{r_2} . Then, the following is valid for $0 < r_1 \leq r \leq r_2$,*

$$\|F\|_{L^\infty(B_r)} \leq \|F\|_{L^\infty(B_{r_1})}^\theta \|F\|_{L^\infty(B_{r_2})}^{1-\theta}, \quad \theta = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}.$$

Lemma 1. *Let f be holomorphic in B_1 , $|f(z)| \leq 1$ in B_1 and E be a measurable set in $[-\frac{1}{5}, \frac{1}{5}]$. Then, there are $N = N(|E|)$ and $\gamma = \gamma(|E|)$ such that*

$$\|f\|_{L^\infty(B_{\frac{1}{2}})} \leq N \|f\|_{L^\infty(E)}^\gamma$$

Proof. For $n \geq 1$, there are $n+1$ points with $-\frac{1}{5} \leq x_0 < x_1 < \dots < x_n \leq \frac{1}{5}$, with $x_i \in \overline{E}$, $i = 0, \dots, n$ and $x_i - x_{i-1} \geq \frac{|E|}{n+1}$, $i = 1, \dots, n$. For example, $x_0 = \inf E$, $x_i = \inf \left(E \cap [x_{i-1} + \frac{|E|}{n+1}, \frac{1}{5}] \right)$. Let

$$P_n(z) = \sum_{i=0}^n f(x_i) \frac{\prod_{j \neq i} (z - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

Then,

$$|P_n(z)| \leq \|f\|_{L^\infty(E)} |E|^{-n} \sum_{i=0}^n \frac{(n+1)^n}{i!(n-i)!} \leq \|f\|_{L^\infty(E)} \left(\frac{3}{|E|} \right)^n, \quad \text{for } |z| \leq \frac{1}{2}.$$

By Cauchy's formula,

$$|f(z) - P_n(z)| = \left| \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)(z-x_0)\dots(z-x_n)}{(\xi-z)(\xi-x_0)\dots(\xi-x_n)} d\xi \right| \leq 2 \left(\frac{7}{8} \right)^n, \quad \text{for } |z| \leq \frac{1}{2}.$$

The last two inequalities give

$$(3.1) \quad \|f\|_{L^\infty(B_{\frac{1}{2}})} \leq \|f\|_{L^\infty(E)} \left(\frac{3}{|E|} \right)^n + 2 \left(\frac{7}{8} \right)^n, \quad \text{for all } n \geq 1,$$

and the minimization in the n -variable of the right hand side of (3.1) implies Lemma 1. \square

Lemma 2. *Let f be analytic in $[0, 1]$, E be a measurable set in $[0, 1]$ and assume there are positive constants M and ρ such that*

$$(3.2) \quad |f^{(k)}(x)| \leq Mk!(2\rho)^{-k}, \quad \text{for } k \geq 0, \quad x \in [0, 1].$$

Then, there are $N = N(\rho, |E|)$ and $\gamma = \gamma(\rho, |E|)$ such that

$$\|f\|_{L^\infty([0,1])} \leq N \|f\|_{L^\infty(E)}^\gamma M^{1-\gamma}.$$

Proof. (3.2) implies that f has a holomorphic extension to $D_\rho = \cup_{0 \leq x \leq 1} B(x, \rho)$, with $|f| \leq 2M$ in D_ρ . Write $[0, 1]$ as a disjoint union of $\frac{5}{2\rho}$ non-overlapping closed intervals of length $\frac{2\rho}{5}$. Among them there is at least one, $I = [x_0 - \frac{\delta}{5}, x_0 + \frac{\delta}{5}]$, such that $|E \cap I| \geq \frac{2\delta|E|}{5}$. Then, $g(z) = f(x_0 + \delta z)/2M$ is holomorphic in B_1 ,

$E_{x_0, \rho} = \rho^{-1}(E \cap I - x_0)$ is measurable in $[-\frac{1}{5}, \frac{1}{5}]$ with measure bounded from below by $\frac{2|E|}{5}$, $\|g\|_{L^\infty(E_{x_0, \rho})} \leq \|f\|_{L^\infty(E)}$ and applying Lemma 1 to g

$$(3.3) \quad \|f\|_{L^\infty(B_{\frac{\rho}{2}}(x_0))} \leq N \|f\|_{L^\infty(E)}^\gamma M^{1-\gamma},$$

with $0 \leq x_0 \leq 1$. Finally, make successive applications of Hadamard's three-circle theorem (a finite number depending on ρ) with a suitable chain of three-circles of radius comparable to ρ and with center at points x in $[0, 1]$ contained in D_ρ , while recalling that on the largest ball $|f| \leq 2M$, to get that

$$(3.4) \quad \|f\|_{L^\infty([0,1])} \leq N \|f\|_{L^\infty(B_{\frac{\rho}{2}}(x_0))}^\theta M^{1-\theta}, \quad \theta = \theta(\rho),$$

and Lemma 2 follows from (3.4) and (3.3). \square

Proof of Theorem 3. We may assume $R = 1$. Let $x \in B_{\frac{1}{2}}$. Using spherical coordinates centered at x ,

$$|E| \leq \int_{S^{n-1}} |\{t \in [0, 1] : x + tz \in E\}| dz,$$

and there is at least one $z \in S^{n-1}$ with $|\{t \in [0, 1] : x + tz \in E\}| \geq |E|/(2\omega_n)$, with ω_n the surface measure on S^{n-1} . Set $\varphi(t) = f(x + tz)$. From (1.5), φ satisfies (3.2), $\|\varphi\|_{L^\infty(E_z)} \leq \|f\|_{L^\infty(E)}$ and Lemma 2 gives

$$(3.5) \quad \|f\|_{L^\infty(B_{\frac{1}{2}})} \leq N \|f\|_{L^\infty(E)}^\gamma M^{1-\gamma}.$$

Finally, setting

$$\tilde{E} = \{x \in E : |f(x)|/2 \leq \int_E |f| dx\},$$

Chebyshev's inequality shows that

$$|\tilde{E}| \geq |E|/2, \quad \|f\|_{L^\infty(\tilde{E})} \leq 2 \int_E |f| dx,$$

and Theorem 3 follows after replacing E by \tilde{E} in (3.5). \square

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(J. Apraiz) UNIVERSIDAD DEL PAÍS VASCO/EUSKAL HERRIKO UNIBERTSITATEA, DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA UNIVERSITARIA POLITÉCNICA DE DONOSTIA-SAN SEBASTIÁN, PLAZA DE EUROPA 1, 20018 DONOSTIA-SAN SEBASTIÁN, SPAIN.

E-mail address: `jone.apraiz@ehu.es`

(L. Escauriaza) UNIVERSIDAD DEL PAÍS VASCO/EUSKAL HERRIKO UNIBERTSITATEA, DPTO. DE MATEMÁTICAS, APTO. 644, 48080 BILBAO, SPAIN.

E-mail address: `luis.escauriaza@ehu.es`