TRANSPORT EQUATIONS WITH FRACTAL NOISE - EXISTENCE, UNIQUENESS AND REGULARITY OF THE SOLUTION

ELENA ISSOGLIO

ABSTRACT. The main result of the present paper is a statement on existence, uniqueness and regularity for mild solutions to a parabolic transport diffusion type equation that involves a non-smooth coefficient. We investigate related Cauchy problems on bounded smooth domains with Dirichlet boundary conditions by means of semigroup theory and fixed point arguments. Main ingredients are the definition of a product of a function and a (not too irregular) distribution as well as a corresponding norm estimate. As an application, transport stochastic partial differential equations driven by fractional Brownian noises are considered in the pathwise sense.

1. INTRODUCTION

We consider the following transport equation on a domain $D \subset \mathbb{R}^d$ with initial and Dirichlet boundary conditions:

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \langle \nabla u, \nabla Z \rangle (t,x), & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases}$$

where D is a bounded open set of $\mathbb{R}^d$ with C^∞ boundary, u_0 is a given function in some appropriate space, Z is a given non-differentiable function on $\mathbb{R}^d$ and the derivative is taken in the distributional sense. The gradient ∇ as well as the Laplacian Δ refer to the space variables. The precise definition of the product $\langle \nabla u, \nabla Z \rangle (t,x)$ will be given below, and it is set by use of the Fourier transform.

The aim of this paper is to give a meaning to the formal problem (1) and to investigate existence, uniqueness and regularity of corresponding solutions.
We rewrite problem (1) in the abstract Cauchy setting, namely we interpret all mappings as functions of time t taking values in some suitable function space X (real function space on $\mathbb{R}^d$, our choice will be specified later). Set $\underline{u} : [0,T] \to X, t \mapsto \underline{u}(t) \in X$ and $(\underline{u}(t))(\cdot) := u(t,\cdot)$. The Dirichlet initial value problem becomes the following abstract Cauchy problem

$$(2) \qquad \begin{cases} \frac{d}{dt}\underline{u} = \Delta_D \underline{u} + \langle \nabla \underline{u}, \nabla Z \rangle, & t \in (0,T] \\ \underline{u} = u_0, & t = 0 \end{cases}$$

where Δ_D stands for the Dirichlet-Laplace operator.
Note that we need some care to give an appropriate definition for the product term $\langle \nabla \underline{u}, \nabla Z \rangle$: In the cases we consider the components of ∇Z will be distributions.

This is not covered by results in the standard literature for partial differential equations (PDEs) (see for instance [5, 12]). We use a priori estimates on this product which lead to optimal regularity results. To our knowledge, this has not been considered anywhere else.

There is a rich literature regarding stochastic PDEs (SPDEs) (see for instance [1, 2, 8] and references therein). In these references the noise is assumed to be of Brownian (or semimartingale) type.

There are also results on SPDEs involving fractional Brownian (or general non-semimartingale) type noises (see for instance [4, 6, 7, 9, 13, 18]) but it seems that there are few results on transport diffusion equations with random non-smooth drift of the form (2).

To our knowledge, the only study regarding this problem is due to Russo and Trutnau [16] where they investigate a stochastic equation like (16) (which is the stochastic analog of (1)) but in space dimension one. The authors proceed by freezing the realization of the noise for each ω and overcome the problem of defining the product between a function and a distribution by means of a probabilistic representation: They express the parabolic PDE probabilistically through the associated diffusion which is the solution of a stochastic differential equation with generalized drift.

In the present paper Fourier analysis is used to define *pointwise* products that work for any space dimension (see Proposition 2.5). The product itself will be a distribution.

We proceed as follows: In Section 2, after having introduced the framework and the notion of (mild) solution, we define an integral operator I. The product together with the action of the semigroup and an integration with respect to time will define the integral operator.

In Section 3 we first collect some useful a priori estimates and bounds, then we state the key Theorem 3.4 dealing with the mapping property of the integral operator in the spaces $C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ and finally we state the main result in Theorem 3.5. By a contraction argument and under suitable conditions on the papameters $\gamma, \delta > 0$, on the noise and on the initial condition we find a unique (mild) solution for (1) in the above-mentioned space. Of interest is the fact that the solution is actually a function, even though we make use of fractional Sobolev spaces of negative index (spaces of distributions) while proving the desired result.

In Section 4 we conclude the paper presenting some applications to stochastic PDEs. We are namely able to solve a class of SPDEs where the noise is, for instance, a temporally homogeneous fractional Brownian field. Moreover combining it with a results of Hinz and Zähle [7] we can treat the more general (stochastic) transport equation of the form

$$
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \langle \nabla u, \nabla Z \rangle(t,x) + \langle F, \frac{\partial}{\partial t} \nabla V \rangle(t,x), & t \in (0,T], x \in D \\
u(t,x) = 0, & t \in (0,T], x \in \partial D \\
u(0,x) = u_0(x), & x \in D
\end{cases}
$$

where F is a given vector and $V = V(t,x)$ is a given non-differentiable function. Throughout the whole paper c denotes a finite positive constant whose exact value is not important and may change from line to line.

2. Preliminaries

2.1. **Framework.** Recall the definition of fractional Sobolev spaces (Bessel potential spaces) on $\mathbb{R}^d$. For $\alpha \in \mathbb{R}$ and $1 < p < \infty$ set

$$H_p^\alpha(\mathbb{R}^d; \mathbb{C}) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C}) : ((1 + |\xi|^2)^{\frac{\alpha}{2}} \hat{f})^\vee \in L_p(\mathbb{R}^d, \mathbb{C}) \right\},$$

equipped with the norm $\|f|H_p^\alpha(\mathbb{R}^d; \mathbb{C})\| = \|((1 + |\xi|^2)^{\frac{\alpha}{2}} \hat{f})^\vee | L_p(\mathbb{R}^d; \mathbb{C})\|$, where $\hat{f}$ stands for the Fourier transform of f on $\mathbb{R}^d$ and $(\cdot)^\vee$ denotes the inverse Fourier transform. We are interested only in real valued distributions (and functions) so we follow [15] and define $\mathcal{S}'(\mathbb{R}^d; \mathbb{R}) := \{ f \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C}) : \bar{f} = f \}$ where $\bar{f}$ is defined by $\bar{f}(\phi) = f(\bar{\phi})$ for all $\phi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$. For $1 < p < \infty$ and $\alpha \in \mathbb{R}$ we define $H_p^\alpha(\mathbb{R}^d; \mathbb{R}) := H_p^\alpha(\mathbb{R}^d; \mathbb{C}) \cap \mathcal{S}'(\mathbb{R}^d; \mathbb{R})$.

For simplicity of notation we omit the writing of the codomain when it is $\mathbb{R}$.

The corresponding Sobolev spaces on D, suitable for our purposes, are defined for all $\alpha > -\frac{1}{2}$ as

$$\tilde{H}_p^\alpha(D) := \left\{ f \in H_p^\alpha(\mathbb{R}^d) : \operatorname{supp}(f) \subset \bar{D} \right\}$$

equipped with the norm in $H_p^\alpha(\mathbb{R}^d)$. Observe that if $\alpha = 0$ then the space $\tilde{H}_p^0(D)$ is simply $L_p(D)$. Such spaces are embedded in each other in the following way: For all $\alpha > \beta$, $H_p^\alpha(\mathbb{R}^d) \subset H_p^\beta(\mathbb{R}^d)$. An analogous relation holds for the spaces on domain D for all $\alpha > \beta > -\frac{1}{2}$.

We omit the subscript index p if $p = 2$. In this case the norm in $H^\alpha(\mathbb{R}^d)$ is denoted by $\| \cdot \|_\alpha$. Moreover when we have a vector (like ∇Z) we write $\nabla Z \in H_p^\alpha(\mathbb{R}^d)$ (and similarly for spaces on D) to intend that every component of the vector ∇Z belongs to such space. The norm of a d-dimensional vector in the space $(H_p^\alpha(\mathbb{R}^d))^d$ is defined as the square root of the sum of the squared norm of each component in $H_p^\alpha(\mathbb{R}^d)$. For simplicity we will indicate it with the same notation.

Consider now the Dirichlet Laplacian Δ_D as the infinitesimal generator of the Dirichlet heat semigroup acting on $L_2(D)$ (see e. g. [22] Section 4.1, [5] Section 7.4.3). Throughout the whole paper we will indicate it with $\Delta_D = -A$. More precisely $-A$ generates a compact $\mathscr{C}_0$ semigroup of contractions $(P_t)_{t\geq 0}$ in $L_2(D)$ (see [22], Theorem 7.2.5). The semigroup is of negative type and symmetric. Moreover it is known that if the semigroup is contractive and symmetric it is also analytic (see [3], Theorem 1.4.1 or [17], Chapter III), thus one can define fractional powers of A of any order (see for instance [14]).

It can be shown (see [21] equations (27.50) and (27.51) or [20] Section 4.9.2) that for all $\gamma, \alpha \in \mathbb{R}$ such that $-\frac{1}{2} < \gamma, \gamma - \alpha < \frac{3}{2}$ the fractional power $A^{\frac{\alpha}{2}}$ maps isomorphically $\tilde{H}^\gamma(D)$ onto $\tilde{H}^{\gamma - \alpha}(D)$, hence there exist $c_1 > 0$ and $c_2 > 0$ such that for all $f \in \tilde{H}^\gamma(D)$

$$(3) \qquad \left\| A^{\frac{\alpha}{2}} f \right\|_{\gamma - \alpha} \leq c_1 \|f\|_\gamma \leq c_2 \left\| A^{\frac{\alpha}{2}} f \right\|_{\gamma - \alpha}.$$

Furthermore one can prove that $\mathcal{D}(A^{\frac{\alpha}{2}}) = \tilde{H}^\alpha(D)$ for all $0 < \alpha < \frac{3}{2}, \alpha \neq \frac{1}{2}$ (for details see [20]).

We recall now a standard result on semigroups, for a proof we refer to [14] Theorem II.6.13 or [22] Theorem 7.7.2.

Theorem 2.1. *Let $-\mathcal{A}$ be the infinitesimal generator of an analytic semigroup T_t on a Banach space $(X, \| \cdot \|_X)$. If for each $t \geq 0$ holds $\|T_t\| \leq M e^{-\omega t}$ with $M \geq 1$ and $\omega > 0$ then*

(a) $T_t : X \to \mathcal{D}(\mathcal{A}^\alpha)$ *for every $t > 0, \alpha \geq 0$;*

(b) *for every $\alpha \geq 0$ and for every $x \in \mathcal{D}(\mathcal{A}^\alpha)$, $T_t \mathcal{A}^\alpha x = \mathcal{A}^\alpha T_t x$;*

(c) *for every $t > 0$ and for every $\alpha \geq 0$ the operator $\mathcal{A}^\alpha T_t$ is bounded and linear and there exist constants M_α (which depends only on α) and $\theta \in (0, \omega)$ such that*

$$\|\mathcal{A}^\alpha T_t\|_{\mathscr{L}(X)} \leq M_\alpha e^{-\theta t} t^{-\alpha};$$

(d) *for each $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that $\forall t > 0$ and for each $x \in \mathcal{D}(\mathcal{A}^\alpha)$ we have*

$$\|T_t x - x\|_X \leq C_\alpha t^\alpha \|\mathcal{A}^\alpha x\|_X.$$

As a consequence of this theorem and of relation (3) we get the following result.

Corollary 2.2. *Let $(P_t)_{t \geq 0}$ be the Dirichlet heat semigroup on $L_2(D)$. Then for all positive t and for any $-\frac{1}{2} < \rho, \gamma, \rho + \gamma < \frac{3}{2}$ we have*

(4) $$P_t : \tilde{H}^\gamma(D) \to \tilde{H}^{\rho+\gamma}(D).$$

In particular if $f \in \tilde{H}^\gamma(D)$ then $\mathrm{supp}(P_t f) \subset \bar{D}$.

Proof. Consider first the case when $\gamma > 0$. Let $f \in \tilde{H}^\gamma(D)$ so by (3) we have $g := A^{\frac{\gamma}{2}} f \in L_2(D)$. We write $P_t f = P_t A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} f = P_t A^{-\frac{\gamma}{2}} g = A^{-\frac{\gamma}{2}} P_t g$ and by Theorem 2.1 (a) we know that $P_t g \in \mathcal{D}(A^\rho)$ for any $\rho \geq 0$. Moreover recall that $\mathcal{D}(A^2 \rho) = \tilde{H}^\rho(D)$ for all $0 \leq \rho < \frac{3}{2}, \rho \neq \frac{1}{2}$, so for this choice of ρ and using (3) we get $P_t f = A^{-\frac{\gamma}{2}} P_t g \in \tilde{H}^{\rho+\gamma}(D)$. Observe that this fact is true also if $\rho = \frac{1}{2}$ since $\tilde{H}^{\rho+\gamma}(D) \subset \tilde{H}^{\frac{1}{2}+\gamma}(D)$ for all $\rho > \frac{1}{2}$.

The case when $\gamma < 0$ is proven in the same way, simply write $A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} P_t f$ instead of $P_t A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} f$. $\square$

2.2. **Mild solutions.** A function $\underline{u}$ is a *mild solution* of (2) if it satisfies the following integral equation

(5) $$\underline{u}(t) = P_t u_0 + \int_0^t P_{t-r} \langle \nabla \underline{u}(r), \nabla Z \rangle \, \mathrm{d}r$$

To give a formal meaning to the product $\langle \cdot, \cdot \rangle$ we make use of the so called *para-product*, see e. g. [15]. We shortly recall the definition and some useful properties. Suppose we are given $f \in \mathcal{S}'(\mathbb{R}^d)$. Choose a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \leq \psi(x) \leq 1$ for every $x \in \mathbb{R}^d$, $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq \frac{3}{2}$. Then consider the following approximation of f

$$S^j f(x) := \left(\psi \left(\frac{\xi}{2^j} \right) \hat{f} \right)^\vee (x)$$

that is in fact the convolution of f with a smoothing function. This approximation is used to define the product of two distributions fg as follows:

$$fg := \lim_{j \to \infty} S^j f S^j g$$

if the limit exists in $\mathcal{S}'(\mathbb{R}^d)$. The convergence in the case we are interested in is part of the assertion below (see [7] appendix C.4, [15] Theorem 4.4.3/1).

Lemma 2.3. *Let $1 < p, q < \infty$ and $0 < \beta < \delta$ and assume that $q > \max(p, \frac{d}{\delta})$. Then for every $f \in H_p^\delta(\mathbb{R}^d)$ and $g \in H_q^{-\beta}(\mathbb{R}^d)$ we have*

(6) $$\|fg|H_p^{-\beta}(\mathbb{R}^d)\| \leq c\|f|H_p^\delta(\mathbb{R}^d)\| \cdot \|g|H_q^{-\beta}(\mathbb{R}^d)\|.$$

The following Lemma regarding a *locality-preserving* property will be used while shifting the properties of the product fg from the whole $\mathbb{R}^d$ to the domain D. For the proof see [15] Lemma 4.2.

Lemma 2.4. *If $f, g \in \mathcal{S}'(\mathbb{R}^d)$ and $\mathrm{supp}(f) \subset \bar{D}$ then also $\mathrm{supp}(fg) \subset \bar{D}$.*

Our aim now is to apply such product to $\nabla \underline{u}(s)$ and ∇Z. We will denote by $\langle \cdot, \cdot \rangle$ the pointwise product combined with the scalar product in $\mathbb{R}^d$.

Proposition 2.5. *Let $\underline{u}(s) \in \tilde{H}_p^{1+\delta}(D)$, $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for $1 < p, q < \infty$, $q > \max(p, \frac{d}{\delta})$, $0 < \beta < \frac{1}{2}$ and $\beta < \delta$. Then the pointwise multiplication $\langle \nabla \underline{u}(s), \nabla Z \rangle$ is well defined, it belongs to the space $\tilde{H}_p^{-\beta}(D)$ and we have the following bound*

$$(7) \qquad \|\langle \nabla \underline{u}(s), \nabla Z \rangle | \tilde{H}_p^{-\beta}(D)\| \leq c \|\nabla \underline{u}(s) | \tilde{H}_p^{\delta}(D)\| \cdot \|\nabla Z | H_q^{-\beta}(\mathbb{R}^d)\|.$$

Proof. The idea is to apply first Lemma 2.3 to define the product as an element of $H_p^{-\beta}(\mathbb{R}^d)$ and then restrict it to $\tilde{H}_p^{-\beta}(D)$ with the help of Lemma 2.4.

Let $f = \nabla \underline{u}(s)$ and $g = \nabla Z$. We should check the conditions in Lemma 2.3. Clearly $g \in H_q^{-\beta}(\mathbb{R}^d)$ because $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ and it is easy to show that $(\nabla)_i$ is bounded form $H^\gamma(\mathbb{R}^d)$ to $H^{\gamma-1}(\mathbb{R}^d)$ for every $\gamma \in \mathbb{R}$ and for all $i = 1, \ldots, d$. The fact that $f \in H_p^{\delta}(\mathbb{R}^d)$ is also clear since $\tilde{H}_p^{1+\delta}(D) \subset H_p^{1+\delta}(\mathbb{R}^d)$.

Denote $m(s) := \langle \nabla \underline{u}(s), \nabla Z \rangle \in H_p^{-\beta}(\mathbb{R}^d)$ and by Lemma 2.3 we get

$$\|m(s) | H_p^{-\beta}(\mathbb{R}^d)\| \leq c \|\nabla \underline{u}(s) | H_p^{\delta}(\mathbb{R}^d)\| \cdot \|\nabla Z | H_q^{-\beta}(\mathbb{R}^d)\| < \infty.$$

Since $\mathrm{supp}\, \underline{u}(s) \subset \bar{D}$ then $\mathrm{supp}\, \nabla \underline{u}(s) \subset \bar{D}$ and so by Lemma 2.4 it follows $\mathrm{supp}\, m(s) \subset \bar{D}$ and so $m(s) \in \tilde{H}_p^{-\beta}(D)$ since $\beta < \frac{1}{2}$. Moreover,

$$\|\langle \nabla \underline{u}(s), \nabla Z \rangle | \tilde{H}_p^{-\beta}(D)\| = \|\langle \nabla \underline{u}(s), \nabla Z \rangle | H_p^{-\beta}(\mathbb{R}^d)\|$$
$$\leq c \|\nabla \underline{u}(s) | H_p^{\delta}(\mathbb{R}^d)\| \cdot \|\nabla Z | H_q^{-\beta}(\mathbb{R}^d)\| = c \|\nabla \underline{u}(s) | \tilde{H}_p^{\delta}(D)\| \cdot \|\nabla Z | H_q^{-\beta}(\mathbb{R}^d)\|.$$

$\square$

The notion of mild solution is now formalized. Next we check the convergence of the integral, so for any fixed $\underline{u}(r) \in \tilde{H}^{1+\delta}(D)$ define the integral operator I by

$$(8) \qquad I_t(\underline{u}) := \int_0^t P_{t-r} \langle \nabla \underline{u}(r), \nabla Z \rangle \, \mathrm{d}r$$

for any $t \in [0, T]$. We consider this operator acting on the Hölder space $C^\gamma([0,T]; X)$ into itself (this mapping property will be proven later, see Theorem 3.4) for some suitable γ and for some infinite dimensional Banach space X. The Hölder space is defined as

$$C^\gamma([0,T]; X) := \{h : [0,T] \to X \text{ s.t. } \|h\|_{\gamma, X} < \infty\}$$

where

$$\|h\|_{\gamma, X} := \sup_{t \in [0,T]} \|h(t)\|_X + \sup_{s < t \in [0,T]} \frac{\|h(t) - h(s)\|_X}{(t-s)^\gamma}.$$

When $X = \tilde{H}^{1+\delta}(D)$ the norm will be indicated by $\|\cdot\|_{\gamma, 1+\delta}$. Next we introduce a family of equivalent norms $\|\cdot\|_{\gamma, X}^{(\rho)}, \rho \geq 1$ defined by

$$\|f\|_{\gamma, X}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \left(\|f(t)\|_X + \sup_{0 \leq s < t} \frac{\|f(t) - f(s)\|_X}{(t-s)^\gamma} \right).$$

3. The main result

In this section we prove the contractivity of the operator I in the Hölder space $C^{\gamma}([0,T];\tilde{H}^{1+\delta}(D))$.

3.1. **Mapping property of I.** Recall that $m(r) := \langle \nabla \underline{u}(r), \nabla Z \rangle$ for all $0 \leq r \leq T$.

Proposition 3.1. *Let $0 < \beta < \frac{1}{2}$ and $\beta < \delta$ and fix a function $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for some $q > \max(2,\frac{d}{\delta})$. Then for all $0 \leq r \leq t \leq T$ and $\underline{u}(t) \in \tilde{H}^{1+\delta}(D)$ we have*

(1) $\|m(r)|\tilde{H}^{-\beta}(D)\| \leq c\|\underline{u}(r)|\tilde{H}^{1+\delta}(D)\|;$

(2) $\|m(t) - m(r)|\tilde{H}^{-\beta}(D)\| \leq c\|\underline{u}(t) - \underline{u}(r)|\tilde{H}^{1+\delta}(D)\|.$

Proof. To see (1), observe that by definition $\nabla \underline{u}(r) \in \tilde{H}^{\delta}(D)$ means that $\nabla \underline{u}(r) \in H^{\delta}(\mathbb{R}^d)$ and $\text{supp}(\nabla \underline{u}(r)) \subset \bar{D}$. Also $(\nabla)_j : H^{1+\delta}(\mathbb{R}^d) \to H^{\delta}(\mathbb{R}^d)$ is bounded for all δ, i.e. for all $f \in H^{1+\delta}(\mathbb{R}^d)$ there exists $c > 0$ such that $\|\nabla f\|_{\delta} \leq c\|f\|_{1+\delta}$. These results combined with Proposition 2.5 (where $p = 2$) lead to (1).

Now we prove (2). Since $\tilde{H}^{-\beta}(D)$ is a linear space then $m(t) - m(r) \in \tilde{H}^{-\beta}(D)$. The pointwise product and the operator ∇ are linear so we can write $m(t) - m(r) = \langle \nabla \underline{u}(t) - \nabla \underline{u}(r), \nabla Z \rangle = \langle \nabla(\underline{u}(t) - \underline{u}(r)), \nabla Z \rangle$. Clearly $\underline{u}(t) - \underline{u}(r)$ is an element of $\tilde{H}^{\delta}(D) \subset H^{\delta}(\mathbb{R}^d)$ so we proceed in the same way as for (1) and we get the wanted result. $\square$

Proposition 3.2. *Let $0 < \beta < \delta < \frac{1}{2}$ and $w \in \tilde{H}^{-\beta}(D)$. Then $P_t w \in \tilde{H}^{1+\delta}(D)$ for any $t > 0$ and moreover there exists a positive constant c such that*

(9) $$\|P_t w\|_{1+\delta} \leq c \|w\|_{-\beta} \, t^{-\frac{1+\delta+\beta}{2}}.$$

Proof. Let $w \in \tilde{H}^{-\beta}(D)$. By (3) we have

$$\|P_t w\|_{1+\delta} \leq c\|A^{\frac{1+\delta}{2}}P_t w\|_0 = c\|A^{\frac{1+\delta}{2}}A^{\frac{\beta}{2}}A^{-\frac{\beta}{2}}P_t w\|_0 = c\|A^{\frac{1+\delta+\beta}{2}}P_t A^{-\frac{\beta}{2}}w\|_0.$$

Since $w \in \tilde{H}^{-\beta}(D)$ then by (3) we have also $A^{-\frac{\beta}{2}}w \in L_2(D)$ and Theorem 2.1 part (c) ensures that the following bound holds for all $t > 0$

$$\|A^{\frac{1+\delta+\beta}{2}}P_t\|_{\mathscr{L}(L_2(D))} \leq M e^{-\theta t}t^{-\frac{1+\delta+\beta}{2}}.$$

This fact together with the previous bound implies

$$\|P_t w\|_{1+\delta} \leq ct^{-\frac{1+\delta+\beta}{2}}\|A^{-\frac{\beta}{2}}w\|_0 \leq ct^{-\frac{1+\delta+\beta}{2}}\|w\|_{-\beta} < \infty,$$

having used in the last inequality again equation (3). $\square$

These two properties can be generalized to a wider range of parameters δ and β (for more details see [20]).

The following integral bounds will be used later. The proof makes use of the Gamma and the Beta functions together with some basic integral estimates.

Lemma 3.3. *If $0 \leq s < t \leq T < \infty$ and $0 \leq \theta < 1$ then for any $\rho \geq 1$ it holds*

(10) $$\int_s^t e^{-\rho r} r^{-\theta} \mathrm{d}r \leq \Gamma(1-\theta)\rho^{\theta-1}.$$

Moreover if $\gamma > 0$ is such that $\theta + \gamma < 1$ then for any $\rho \geq 1$ there exists a positive constant C such that

(11) $$\int_0^t e^{-\rho(t-r)}(t-r)^{-\theta} r^{-\gamma} \mathrm{d}r \leq C\rho^{\theta-1+\gamma}.$$

In what follows we state and give the proof of the main mapping property of the integral operator: It is a contraction on a Banach space of function with Hölder-type regularity in time and fractional Sobolev-type regularity in space.

Theorem 3.4. *Let $0 < \beta < \delta < \frac{1}{2}$ and $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for $q > \max(2, \frac{d}{\delta})$. Then for any γ such that $0 < 2\gamma < 1 - \delta - \beta$ it holds*

$$I : C^\gamma([0,T]; \tilde{H}^{1+\delta}(D)) \to C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$$

and the following estimate holds for any fixed $\underline{u} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$

$$(12) \qquad \|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta}^{(\rho)} \le c(\rho) \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$$

where $c(\rho)$ is a function of ρ not depending on $\underline{u}$ nor T and such that

$$\lim_{\rho \to \infty} c(\rho) = 0.$$

Proof. Given any $\underline{u} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ our goal is to bound

$$(13) \qquad \|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta}^{(\rho)} = \sup_{0 \le t \le T} \left(e^{-\rho t} \|I_t(\underline{u})\|_{1+\delta} + \right.$$

$$\left. + e^{-\rho t} \sup_{0 \le s < t} \frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma} \right) =: \sup_{0 \le t \le T} ((A) + (B))$$

using the (ρ)-norm of $\underline{u}$, namely using $\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$.

Step 1: Consider part (A). Fix $t \in [0,T]$.

$$e^{-\rho t} \|I_t(\underline{u})\|_{1+\delta} = e^{-\rho t} \left\| \int_0^t P_{t-r} m(r) \, dr \right\|_{1+\delta}$$

$$\le e^{-\rho t} \int_0^t \|P_{t-r} m(r)\|_{1+\delta} \, dr$$

apply Proposition 3.2 with $w = m(s) \in \tilde{H}^{-\beta}(D)$ and afterwards Proposition 3.1 (1) and obtain

$$(A) \le e^{-\rho t} \int_0^t \|m(r)\|_{-\beta} (t-r)^{-\frac{1+\delta+\beta}{2}} \, dr$$

$$\le c e^{-\rho t} \int_0^t \|\underline{u}(r)\|_{1+\delta} (t-r)^{-\frac{1+\delta+\beta}{2}} \, dr.$$

Observe that for any $0 \le r \le t \le T$

$$e^{-\rho r} \|\underline{u}(r)\|_{1+\delta} \le \sup_{0 \le r \le T} e^{-\rho r} \|\underline{u}(r)\|_{1+\delta} \le \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$$

and then we obtain

$$(A) = e^{-\rho t} \|I_t(\underline{u})\|_{1+\delta} \le c \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \int_0^t e^{-\rho(t-r)} (t-r)^{-\frac{1+\delta+\beta}{2}} \, dr$$

$$= c \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \int_0^t e^{-\rho r} r^{-\frac{1+\delta+\beta}{2}} \, dr \le c \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \rho^{\frac{1+\delta+\beta}{2}-1}$$

having used estimate (10) of Lemma 3.3 in the last line. Clipping the result together we can state that

$$(A) = e^{-\rho t} \|I_t(\underline{u}) | \tilde{H}^{1+\delta}(D)\| \le c(\rho) \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$$

where $c(\rho) = c\rho^{\frac{\delta+\beta-1}{2}}$ and since $\frac{\delta+\beta-1}{2} < 0$ we have $c(\rho) \to 0$ as $\rho \to \infty$.

Step 2: Consider part (B). Let for the moment fix our attention only on the argument inside the norm in the numerator of (B). Recall that $0 \le s < t \le T$. We make a change of variable in the middle integral $r' = r - t + s$ and we obtain

$$\int_0^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r$$

$$= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_{t-s}^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r$$

$$= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_0^s P_{s-r}m(r+t-s)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r$$

$$= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_0^s P_{s-r}(m(r+t-s)-m(r))\,\mathrm{d}r$$

These computations enable us to write

$$(B) = e^{-\rho t} \sup_{0 \le s < t} \frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma}$$

$$= e^{-\rho t} \sup_{0 \le s < t} \frac{\|\int_0^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}$$

$$\le e^{-\rho t} \sup_{0 \le s < t} \frac{\|\int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}$$

$$+ e^{-\rho t} \sup_{0 \le s < t} \frac{\|\int_0^s P_{s-r}(m(r+t-s)-m(r))\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} := (C) + (D).$$

Step 3: Consider term (C).

The numerator is similar to the term (A) and therefore we proceed as we did in Step 1. We have

$$(C) = e^{-\rho t} \sup_{0 \le s < t} \frac{\|\int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}$$

$$\le e^{-\rho t} \sup_{0 \le s < t} \frac{\int_0^{t-s} c\|\underline{u}(r)\|_{1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r}{(t-s)^\gamma}$$

$$\le \sup_{0 \le s < t} \int_0^{t-s} e^{-\rho(t-r)} c\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}(t-r)^{-\frac{1+\delta+\beta}{2}}(t-s)^{-\gamma}\,\mathrm{d}r$$

$$\le c\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \sup_{0 \le s < t} \int_0^{t-s} e^{-\rho(t-r)}(t-r)^{-\frac{1+\delta+\beta}{2}}r^{-\gamma}\,\mathrm{d}r$$

$$= c\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \int_0^t e^{-\rho(t-r)}(t-r)^{-\frac{1+\delta+\beta}{2}}r^{-\gamma}\,\mathrm{d}r$$

apply estimate (11) in Lemma 3.3 with $\theta = \frac{1+\delta+\beta}{2}$: since by hypothesis $2\gamma < 1-\delta-\beta$ then $\gamma + \theta < 1$. We obtain

$$(C) \le c\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \rho^{\frac{1+\delta+\beta+2\gamma}{2}-1} \le c\|\underline{u}\|_{\gamma,1+\delta}^{(\rho)} \rho^{\frac{\delta+\beta+2\gamma-1}{2}}.$$

Clipping the result together

$$(C) = e^{-\rho t} \sup_{0 \le s < t} \frac{\| \int_0^{t-s} P_{t-r} m(r) \, \mathrm{d}r \|_{1+\delta}}{(t-s)^\gamma} \le c_1 \|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta} \rho^{\frac{\delta + \beta + 2\gamma - 1}{2}}.$$

Step 4: Consider term (D).

First apply Proposition 3.2 to $w = m(r + t - s) - m(r)$ which is an element of $\tilde{H}^{-\beta}(D)$ thanks to Property 2.5. Then apply Proposition 3.1, (2).

$$(D) = e^{-\rho t} \sup_{0 \le s < t} \frac{\| \int_0^s P_{s-r}(m(r + t - s) - m(r)) \, \mathrm{d}r \|_{1+\delta}}{(t-s)^\gamma}$$

$$\le e^{-\rho t} \sup_{0 \le s < t} \frac{\int_0^s \|m(r + t - s) - m(r)\|_{-\beta}(s-r)^{-\frac{1+\delta+\beta}{2}}}{(t-s)^\gamma} \, \mathrm{d}r$$

$$\le c e^{-\rho t} \sup_{0 \le s < t} \int_0^s \frac{e^{-\rho(r+t-s)}}{e^{-\rho(r+t-s)}} \frac{\|\underline{u}(r + t - s) - \underline{u}(r)\|_{1+\delta}(s-r)^{-\frac{1+\delta+\beta}{2}}}{(t-s)^\gamma} \, \mathrm{d}r$$

$$\le c \sup_{0 \le s < t} \int_0^s e^{-\rho(s-r)} e^{-\rho(r+t-s)} \frac{\|\underline{u}(r + t - s) - \underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma} (s-r)^{-\frac{1+\delta+\beta}{2}} \, \mathrm{d}r.$$

Fix the attention on the term $e^{-\rho(r+t-s)} \frac{\|\underline{u}(r+t-s) - \underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma}$ and set $h = t - s$: we obtain

$$(14) \qquad\qquad\qquad e^{-\rho(r+h)} \frac{\|\underline{u}(r + h) - \underline{u}(r)\|_{1+\delta}}{h^\gamma}.$$

Moreover observe that

$$\|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta} = \sup_{0 \le t \le T} e^{-\rho t} \|\underline{u}(t)\|_{1+\delta} + \sup_{0 \le r < t \le T} e^{-\rho t} \frac{\|\underline{u}(t) - \underline{u}(r)\|_{1+\delta}}{(t-r)^\gamma}$$

and in particular, setting again $t - r = h$, the second summand can be rewritten as

$$\sup_{0 < h \le r+h \le T} e^{-\rho(r+h)} \frac{\|\underline{u}(r + h) - \underline{u}(r)\|_{1+\delta}}{h^\gamma}.$$

Therefore we can bound (14) by $\|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta}$ (since the parameters r and h are such that $0 < h \le r + h \le T$) and applying once more estimate (10) in Lemma 3.3 the upper bound for (D) becomes

$$(D) \le c \|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta} \sup_{0 \le s < t} \int_0^s e^{-\rho(s-r)} (s-r)^{-\frac{1+\delta+\beta}{2}} \, \mathrm{d}r$$

$$\le c_2 \|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta} \rho^{\frac{\delta+\beta-1}{2}} \Gamma\left(\frac{\delta + \beta - 1}{2} \right).$$

Clipping the result for part (B) we obtain

$$(15) \qquad (B) = (C) + (D) = e^{-\rho t} \sup_{0 \le s < t} \frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma} \le c'(\rho) \|\underline{u}\|^{(\rho)}_{\gamma, 1+\delta}$$

where $c'(\rho) = c_1 \rho^{\frac{\delta+\beta+2\gamma-1}{2}} + c_2 \rho^{\frac{\delta+\beta-1}{2}}$ and since $\frac{\delta+\beta+2\gamma-1}{2}$ and $\frac{\delta+\beta-1}{2}$ are negative we have $c'(\rho) \to 0$ as $\rho \to \infty$.

Finally observe that the bound for $(A) + (B)$ does not depend on t and then the supremum over $0 \leq t \leq T$ of $(A) + (B)$ is simply bounded by

$$\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta}^{(\rho)} = \sup_{0 \leq t \leq T} \left((A) + (B) \right) \leq (c(\rho) + c'(\rho)) \|\underline{u}\|_{\gamma,1+\delta}^{(\rho)}$$

that is the thesis. $\square$

3.2. **Theorem of existence and uniqueness.** Now we prove existence and uniqueness of a global mild solution.

Theorem 3.5. *Let* $0 < \beta < \delta < \frac{1}{2}$ *and* $0 < 2\gamma < 1 - \beta - \delta$. *Fix* $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ *for some* $q > \max(2, \frac{d}{\delta})$. *Then for any initial condition* $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ *and for any positive finite time* T *there exists a unique mild solution* $\underline{u}$ *in* $C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$ *for (2) satisfying the integral equation* $\underline{u}(t) = P_t u_0 + I_t(\underline{u})$.

Proof. From Theorem 3.4 we know that if $\underline{u} \in C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$ then $I_{(\cdot)}(\underline{u}) \in C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$.
Now we should ensure that for $u_0 \in \tilde{H}^\sigma(D)$, with $\sigma \geq 1 + \delta + 2\gamma$ then $P_{(\cdot)} u_0 \in C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$ too, using the (ρ)-norm, namely we should check that

$$\sup_{0 \leq t \leq T} e^{-\rho t} \left(\|P_t u_0\|_{1+\delta} + \sup_{0 \leq s < t} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma} \right) < \infty.$$

Recall that P_t is a bounded linear operator on $\tilde{H}^\sigma(D)$ for $-\frac{1}{2} < \sigma$, therefore for every $x \in \tilde{H}^\sigma(D)$, $\|P_t x | \tilde{H}^\sigma(D)\| \leq \|P_t\| \cdot \|x\|_\sigma < \infty$. Since $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D) \subset \tilde{H}^{1+\delta}(D)$ then

$$\sup_{0 \leq t \leq T} e^{-\rho t} \|P_t u_0\|_{1+\delta} \leq c \sup_{0 \leq t \leq T} \|P_t u_0\|_{1+\delta} < \infty.$$

For the second summand use part (d) of Theorem 2.1 and relation (3) to obtain

$$\begin{aligned}
\|P_t u_0 - P_s u_0\|_{1+\delta} &= \|P_s (P_{t-s} - I) u_0\|_{1+\delta} \\
&\leq c \|P_s\| \|(P_{t-s} - I) u_0\|_{1+\delta} \leq c \|P_s\| (t-s)^\alpha \|A^\alpha u_0\|_{1+\delta} \\
&\leq c \|P_s\| (t-s)^\alpha \|u_0\|_{1+\delta+2\alpha} \leq c M e^{-\omega s} (t-s)^\alpha \|u_0\|_{1+\delta+2\alpha}
\end{aligned}$$

for any $0 < \alpha < 1$. Therefore the second summand becomes

$$\sup_{0 \leq t \leq T} e^{-\rho t} \sup_{0 \leq s < t} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma} \leq \sup_{0 \leq s < t \leq T} e^{-\rho t} c_s (t-s)^\alpha \frac{\|u_0\|_{1+\delta+2\alpha}}{(t-s)^\gamma}$$

and if we choose $\alpha = \gamma$ then

$$\sup_{0 \leq t \leq T} e^{-\rho t} \sup_{0 \leq s < t} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma} \leq \sup_{0 \leq s < t \leq T} e^{-\rho t} c_s \|u_0\|_{1+\delta+2\gamma}$$

that is a finite quantity if $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$.
So for any fixed $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ the operator $J_{(\cdot)} := P_{(\cdot)} u_0 + I_{(\cdot)}$ is mapping $C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$ into itself. It is left to prove that $J_{(\cdot)}$ is a contraction, namely that there exists a constant $k < 1$ such that for all $\underline{u}, \underline{v} \in C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$

$$\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta}^{(\rho)} \leq k \|\underline{u} - \underline{v}\|_{\gamma,1+\delta}^{(\rho)}.$$

For this aim observe that

$$\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta}^{(\rho)} = \|P_t u_0 + I_{(\cdot)}(\underline{u}) - P_t u_0 - I_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta}^{(\rho)}$$

$$= \left\| \int_0^\cdot P_{\cdot-r} \langle \nabla \underline{u}(r), \nabla Z \rangle \mathrm{d}r - \int_0^\cdot P_{\cdot-r} \langle \nabla \underline{v}(r), \nabla Z \rangle \mathrm{d}r \right\|_{\gamma,1+\delta}^{(\rho)}$$

$$\leq \left\| \int_0^\cdot P_{\cdot-r} \left(\langle \nabla(\underline{u}(r) - \underline{v}(r)), \nabla Z \rangle \mathrm{d}r \right) \right\|_{\gamma,1+\delta}^{(\rho)} \leq \|I_{(\cdot)}(\underline{u} - \underline{v})\|_{\gamma,1+\delta}^{(\rho)}.$$

We clearly have $\underline{w} := \underline{u} - \underline{v} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ and then it suffices to apply the result of Theorem 3.4 with $\underline{w}$ instead of $\underline{u}$ and choose ρ big enough such that the constant $c(\rho)$ appearing in (12) is less than 1. $\qquad\square$

4. APPLICATIONS

In this section we will apply the previous results to some stochastic PDEs.

4.1. The stochastic transport equation.
Consider the stochastic transport equation given by

$$(16) \quad \begin{cases} \frac{\partial u}{\partial t}(t,x) = \sigma^2 \Delta u(t,x) + \langle \nabla u(t,x), \nabla Y(x,\omega) \rangle, & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases}$$

where $Y = \{Y(x,\omega)\}_{x \in \mathbb{R}^d}$ is a stochastic field defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. One suitable example for the noise Y is the *Levy fractional Brownian motion* $\{B^H(x)\}_{x \in \mathbb{R}^d}$ which is the isotropic generalization of the fractional Brownian motion (see [11]). This field is defined to be a centered Gaussian field on $\mathbb{R}^d$ of covariance function

$$\mathbb{E}[B^H(x)B^H(y)] = \frac{1}{2}(|x|_d^{2H} + |y|_d^{2H} - |x-y|_d^{2H})$$

where $|\cdot|_d$ stands for the Euclidean norm in $\mathbb{R}^d$. The parameter $0 < H < 1$ is called Hurst parameter. In case when $H = \frac{1}{2}$ we recover the *Levy Brownian motion*, whereas if $d = 1$ we get the fractional Brownian motion. Using a Kolmogorov continuity theorem suitable for stochastic fields (see for instance [10], Theorem 1.4.1) and basic properties of Gaussian random variables one can show that there exist $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and a modification of $B^H(x), x \in D$ (for simplicity we call it again $B^H(x)$) with $D \subset \mathbb{R}^d$ arbitrary bounded domain of $\mathbb{R}^d$ such that for every $\omega \in \Omega_1$ and for every $x, y \in D$ we have

$$|B^H(x,\omega) - B^H(y,\omega)| \leq K_\omega |x-y|_d^\alpha, \ \forall \alpha < H$$

where K is a positive random variable with finite moments of every order.
In other words, for almost every realization ω the field is α-Hölder continuous on D of any order $\alpha < H$. This fact together with the following property enable us to apply the results presented in the previous section to equation (16) in a pathwise sense.

Proposition 4.1. *Let h be a compactly supported real valued α-Hölder continuous function on $\mathbb{R}^d$ for some $0 < \alpha < 1$. Then for any $\alpha' < \alpha$ we have $h \in H_p^{\alpha'}(\mathbb{R}^d)$ for all $2 \leq p < \infty$.*

The proof makes use of the equivalent norm

$$\|h\|_{L^p} + \left(\int_{|y| \leq 1} \frac{\|h(\cdot + y) - h(\cdot)\|_{L^p}^2}{|y|^{d+2\alpha'}} \mathrm{d}y \right)^{\frac{1}{2}}$$

for the Besov spaces $B_{p,2}^{\alpha'}(\mathbb{R}^d)$ and of embedding properties between Besov and Sobolev spaces (see [19] for more details).

In order to apply this to (almost every) path of B^H we should ensure the compactness of the support. This is not true in general. Instead, since (16) is considered only on the domain D, let $\psi(x), x \in \mathbb{R}^d$ be a C^∞-function with compact support and such that $\psi(x) = 1 \ \forall x \in \bar{D}$. Then for almost every $\omega \in \Omega$ the function $\psi(x)B^H(\omega, x)$ is α-Hölder continuous we have that for all $1 < q < \infty$ and for all $\alpha' < \alpha < H$, $\psi(\cdot)B^H(\omega, \cdot) \in H_q^{\alpha'}(\mathbb{R}^d)$. For consistency of notation call $1 - \beta := \alpha'$, and so $1 - \beta < H$. In order to match the conditions on the parameter β we have to choose $\frac{1}{2} < H < 1$. Then for every $\omega \in \Omega_1$ we set $Z(x) := \psi(x)B^H(\omega, x)$ and so Theorem 3.5 ensures existence and uniqueness of a function solution to the stochastic Dirichlet problem (16) with $Y = B^H$.

4.2. **A (more) general stochastic transport equation.** We combine in this section the main result obtained in this paper with a result obtained in [7].
Recall Definition 2.1 in [7] (we only need the case $k = 1$) where the authors define an integral operator of the type $I_t^\alpha(F, \frac{\partial}{\partial t}\nabla V)$ for some given $F \in \mathbb{R}^d$ and $V = V(t, x_1, \ldots, x_d)$. Their idea is to use Fourier transform to perform the integration with respect to the space variable x and fractional derivatives to give a meaning to the derivative with respect to time and then perform the integration. Moreover they exploited the regularity of this integral, and they proved in Proposition 7.1 that if $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \gamma < 1$ and $2\gamma + \tilde{\delta} < 2 - 2\alpha - \beta$ then the integral $I_{(\cdot)}^\alpha(F, \frac{\partial}{\partial t}\nabla V)$ (which in fact does not depend on α) belongs to the space $C^\gamma([0, T]; \tilde{H}^{\tilde{\delta}}(D))$ for any given function $V \in C^{1-\alpha}([0, T]; H^{1-\beta}(\mathbb{R}^d))$ and vector $F \in \mathbb{R}^d$.

Taking this into account we are able to give the following existence and uniqueness result.

Corollary 4.2. *Let $T > 0$ be fixed, choose $0 < \beta < \delta < \frac{1}{2}$ and $0 < 2\gamma < 1 - \beta - \delta$. Fix $F \in \mathbb{R}^d$, $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ and $V \in C^{1-\alpha}([0, T]; H_q^{1-\beta}(\mathbb{R}^d))$ for some $q > \max(2, \frac{d}{\delta})$ and for some $0 < \alpha < 1$ such that $\alpha + \gamma < 1$. Then given any initial condition $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ there exists a unique global mild solution $u(t, x)$ in the Hölder space $C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$ for the problem*

$$(17) \qquad \begin{cases} \frac{\partial u}{\partial t}(t, x) = \sigma^2 \Delta u(t, x) + \langle \nabla u(t, x), \nabla Z(x) \rangle \\ \qquad\qquad\qquad + \langle F, \frac{\partial}{\partial t}\nabla V(t, x) \rangle, & t \in (0, T], x \in D \\ u(t, x) = 0, & t \in (0, T], x \in \partial D \\ u(0, x) = u_0(x), & x \in D \end{cases}$$

and the solution is given by

$$u(t, \cdot) = P_t u_0 + I_t(\underline{u}) + I_t^\alpha(F, \frac{\partial}{\partial t}\nabla V).$$

Proof. Set $\tilde{\delta} := 1 + \delta$. Since $2\gamma < 1 - \delta - \beta$ then $2\gamma + \tilde{\delta} < 2 - \beta$ and if one chooses a positive α such that $2\gamma + \tilde{\delta} < 2 - \beta - 2\alpha$ then the condition $\alpha + \gamma < 1$ is satisfied and by Proposition 7.1 in [7] we have $I_{(\cdot)}^\alpha(F, \frac{\partial}{\partial t}\nabla V) \in C^\gamma([0, T]; \tilde{H}^{\tilde{\delta}}(D))$. Finally

apply a contraction principle as applied in the proof of Theorem 3.5 and recover the thesis.   □

With the same technique illustrated in Section 4.1 one can solve (17) in the case when Z and V are substituted by stochastic fields, and then the system is solved in the pathwise sense. See [7], Section 6 for a survey on possible noises in place of V.

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E. Issoglio: Institut für Stochastik, Friedrich-Schiller Universität, Jena, Germany; elena.issoglio@uni-jena.de

School of Mathematics, The University of Manchester, Manchester, United Kingdom; elena.issoglio@manchester.co.uk