

# ALMOST COMMUTING UNITARY MATRICES RELATED TO TIME REVERSAL

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ABSTRACT. The behavior of fermionic systems depends on the geometry of the system and the symmetry class of the Hamiltonian and observables. Almost commuting matrices arise from band-projected position observables in such systems. One expects the mathematical behavior of almost commuting Hermitian matrices to depend on two factors. One factor will be the approximate polynomial relations satisfied by the matrices. The other factor is what algebra the matrices are in, either  $\mathbf{M}_n(\mathbb{A})$  for  $\mathbb{A} = \mathbb{R}$ ,  $\mathbb{A} = \mathbb{C}$  or  $\mathbb{A} = \mathbb{H}$ , the algebra of quaternions.

There are potential obstructions keeping  $k$ -tuples of almost commuting operators from being close to a commuting  $k$ -tuple. We consider two-dimensional geometries and so this obstruction lives in  $KO_{-2}(\mathbb{A})$ . This obstruction corresponds to either the Chern number or spin Chern number in physics. We show that if this obstruction is the trivial element in  $K$ -theory then the approximation by commuting matrices is possible.

## 1. INTRODUCTION

1.1. **Approximate representations.** Consider the two sets of relations

$\mathcal{S}_\delta \quad \begin{aligned} X_r^* &= X_r \\ \ X_r X_s - X_s X_r\  &\leq \delta \\ \ X_1^2 + X_2^2 + X_3^2 - I\  &\leq \delta \end{aligned}$	$\mathcal{T}'_\delta \quad \begin{aligned} X_r^* &= X_r \\ \ X_r X_s - X_s X_r\  &\leq \delta \\ \ X_1^2 + X_2^2 - I\  &\leq \delta \\ \ X_3^2 + X_4^2 - I\  &\leq \delta \end{aligned}$
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that we call the soft sphere relations,  $\mathcal{S}_\delta$ , in matrix unknowns  $X_1, X_2, X_3$ , and the soft torus relations,  $\mathcal{T}'_\delta$ , in matrix unknowns  $X_1, X_2, X_3, X_4$ . Our main results concern the operator-norm distance from a representation of  $\mathcal{S}_\delta$  or  $\mathcal{T}'_\delta$  to representations of  $\mathcal{S}_0$  or  $\mathcal{T}'_0$ . We show that the distance one must move away from the  $X_r$  to find Hermitian matrices that are commuting goes to zero as the commutator norm goes to zero.

We show that when the  $X_r$  are taken as variables in  $\mathbf{M}_n(\mathbb{R})$ , then representations of  $\mathcal{S}_\delta$  are, in a uniform fashion, close to representations

of  $\mathcal{S}_0$ . When the  $X_r$  are taken as variables in  $\mathbf{M}_n(\mathbb{C})$ , there is an obstruction in  $\mathbb{Z}$  that dictates the possibility of such an approximation. When the  $X_r$  are taken as variables in  $\mathbf{M}_n(\mathbb{H})$ , (the algebra of quaternions) there is again an obstruction, but now in  $\mathbb{Z}_2$ . For  $X_r \in \mathbf{M}_n(\mathbb{A})$  in these three cases, the obstruction is formally defined to be in  $KO_{-2}(\mathbb{A})$  and we prove this is the only obstruction to the desired approximation by commuting Hermitian matrices.

The complex case of our results were proven in [17, Cor. 13] and [4, Cor. 6.15], using the techniques of semiprojectivity and  $K$ -theory for  $C^*$ -algebras. The connection of these matrix results to condensed matter physics was not noticed until many years later [11, 9, 10, 18]. Of course the relevance of the  $K$ -theory of  $C^*$ -algebras to condensed matter physics was known earlier [1].

**1.2. Structured complex matrices.** Quaternionic matrices arrive in disguise in physics via the isometric embedding  $\chi : \mathbf{M}_N(\mathbb{H}) \rightarrow \mathbf{M}_{2N}(\mathbb{C})$  defined as

$$\chi \left( A + B\hat{j} \right) = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}$$

for complex matrices  $A$  and  $B$ . The image of  $\chi$  can be described as the matrices that commute with the antiunitary operator

$$\mathcal{T}\xi = -Z\bar{\xi}$$

where  $\xi$  is in  $\mathbb{C}^{2N}$  and

$$Z = Z_N = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

In a finite model,  $\mathcal{T}$  is typically playing the role of time-reversal. From a purely mathematical standpoint, it is generally easier to think in terms of the *dual operation*

$$(1.1) \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\sharp = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}$$

alternately defined as

$$(1.2) \quad X^\sharp = -ZX^T Z.$$

The image of  $\chi$  is the set of matrices with  $X^* = X^\sharp$ . We are using mathematical notation, so  $X^*$  refers to the conjugate-transpose of  $X$ .

Similarly, we think of a real matrix  $X$  as a complex matrix for which  $X^* = X^T$ .

A good survey paper regarding the equivalence of matrices of quaternions and structured complex matrices is [5]. This does not address norms on  $\mathbf{M}_N(\mathbb{H})$ .

The norm we consider here is that induced by the operator norm on  $\mathbf{M}_{2N}(\mathbb{C})$ . Thus we simply use  $\|A\|$  to denote the operator norm (a.k.a. the spectral norm) of a complex matrix and define for  $X$  in  $\mathbf{M}_N(\mathbb{H})$  the norm to be

$$\|X\| = \|\chi(X)\|.$$

This norm can be seen to be the norm induced by the action of  $X$  on  $\mathbb{H}^N$ , but that is not important. In the same way, we could think about the norm of a real matrix induced by its action on  $\mathbb{R}^n$  but prefer to consider this norm as being defined via its action on  $\mathbb{C}^n$ .

All theorems will be stated in terms of complex matrices, possibly respecting an additional symmetry such as being self-dual. This allows us the freedom to combine two Hermitian matrices  $X_1$  and  $X_2$  into one matrix

$$U = X_1 + iX_2.$$

If  $X_1$  and  $X_2$  are self-dual, then they are in  $\chi(\mathbf{M}_N(\mathbb{H}))$ , whereas  $U$  is self-dual but most likely  $U^* \neq U^\sharp$ . That is,  $U$  can't be assumed to be in  $\chi(\mathbf{M}_N(\mathbb{H}))$ . We abandon  $\mathcal{T}'_\delta$  in favor of the relations  $\mathcal{T}_\delta$

$$\boxed{\begin{array}{l} \mathcal{T}_\delta \quad U_r^* U_r = U_r U_r^* = I \\ \quad \quad \|U_1 U_2 - U_2 U_1\| \leq \delta \end{array}}$$

which we no longer apply elements of  $\mathbf{M}_N(\mathbb{H})$ . (They *can* be applied there, but doing so leads to different questions that have less obvious connections to physics.) Instead we apply them to complex matrices  $U_1$  and  $U_2$  and then add as appropriate  $U_r^\sharp = U_r$  or  $U_r^T = U_r$ .

*Beyond this point, matrices are assumed to be complex.*

**1.3. Real  $C^*$ -algebras.** Most of our theorems are statements about Real  $C^*$ -algebras. More specifically, we consider  $C^{*,\tau}$ -algebras. This is an ordinary (so complex)  $C^*$ -algebra  $A$  with the additional structure of an *anti-multiplicative*,  $\mathbb{C}$ -linear map  $\tau : A \rightarrow A$  for which  $\tau(\tau(a)) = a$  and  $\tau(a^*) = \tau(a)^*$ .

We prefer the notation  $a \mapsto a^\tau$  to keep close to our essential examples  $(\mathbf{M}_n(\mathbb{C}), T)$  and  $(\mathbf{M}_n(\mathbb{C}), \sharp)$ . For background of this perspective see [10]. For a reference that uses more traditional notation, see [13]. Most importantly, our proofs rely on many results from our previous paper [19]. This deals with the same approximation-by-commuting question, but with the equations  $\mathcal{D}_\delta$

$$\boxed{\begin{array}{l} \mathcal{D}_\delta \quad X_r^* = X_r \\ \quad \quad \|X_1 X_2 - X_2 X_1\| \leq \delta \\ \quad \quad \|X_1\|, \|X_2\| \leq 1 \end{array}}$$

in two matrix variables. The underlying geometry is the disk and so the potential for an obstruction in  $K$ -theory is eliminated. Hermitian almost commuting matrices are always close to commuting Hermitian matrices. For complex matrices, this is Lin's theorem [14], and the result stays true in the real or self-dual case.

Our approach is to move from the disk to the sphere along the lines of [17]. This involves reformulation the problem as a lifting problem. Then we move from the sphere to the torus using the interplay between push-out diagrams and extensions, generalizing results from [4].

**1.4. Band-projected position matrices.** Consider a lattice model with Hamiltonian  $H = H^*$ , for a single particle, tight-binding model on a surface in  $d$ -space determined by some equations  $p(x_1, \dots, x_d) = 0$ . The position matrices in the model will be diagonal, so matrices  $\hat{X}_r$  with  $[\hat{X}_r, \hat{X}_s] = 0$  and  $p(\hat{X}_1, \dots, \hat{X}_d) = 0$ . Under some assumptions, essentially that we have a spectral gap and local interactions, the Hamiltonian will approximately commute with the position matrices. Let  $P$  denote the projection onto the states below the Fermi energy. When we form the band-projected position matrices  $X_r = P\hat{X}_rP$  we arrive at matrices that are only almost commuting. Depending on the universality class [20] of the system, both  $H$  and  $\hat{X}_r$  will have extra symmetries, as will the  $X_r$ . For example, time reversal invariance will result in these matrices being self-dual.

There are important details [10] needed to correct for the fact that the  $X_r$  do not have full rank and that the equations need to change to allow for larger physical size of the lattice when the number of lattice size increases. The result in many interesting cases is matrices  $X_1^{(n)}, \dots, X_d^{(n)}$  of increasing size with

$$\|[X_r^{(n)}, X_s^{(n)}]\| \rightarrow 0$$

and

$$\|p(X_1, \dots, X_d)\| \rightarrow 0.$$

When the lattice geometry is the two-torus, the resulting obstruction to the matrix approximation problem is a integer that corresponds to the Chern number. When time reversal invariance is assumed, the extra symmetry leads to an obstruction in  $\mathbb{Z}_2$ . This obstruction, for the self-dual matrix approximation problem, corresponds to the spin Chern number used to detect two-dimensional topological insulators. Changing the geometry to a three-torus leads to an obstruction in  $KO_{-3}(\mathbb{H}) \cong \mathbb{Z}_2$ . There is numerical evidence [10], and the  $K$ -homology arguments of [12] and [6, §III], that this obstruction will be useful for detecting three-dimensional topological insulators.

**1.5. The obstructions.** Consider  $H_1, H_2$  and  $H_3$  that are a representation of  $\mathcal{S}_\delta$  with  $\delta < \frac{1}{4}$ . We define

$$B(H_1, H_2, H_3) = \begin{bmatrix} H_3 & H_1 + iH_2 \\ H_1 - iH_2 & -H_3 \end{bmatrix}$$

which will be Hermitian and invertible [16, Lemma 3.2]. The obstruction  $\text{Bott}(H_1, H_2, H_3)$  to being close to commuting is the  $K$ -theory class in

$$[\text{polar}(B(H_1, H_2, H_3))] \in K_0(\mathbb{C})$$

A more computable description is to use the signature, meaning half of the number of positive eigenvalues of  $B$  minus half the number of negative eigenvalues. We call  $\text{Bott}(H_1, H_2, H_3)$  the *Bott index*.

(We are off by a minus sign from the definition in [9]. The work in [16] was done without noticing the role of the Pauli spin matrices.)

Here we adopt the convention that  $K_0(A)$  is defined via homotopy classes of self-adjoint unitary elements in  $\mathbf{M}_n(A)$ . This is not the standard view in terms of projections, but is equivalent by a simple shift and rescaling. This is for  $A$  a unital  $C^*$ -algebra.

Given a  $C^{*,\tau}$ -algebra  $(A, \tau)$  we regard  $K_2(A, \tau)$  as defined via classes of invertible in  $\mathbf{M}_{2n}(A)$  with  $x^* = x$  and  $x^\tau = -x$ . In [10] we worked with invertibles with  $x^* = -x$  and  $x^\tau = -x$ . These are equivalent and the conversion from one picture to the other is done simply by multiplying by  $i$ .

When  $H_r = H_r^\sharp$  for all  $r$ , or  $H_r = H_r^\top$ , the Bott index vanishes so it is possible to approximate by commuting Hermitian matrices. Notice these nearby commuting matrices will be only approximately self-dual or symmetric. In the self-dual case a new obstruction arises when we try to approximate by matrices that are at once commuting, Hermitian and self-dual. Here the larger matrix  $B(H_1, H_2, H_3)$  satisfies

$$(B(H_1, H_2, H_3))^{\sharp \otimes \sharp} = B(H_1, H_2, H_3).$$

Noticing that

$$K_2(\mathbf{M}_{4N}(\mathbb{C}), \sharp \otimes \sharp) \cong \mathbb{Z}_2,$$

the first named author and Hastings went on [10] to define the *Pfaffian-Bott index*, denoted

$$\text{Pf-Bott}(H_1, H_2, H_3),$$

as the  $K$ -theory class

$$[\text{polar}(B(H_1, H_2, H_3))] \in K_2(\mathbf{M}_{4N}(\mathbb{C}), \sharp \otimes \sharp) \cong \mathbb{Z}_2.$$

When  $X$  is any invertible matrix, we define  $\text{polar}(X)$  as the unitary in the polar decomposition. That is, it is the *polar part* of  $X$  or, in

terms of the functional calculus,

$$\text{polar}(X) = X (X^* X)^{-\frac{1}{2}}.$$

Much of the work in [18] was in demonstrating that the Pfaffian-Bott index can be efficiently computed numerically using a Pfaffian. The Pfaffian cannot be applied directly, but conjugating  $B(H_1, H_2, H_3)$  by a fixed unitary leads to a purely imaginary, skew-symmetric matrix. The sign of that Pfaffian of that tells us which  $K_2$ -class contains  $B(H_1, H_2, H_3)$ .

**1.6. Main theorems.** We state now two of our four main theorems, along with the complex version. For consistency with [18], we regard the Bott index as an element of  $\mathbb{Z}$  and the Pfaffian-Bott index as an element of the multiplicative group  $\{\pm 1\}$ .

**Theorem 1.1.** ([17, Cor. 13]) *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever matrices  $H_1, H_2, H_3$  form a representation of  $\mathcal{S}_\delta$ , and*

$$\text{Bott}(H_1, H_2, H_3) = 0,$$

*there are matrices  $K_1, K_2, K_3$  that form a representation of  $\mathcal{S}_0$  and so that*

$$\|K_r - H_r\| \leq \epsilon \quad (r = 1, 2, 3).$$

**Theorem 1.2.** *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever  $H_1, H_2, H_3$  are complex symmetric matrices that form a representation of  $\mathcal{S}_\delta$ , there are complex symmetric matrices  $K_1, K_2, K_3$  that form a representation of  $\mathcal{S}_0$  and so that*

$$\|K_r - H_r\| \leq \epsilon \quad (r = 1, 2, 3).$$

**Theorem 1.3.** *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever  $H_1, H_2, H_3$  are self-dual matrices that form a representation of  $\mathcal{S}_\delta$ , and*

$$\text{Pf-Bott}(H_1, H_2, H_3) = 1,$$

*there are self-dual matrices  $K_1, K_2, K_3$  that form a representation of  $\mathcal{S}_0$  and so that*

$$\|K_r - H_r\| \leq \epsilon \quad (r = 1, 2, 3).$$

Theorems 1.2 and 1.3 settle Conjectures 3 and 4 from [9, §VI.C], while Conjectures 1 and 2 from that paper were settled in our earlier paper.

To get theorems about unitary matrices, we utilize an old trick [16] to turn a representation of  $\mathcal{T}_\delta$  into a representation of  $\mathcal{S}_\delta$ , but modified as in [10] to account to the additional symmetry. We define nonnegative

real-valued functions on the circle  $f$ ,  $g$  and  $h$  so that  $f^2 + g^2 + h^2 = 1$  and  $gh = 0$  and so that

$$(z, w) \mapsto (f(w), g(w) + \frac{1}{2} \{h(w), z\})$$

is a degree-one mapping of the two torus in  $\mathbb{C}^2$  to the unit sphere in  $\mathbb{R} \times \mathbb{C}$ . The exact choice only effects the relation of  $\delta$  to  $\epsilon$  in the theorems below.

Given  $U_1$  and  $U_2$  that form a representation of  $\mathcal{T}_\delta$  we define

$$\begin{aligned} H_1 &= f(U_2) \\ H_2 &= g(U_2) + \frac{1}{4} \{h(U_2), U_1^*\} + \frac{1}{4} \{h(U_2), U_1\} \\ H_3 &= \frac{i}{4} \{h(U_2), U_1^*\} - \frac{i}{4} \{h(U_2), U_1\} \end{aligned}$$

which then is a representation for  $S_\eta$  where  $\eta$  can be taken small when  $\delta$  is small. The anticommutator  $\{-, -\}$  is used to ensure that  $U_r = U_r^T$  or  $U_r = U_r^\sharp$  propogates to the same symmetry in the  $H_r$ . Now we define

$$\text{Bott}(U_1, U_2) = \text{Bott}(H_1, H_2, H_3)$$

and

$$\text{Pf-Bott}(U_1, U_2) = \text{Pf-Bott}(H_1, H_2, H_3).$$

It is possible to define these invariants when the  $U_r$  are only approximately unitary. A simple approach is to compute the invariant as defined above but using  $\text{polar}(U_r)$  in place of  $U_r$ .

It is not hard to see that when these indices are nontrivial the approximation by commuting matrices of the required form is not possible. See [10]. What is not so apparent is that these indices can be non-trivial. In the case of almost commuting matrices that are unitary, and with no other restrictions, we have the example first considered by Voiculescu [27], with  $A_n$  the cyclic shift on the basis of  $\mathbb{C}^n$  and  $B_n$  a diagonal unitary. Specifically, when

$$A_n = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} e^{2\pi i/n} & & & & \\ & e^{4\pi i/n} & & & \\ & & \ddots & & \\ & & & e^{-2\pi i/n} & \\ & & & & 1 \end{bmatrix}$$

we have

$$\text{Bott}(A_n, B_n) = 1$$

while  $\|[A_n, B_n]\| \rightarrow 0$ . For a self-dual example, we pair this example with its transpose. Using Theorem 2.8 of [10] and the fact the transpose

commutes with the functional calculus, one can show

$$\text{Pf-Bott} \left( \left[ \begin{array}{c|c} A_n & \\ \hline & A_n^\top \end{array} \right], \left[ \begin{array}{c|c} B_n & \\ \hline & B_n^\top \end{array} \right] \right) = -1$$

while the Bott index here is trivial. This is an example of self-dual, almost commuting unitaries that are close to commuting unitaries, but far from commuting self-dual unitaries.

Here then are our other two main theorems, along with the complex version.

**Theorem 1.4.** ([4, Cor. 6.15]) *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever matrices  $U_1, U_2$  form a representation of  $\mathcal{T}_\delta$ , and*

$$\text{Bott}(U_1, U_2) = 0,$$

*there are matrices  $V_1, V_2$  that form a representation of  $\mathcal{T}_0$  and so that*

$$\|U_r - V_r\| \leq \epsilon \quad (r = 1, 2).$$

**Theorem 1.5.** *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever  $U_1, U_2$  are complex symmetric matrices that form a representation of  $\mathcal{T}_\delta$ , there are complex symmetric matrices  $V_1, V_2$  that form a representation of  $\mathcal{T}_0$  and so that*

$$\|U_r - V_r\| \leq \epsilon \quad (r = 1, 2).$$

**Theorem 1.6.** *For every  $\epsilon > 0$  there exists a  $\delta > 0$  so that whenever  $U_1, U_2$  are self-dual matrix representations of  $\mathcal{T}_\delta$ , and*

$$\text{Pf-Bott}(U_1, U_2) = 1,$$

*there are self-dual matrices  $V_1, V_2$  that are representation of  $\mathcal{T}_0$  and so that*

$$\|U_r - V_r\| \leq \epsilon \quad (r = 1, 2).$$

## 2. BLOCK SYMMETRIES IN UNSTABLE $K$ -THEORY

Recall  $\text{Bott}(H_1, H_2, H_3)$  lives in a matrix algebra twice as big as the  $H_r$ , and it can be written

$$\text{Bott}(H_1, H_2, H_3) = \sum H_r \otimes \sigma_r$$

where  $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ ,  $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . These matrices, up to sign and scaling the Pauli spin matrices, are self-dual. When the  $H_r$  are complex symmetric,  $\text{Bott}(H_1, H_2, H_3)$  is self-dual. When the  $H_r$  are self-dual,  $\text{Bott}(H_1, H_2, H_3)$  can be conjugated to complex symmetric, but we need to be working in the original formation and so deal with the operation  $\sharp \otimes \sharp$ .



Recall that there are, up to isomorphism, just two  $\tau$ -structures [13, §10.1] that can be put on  $\mathbf{M}_n(\mathbb{C})$ , the transpose or, when  $n$  is even, the dual. We need a result about how symmetries on a matrix in  $\mathbf{M}_n(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C})$  can force symmetries on one of its blocks. This is Lemma 2.3.

**Lemma 2.1.** *Suppose  $(\mathbf{M}_n(\mathbb{C}), \tau)$  is a  $C^{*,\tau}$ -algebra, and consider the larger  $C^{*,\tau}$ -algebra*

$$(\mathbf{M}_{2n}(\mathbb{C}), \tau \otimes \sharp),$$

meaning that on  $\mathbf{M}_{2n}(\mathbb{C})$  we are using the  $\tau$ -operation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\tau \otimes \sharp} = \begin{bmatrix} D^\tau & -B^\tau \\ -C^\tau & A^\tau \end{bmatrix}.$$

Within the group

$$\{W \in \mathbf{M}_{2n}(\mathbb{C}) \mid \det(W) = 1, W^* = W^{-1} = W^{\tau \otimes \sharp}\}$$

the set of

$$W = \begin{bmatrix} A & B \\ -B^{*\tau} & A^{\tau*} \end{bmatrix}$$

for which both  $A$  and  $B$  are invertible, is a dense open subset.

*Proof.* Up to isomorphism of  $C^{*,\tau}$ -algebras, there are two cases,  $\tau = \mathsf{T}$  and  $\tau = \sharp$ . In both cases, the openness is clear.

Suppose  $\tau = \mathsf{T}$ , which means  $\tau \otimes \sharp = \sharp$ . First we note that the invertibility of  $A$  and  $B$  holds in one case, specifically

$$(2.1) \quad W_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$$

which is a symplectic unitary, i.e.  $W^\sharp = W^* = W^{-1}$ .

All symplectic matrices have determinant 1, so we can ignore the determinant condition. We know from Lie theory that every symplectic unitary is  $e^H$  for a matrix  $H$  with  $H^* = H^\sharp = -H$ . Given any symplectic unitary

$$W_1 = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}$$

we can find a matrix  $H$  with  $H^* = H^\sharp = -H$  such that  $W_1 = e^H W_0$ . Thus we have an analytic path  $W_t = e^{tH} W_0$  from  $W_0$  to  $W_1$ , with  $W_t$  a symplectic unitary at every  $t$ . Let the upper blocks of  $W_t$  be  $A_t$  and  $B_t$ . Considering the power series for  $e^{tH}$  we find convergent power series for the scalar paths  $\det(A_t)$  and  $\det(B_t)$ . As these analytic paths are nonzero around  $t = 1$ , neither can vanish on any open interval. We can choose  $t$  close to 0 to find  $W_t$  with  $A_t$  and  $B_t$  both invertible.

We prove the  $\tau = \sharp$  case similarly, starting with the same example,  $W_1$  as in (2.1), but now with  $n = 2N$ . A real orthogonal matrix of determinant one is  $e^H$  for a matrix with  $H^* = H^T = -H$ . Translating this fact via the isomorphism  $\Phi$  from  $(\mathbf{M}_{4N}(\mathbb{C}), \sharp \otimes \sharp)$  to  $(\mathbf{M}_{4N}(\mathbb{C}), \mathbb{T})$ , which is just conjugation by a unitary (see [10, Lemma 1.3] or equivalently (3.2), we see that when  $W$  is a unitary with determinant one and  $W^* = W^{\sharp \otimes \sharp}$ , there is a matrix  $H$  with  $H^* = H^{\sharp \otimes \sharp} = -H$  so that  $e^H = W$ . Therefore if we are given  $W_1$  a unitary with  $W^* = W^{\sharp \otimes \sharp}$  we can find  $H$  with  $H^* = H^{\sharp \otimes \sharp} = -H$  such that  $W_1 = e^H W_0$ , and so we have an analytic path  $W_t = e^{tH} W_0$  of unitaries, and one can check directly, or using  $\Phi$ , that  $W_t^* = W_t^{\sharp \otimes \sharp}$ . The rest of the argument is exactly as in the previous case.  $\square$

**Lemma 2.2.** *If  $a$  and  $b$  are invertible elements of a unital  $C^*$ -algebra and  $aa^* + bb^* = 1$  then*

$$\text{polar}(a^*b) = (\text{polar}(a))^* \text{polar}(b).$$

*Proof.* We use the formula  $\text{polar}(x) = x(x^*x)^{-\frac{1}{2}}$  and compute

$$\begin{aligned} a^*b(b^*aa^*b)^{-\frac{1}{2}} &= a^*b(b^*(1 - bb^*)b)^{-\frac{1}{2}} \\ &= a^*(1 - bb^*)^{-\frac{1}{2}}(bb^*)^{-\frac{1}{2}}b \\ &= (a^*a)^{-\frac{1}{2}}a^*b(b^*b)^{-\frac{1}{2}}. \end{aligned} \quad \square$$

**Lemma 2.3.** *If*

$$W = \begin{bmatrix} A & B \\ -B^*\tau & A^{\tau*} \end{bmatrix}$$

*is a unitary in  $\mathbf{M}_{2n}(\mathbb{C})$  with  $W^* = W^{\tau \otimes \sharp}$  and both  $A$  and  $B$  are invertible, then  $(\text{polar}(A))^* \text{polar}(B)$  is fixed by  $\tau$ .*

*Proof.* From the fact that  $W$  is unitary we deduce  $AA^* + BB^* = I$  and  $A^*B = B^{\tau}A^{\tau*}$ . The last lemma applies, so

$$(\text{polar}(A))^* \text{polar}(B) = \text{polar}(A^*B).$$

Also  $(A^*B)^{\tau} = A^*B$ . From the definition of the polar part in terms of functional calculus we find  $(\text{polar}(x))^{\tau} = \text{polar}(x^{\tau})$  and so

$$(\text{polar}(A^*B))^{\tau} = \text{polar}(A^*B). \quad \square$$

### 3. STRUCTURE DIAGONALIZATION AND $KO_2$

**3.1. Hermitian anti-self-dual invertibles.** The  $K$ -theory we will need to track will be  $K_2$ . In the case of symmetric matrices, we will need  $K_2$  of  $\mathbf{M}_n(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C}) \cong \mathbf{M}_{2n}(\mathbb{C})$ . The fact that  $K_2(\mathbf{M}_{2N}(\mathbb{C}), \sharp) = 0$

can be given the concrete realization that for any two Hermitian anti-self-dual matrices that are approximately unitary, there is a symplectic unitary that approximately conjugates one to the other.

**Lemma 3.1.** *Suppose  $X$  in  $\mathbf{M}_{2N}(\mathbb{C})$  is Hermitian and anti-self-dual. Then there exists a symplectic unitary  $W$  and a diagonal matrix  $D$  with nonnegative real diagonal entries so that*

$$X = W \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} W^*.$$

*Proof.* The matrix  $Y = -iX$  has  $Y^\sharp = Y^*$  so we may apply results about matrices of quaternions. Since  $Y$  is normal, indeed skew-Hermitian, the spectral theorem for matrices of quaternions [5] gives us  $W$  so that

$$X = W \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} W^*.$$

We need only adjust  $W$  and  $D$  so that the diagonal of  $D$  ends up non-negative. We can swap the  $j$ -th diagonal element of  $D$  with the  $j$ -th diagonal element of  $-D$  as needed by utilizing the formula

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^* = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix}. \quad \square$$

**Theorem 3.2.** *Suppose  $\|S^2 - I\| < 1$  and  $S^* = S$  and  $S^\sharp = -S$ . Then there is a symplectic unitary  $W$  so that*

$$\left\| S - W \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} W^* \right\| \leq \|S^2 - I\|.$$

*Proof.* Lemma 3.1 tells us

$$S = W^* \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} W$$

for some symplectic unitary  $W$ . Let  $\delta = \|S^2 - I\|$  so  $\delta < 1$  and

$$\|D^2 - I\| = \left\| \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}^2 - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\| = \|S^2 - I\| = \delta$$

and so the diagonal element so of  $D$  are in the set

$$\{\lambda \mid 1 - \delta \leq \lambda^2 \leq 1 + \delta\}.$$

The result now follows easily. □

**3.2. Coupling two dual operations.** In the self-dual case, the  $K$ -theory is calculated in  $\mathbf{M}_{2N}(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C}) \cong \mathbf{M}_{4N}(\mathbb{C})$  and the operation that specifies the symmetry on the Bott matrix is  $\sharp \otimes \sharp$ . In terms of  $4n$ -by- $4n$  matrices,  $\sharp \otimes \sharp$  is  $T$  conjugated by a unitary. There are many identifications of  $\mathbf{M}_{2N}(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C})$  with  $\mathbf{M}_{4N}(\mathbb{C})$  and the one we use operates via

$$B \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}.$$

Of course  $B$  itself will often be written out in terms of four  $N$ -by- $N$  blocks. We are using

$$Z = Z_N = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbf{M}_{2N}(\mathbb{C})$$

and the operation  $\sharp \otimes \sharp$  works out as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\sharp \otimes \sharp} = \begin{bmatrix} D^\sharp & -B^\sharp \\ -C^\sharp & A^\sharp \end{bmatrix}.$$

The symmetry  $X^{\sharp \otimes \sharp} = X^*$  means

$$(3.1) \quad X = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ -\overline{B_{22}} & \overline{B_{21}} & \overline{A_{22}} & -\overline{A_{21}} \\ \overline{B_{21}} & -\overline{B_{11}} & -\overline{A_{21}} & \overline{A_{11}} \end{bmatrix}.$$

The isomorphism [10, Lemma 1.3] we need is

$$(3.2) \quad \Phi : (\mathbf{M}_{4N}(\mathbb{C}), \sharp \otimes \sharp) \rightarrow (\mathbf{M}_{4N}(\mathbb{C}), T)$$

defined by

$$\Phi(X) = UXU^*$$

where  $X$  is  $4N$ -by- $4N$  and

$$U = \frac{1}{\sqrt{2}}(I \otimes I - iZ_N \otimes Z_N).$$

Any  $X$  in the form (3.1) is conjugate to a matrix with real entries, and so  $\det(X) \in \mathbb{R}$ . This is the trick that allows us to get our  $\mathbb{Z}_2$ -invariant as the sign of a determinant of some invertible  $X$  that is full of complex numbers.

**3.3. Hermitian anti-symmetric invertibles.** The  $K_2$ -group of  $(\mathbf{M}_{2n}(\mathbb{C}), \mathbb{T})$  is  $\mathbb{Z}_2$ , and this fact can be given following concrete realization. The hermitian anti-symmetric matrices fall into two classes: those that can be approximately conjugated by an element of  $SL_{2n}(\mathbb{R})$  to

$$S_0 = \begin{bmatrix} 0 & i & & & & \\ -i & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & i & \\ & & & -i & 0 & \end{bmatrix}$$

and those that cannot. It is the Pfaffian that can decide in which class is a matrix  $X$ , depending on the sign of  $i^{-n}\text{Pf}(X)$ .

**Theorem 3.3.** *Suppose  $\|S^2 - I\| < 1$  and  $S^* = S$  and  $S^T = -S$  for some  $S$  in  $\mathbf{M}_{2n}(\mathbb{C})$ . Then there is a real orthogonal  $W$  of determinant one with*

$$\|S - WS_0W^*\| \leq \|S^2 - I\|$$

*if and only if  $i^{-n}\text{Pf}(S)$  is positive.*

*Proof.* The “if” part follows from [10, Theorem 9.4] applied to  $-iS$ .

Suppose there is  $W$  in  $SL_{2n}(\mathbb{R})$  so that

$$\|S - WS_0W^*\| < 1.$$

Then  $S_1 = W^*SW$  satisfies  $S_1^* = S_1$  and  $S_1^T = -S_1$  and  $\|S_1 - S_0\| < 1$ . The path  $S_t = (1-t)S_0 + tS_1$  is within 1 of  $S_0$  and  $S_0$  is a unitary, so each  $S_t$  is invertible and skew-symmetric. Therefore the path  $i^{-n}\text{Pf}(S_t)$  cannot change. Since  $i^{-n}\text{Pf}(S_0) = 1$  we must have  $i^{-n}\text{Pf}(S_t) > 0$  for all  $t$ .  $\square$

**Theorem 3.4.** *Suppose  $\|S^2 - I\| < 1$  and  $S^* = S$  and  $S^{\#\otimes\#} = -S$  for some  $S$  in  $\mathbf{M}_{4n}(\mathbb{C})$ . Then there is a unitary  $W$  with  $W^{\#\otimes\#} = W^*$  with*

$$\left\| S - W \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} W^* \right\| \leq \|S^2 - I\|$$

*if and only if the  $K_2$  class represented by  $S$  is trivial.*

*Proof.* We will derive this from Theorem 3.3 via the isomorphism  $\Phi$  from (3.2). Recall this satisfies  $\Phi(X^{\#\otimes\#}) = (\Phi(X))^T$  and, being a  $*$ -isomorphism, satisfies  $\Phi(X^*) = (\Phi(X))^*$ . A short computation (with blocks in two different sizes) tells us

$$\text{Pf} \left( \Phi \left( \begin{bmatrix} I & \\ & -I \end{bmatrix} \right) \right) = \text{Pf} \begin{bmatrix} 0 & 0 & 0 & iI \\ 0 & 0 & -iI & 0 \\ 0 & iI & 0 & 0 \\ -iI & 0 & 0 & 0 \end{bmatrix} = (-1)^n.$$

We apply Theorem 3.3 to  $\Phi \left( \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right)$  and find a real orthogonal  $W_1$  so that

$$\Phi \left( \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right) = W_1 S_0 W_1^*.$$

The conditions on  $S$  translates to  $\|\Phi(S)^2 - I\| < 1$ ,  $\Phi(S)^* = \Phi(S)$ , and  $\Phi(S)^T = -\Phi(S)$ . If the  $K_2$ -class of  $S$  is trivial then  $i^{-2n} \text{Pf}(\Phi(S))$  is positive and so there is a real orthogonal  $W_2$  with

$$\|\Phi(S) - W_2 S_0 W_2^*\| \leq \|\Phi(S)^2 - I\| = \|S^2 - I\|.$$

The desired unitary is then  $W = \Phi^{-1}(W_2 W_1^*)$ . If on the other hand there is a unitary  $W \in \mathbf{M}_{4n}(\mathbb{C})$  as in the statement of the theorem, then we see that  $\Phi(W)W_1$  almost conjugates  $\Phi(S)$  to  $S_0$ . Hence  $i^{-2n} \text{Pf}(\Phi(S))$  is positive, and so the  $K_2$ -class of  $S$  is trivial.  $\square$

**3.4. From  $K_{-2}$  to  $K_2$ .** Behind our definition of the Bott index and the Pfaffian-Bott index is a specific generator of  $K_{-2}$  of  $(C(S^2), \text{id})$ . The associated  $R^*$ -algebra here is  $C(S^2, \mathbb{R})$ . We skip over the actual definition of  $K_{-2}$  and utilize a very important part of Bott periodicity which is that there is a natural isomorphism [23]

$$K_{-2}(A) \cong K_2(A \otimes \mathbb{H}),$$

for  $R^*$ -algebras, and which in terms of  $C^{*,\tau}$ -algebras is

$$K_{-2}(A, \tau) \cong K_2(A \otimes \mathbf{M}_2(\mathbb{C}), \tau \otimes \sharp).$$

We take then  $K_2(A \otimes \mathbf{M}_2(\mathbb{C}), \tau \otimes \sharp)$  as the definition of  $K_{-2}(A, \tau)$ . With this convention, the generator of  $K_{-2}(C(S^2), \text{id})$  is the the class of skew- $\tau$  invertible Hermitian elements represented by

$$b = \begin{bmatrix} x & y + iz \\ y - iz & -x \end{bmatrix}$$

where  $x$ ,  $y$  and  $z$  are the coordinate functions if we regard  $S^2$  as the unit sphere in  $\mathbb{R}^3$ . A basic fact in  $K$ -theory is that

$$K_{-2}(C(S^2), \text{id}) \cong \mathbb{Z}.$$

If we have an actual  $*\text{-}\tau$ -homomorphism  $\varphi$  from  $(C(S^2), \text{id})$  to some  $(A, \tau)$ , as in the proofs that follow, then calculating  $K_{-2}(\varphi)$  is just a matter of following  $b$  over to where it lands in  $(\mathbf{M}_2(A), \tau \otimes \sharp)$ , meaning

$$\begin{bmatrix} \varphi(x) & \varphi(y) + i\varphi(z) \\ \varphi(y) - i\varphi(z) & -\varphi(x) \end{bmatrix}.$$

4. SPHERICAL TO CYLINDRICAL COORDINATES

We consider various inclusions of commutative  $C^{*,\tau}$ -algebras, all induced by surjections. The first we will encounter is

$$(4.1) \quad \iota_1 : C(S^2, \text{id}) \hookrightarrow C(S^1 \times [-1, 1], \text{id})$$

which is unital, induced by

$$(e^{2\pi i\theta}, t) \mapsto \left( \sqrt{1-t^2} \cos(\theta), \sqrt{1-t^2} \sin(\theta), t \right).$$

We consider  $C(S^1 \times [-1, 1], \text{id})$  as universal for a unitary  $v$  and a positive contraction  $k$  that commute. We also need relations for the  $\tau$  operation to create the identity involution on the cylinder, so we require  $v^\tau = v$  and  $k^\tau = k$ . In term of generators and relations, the inclusion (4.1) operates via

$$\begin{aligned} h_1 &\mapsto (1 - k^2)^{\frac{1}{2}} (v^* + v), \\ h_2 &\mapsto (1 - k^2)^{\frac{1}{2}} (iv^* - iv), \\ h_3 &\mapsto k. \end{aligned}$$

Generators and relations for real  $C^*$ -algebras have only been considered implicitly, as in [7, §II]. The relations we need are rather basic, involving only real  $C^*$ -algebras that are commutative or finite-dimensional, so we will not explore this topic formally here. It will be explored elsewhere.

**Theorem 4.1.** *Suppose  $d_n$  is a sequence of natural numbers and that*

$$\varphi : C(S^2, \text{id}) \rightarrow \prod (\mathbf{M}_{d_n}(\mathbb{C}), \mathbb{T}) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \mathbb{T})$$

*is a unital  $*$ - $\tau$ -homomorphism. Then there is a unital  $*$ - $\tau$ -homomorphism  $\psi$  so that*

$$\begin{array}{ccc} C(S^1 \times [-1, 1], \text{id}) & & \\ \uparrow \iota_1 & \dashrightarrow \psi & \\ C(S^2, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{d_n}(\mathbb{C}), \mathbb{T}) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \mathbb{T}) \end{array}$$

*commutes.*

*Proof.* Let  $H_r$  be the image under  $\varphi$  of the generators  $h_r$  of  $C(S^2)$  and select lifts to the product, so matrices  $H_{n,r} \in \mathbf{M}_{d_n}(\mathbb{C})$ . Taking averages and fiddling with functional calculus, we can assume that these are contractions with

$$(4.2) \quad H_{n,r} = H_{n,r}^* = H_{n,r}^\mathbb{T}.$$

Let  $\pi$  denote the quotient map. Since

$$H_r = \pi(\langle H_{1,r}, H_{2,r}, \dots \rangle)$$

we have that as  $n \rightarrow \infty$

$$(4.3) \quad \|H_{n,r}H_{n,s} - H_{n,s}H_{n,r}\| \rightarrow 0 \quad \text{and} \quad \left\| \sum_r H_{n,r}^2 - I \right\| \rightarrow 0$$

When defining  $\psi$  by where to send generators we can ignore any initial segment. Therefore we assume without loss of generality that

$$(4.4) \quad \|H_{n,r}H_{n,s} - H_{n,s}H_{n,r}\| < \frac{1}{4} \quad \text{and} \quad \left\| \sum_r H_{n,r}^2 - I \right\| < \frac{1}{4}.$$

We apply [9, Lemma 3.2] to

$$S_n = \begin{bmatrix} H_{n,3} & H_{n,1} + iH_{n,2} \\ H_{n,1} - iH_{n,2} & -H_{n,3} \end{bmatrix}$$

to find  $\|S_n^2 - I\| < 1$  for all  $n$ , and  $\|S_n^2 - I\| \rightarrow 0$ . By (4.2) we have  $S_n^* = S_n$  and  $S_n^\sharp = -S_n$ .

Let  $\delta_n$  be some numbers with  $\delta_n \rightarrow 0$  and

$$\|S_n^2 - I\| < \delta_n < 1.$$

Theorem 3.2 provides us with  $A_n$  and  $B_n$  so that

$$W_n = \begin{bmatrix} A_n & B_n \\ -B_n & A_n \end{bmatrix}$$

is a symplectic unitary (so of determinant one) with

$$\left\| S_n - W_n^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} W_n \right\| < \delta_n.$$

Lemma 2.1 allows us to perturb  $A_n$  and  $B_n$  a little so as to keep these conditions and have  $A_n$  and  $B_n$  invertible. Lemmas 2.2 and 2.3 tell us

$$\text{polar}(A_n^* B_n) = (\text{polar}(A_n))^* \text{polar}(B_n)$$

and that

$$(\text{polar}(A_n^* B_n))^T = \text{polar}(A_n^* B_n).$$

Now we follow the procedure in [9], which has a history going back to [15]. However, we work in terms of

$$\begin{aligned} A &= \pi(\langle A_1, A_2, \dots \rangle), \\ B &= \pi(\langle B_1, B_2, \dots \rangle), \\ S &= \begin{bmatrix} H_3 & H_1 + iH_2 \\ H_1 - iH_2 & -H_3 \end{bmatrix} \end{aligned}$$



and

$$W = \begin{bmatrix} A & B \\ -B^{*\tau} & A^{*\tau} \end{bmatrix}$$

as we are not aiming for a quantitative result. We use  $\tau$  to denote the operation in the quotient induced by all the the matrix transpose operations. We can see that  $W$  is a unitary with

$$(4.5) \quad S = W^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W.$$

>From  $WW^* = I$  we derive  $AA^* + BB^* = 1$  and  $AB^\tau = BA^\tau$ . >From (4.5) we derive

$$\begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}H_3 & \frac{1}{2}H_1 + \frac{i}{2}H_2 \\ \frac{1}{2}H_1 - \frac{i}{2}H_2 & \frac{1}{2} - \frac{1}{2}H_3 \end{bmatrix},$$

we so we find

$$\begin{aligned} A^*A &= \frac{1}{2} + \frac{1}{2}H_3, \\ A^*B &= \frac{1}{2}H_1 + \frac{i}{2}H_2, \\ B^*B &= \frac{1}{2} - \frac{1}{2}H_3. \end{aligned}$$

Combining two of these we find  $A^*A + B^*B = I$ .

Let

$$Z = \pi(\langle \text{polar}(A_1), \text{polar}(A_2), \dots \rangle)$$

and

$$V = \pi(\langle \text{polar}(B_1), \text{polar}(B_2), \dots \rangle)$$

so that

$$A = Z(A^*A)^{\frac{1}{2}} = (AA^*)^{\frac{1}{2}}Z$$

and

$$B = V(B^*B)^{\frac{1}{2}} = (BB^*)^{\frac{1}{2}}V.$$

Then

$$V \left( \frac{1}{2} + \frac{1}{2}H_3 \right) V^* = VA^*AV^* = 1 - VB^*BV^* = 1 - BB^* = AA^*$$

and

$$Z^*AA^*Z = Z^*AA^*Z = A^*A = \frac{1}{2} + \frac{1}{2}H_3.$$

Put together, these tell us

$$V^*Z \left( \frac{1}{2} + \frac{1}{2}H_3 \right) Z^*V = \frac{1}{2} + \frac{1}{2}H_3$$

and so the unitary  $U = Z^*V$  commutes with  $H_3$ . Since

$$U = \pi(\langle (\text{polar}(A_1))^* \text{polar}(B_1), (\text{polar}(A_2))^* \text{polar}(B_2), \dots \rangle)$$

we have the additional symmetry  $U^\tau = U$ . This gives us  $\psi$  sending the generators of  $U$  and  $H_3$ . This makes the diagram commute because

$$\begin{aligned}
 U(1 - H_3^2)^{\frac{1}{2}} &= 2Z^*V(B^*B)^{\frac{1}{2}}(A^*A)^{\frac{1}{2}} \\
 &= 2Z^*(1 - BB^*)^{\frac{1}{2}}B \\
 &= 2Z^*(AA^*)^{\frac{1}{2}}B \\
 &= 2A^*B \\
 &= H_1 + iH_2. \quad \square
 \end{aligned}$$

**Theorem 4.2.** *Suppose  $d_n$  is a sequence of natural numbers and that*

$$\varphi : C(S^2, \text{id}) \rightarrow \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \#) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \#)$$

*is a unital  $*$ - $\tau$ -homomorphism. If  $K_{-2}(\varphi) = 0$  then there is a unital  $*$ - $\tau$ -homomorphism  $\psi$  so that*

$$\begin{array}{ccc}
 C(S^1 \times [-1, 1], \text{id}) & & \\
 \uparrow \iota_1 & \searrow \psi & \\
 C(S^2, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \#) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \#)
 \end{array}$$

*commutes.*

*Proof.* The proof begins as before, with  $H_{n,r}$  in  $\mathbf{M}_{2d_n}(\mathbb{C})$  and with (4.2) replaced by

$$H_{n,r} = H_{n,r}^* = H_{n,r}^\#.$$

After making the reduction to get (4.4) the Pfaffian-Bott index is well defined for all  $n$ . The assumption on the  $K$ -theory of  $\varphi$  tells us that this index is zero for large  $n$ . Further truncating the sequences, we can reduce to the case where  $iS_n$  represents the trivial  $K_2$  class for all  $n$ .

Let  $\delta_n$  be some numbers with  $\delta_n \rightarrow 0$  and

$$\|S^2 - I\| < \delta_n < 1.$$

Theorem 3.4 provides us with

$$W_n = \begin{bmatrix} A_n & B_n \\ -B_n & A_n \end{bmatrix}$$

that is a unitary of determinant one with  $W_n^{\#\otimes\#} = W_n^*$  and

$$\left\| S_n - W_n^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} W_n \right\| < \delta_n.$$

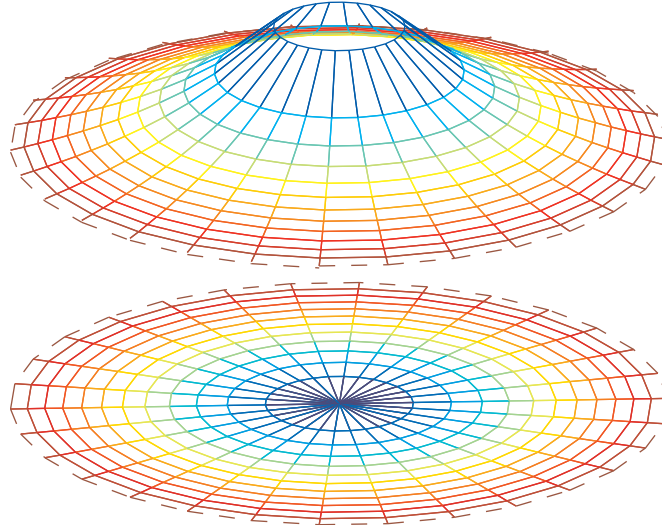


FIGURE 4.1. The map of  $\Omega^{[1]}$  onto  $\Omega$ .

Lemma 2.1 again allows us to reduce to the case where  $A_n$  and  $B_n$  are invertible. Lemmas 2.2 and 2.3 tell us

$$\text{polar}(A_n^* B_n) = (\text{polar}(A_n))^* \text{polar}(B_n)$$

and

$$(\text{polar}(A_n^* B_n))^\sharp = \text{polar}(A_n^* B_n).$$

The rest of the proof goes though unchanged, although  $\tau$  in the quotient is that derived from the sequence of  $\sharp$  operations.  $\square$

We now wish to proceed as in [4, Section 6.3], and so need a way to “poke holes” in open patches. Let  $\Omega$  denote the open unit disk

$$\Omega = \{(x, y) \mid x^2 + y^2 < 1\}$$

and

$$\Omega^{[1]} = \{(x, y) \mid 1 \leq x^2 + y^2 < 2\}.$$

We have a proper surjective continuous map  $\Omega^{[1]} \rightarrow \Omega$  sending

$$(x, y) \mapsto \left( \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} x, \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} y \right)$$

but we think of  $\Omega^{[1]}$  as  $\Omega$  with  $\mathbf{0}$  replaced by a circle. See Figure 4.1. This gives us our map

$$\iota : C_0(\Omega) \hookrightarrow C_0(\Omega^{[1]}).$$

**Corollary 4.3.** *Suppose*

$$\varphi : C_0(\Omega, \text{id}) \rightarrow \prod (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n)$$

*is a  $*$ - $\tau$ -homomorphism with  $\tau_n = \sharp$  for all  $n$  or  $\tau_n = \mathbb{T}$  for all  $n$ . If  $K_{-2}(\varphi) = 0$  then there exists a  $*$ - $\tau$ -homomorphism  $\psi$  so that*

$$\begin{array}{ccc} C(\Omega^{[1]}, \text{id}) & & \\ \uparrow \iota & \dashrightarrow \psi & \\ C(\Omega, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) \end{array}$$

*commutes.*

*Proof.* The inclusion  $C_0(\Omega, \text{id}) \hookrightarrow (C_0(\Omega, \text{id}))^\sim$  into the unitization is an isomorphism on  $K_{-2}$ , (essentially by definition since  $K_{-2}(\mathbb{R}) = 0$ ). Therefore it suffices to prove the equivalent unital extension problem where we unitize the two commutative  $C^{*,\tau}$ -algebras:

$$\begin{array}{ccc} C(\mathbb{D}, \text{id}) & & \\ \uparrow \iota_2 & \dashrightarrow \psi & \\ C(S^2, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) \end{array}$$

Here  $\iota_2$  comes from the degree-one map  $\mathbb{D} \rightarrow S^2$  that sends the origin to the north pole and the boundary circle to the south pole. We have the factorization

$$\begin{array}{ccccc} C(S^2) & \xrightarrow{\iota_2} & C(\mathbb{D}, \text{id}) & \longrightarrow & C(S^1 \otimes [-1, 1], \text{id}) \\ & & \searrow & \nearrow & \\ & & & \iota_0 & \end{array}$$

and so this follows from Theorems 4.1 and 4.2.  $\square$

To prove our results for the torus geometry, we need to know about amalgamated products of  $C^{*,\tau}$ -algebras. For the sphere geometry, we are ready to give a proof. That is, we now prove Theorems 1.2 and 1.3.

Given

$$\varphi : C(S^2) \rightarrow \prod (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n)$$

with trivial  $K_2$  we have, as we saw in the proof of Corollary 4.3, an extension to a  $*$ - $\tau$ -homomorphism from  $C(\mathbb{D})$ . This we can lift to the product by the main result in [19]. This solves the lifting problem, at least when the  $K$ -theory allows it, for the map from  $C(S^2)$ . We leave to the reader the usual conversion of Theorems 1.2 and 1.3 into a lifting problem via generators and relations.

5. AMALGAMATED PRODUCTS OF  $C^{*,\tau}$ -ALGEBRAS

Define  $\Upsilon_A : A \rightarrow A^{\text{op}}$  to be the identity map on the underlying set, so a  $*$ -anti-homomorphism. We have  $(A^{\text{op}})^{\text{op}} = A$ .

**Lemma 5.1.** *Suppose  $C$ ,  $A_1$  and  $A_2$  are  $C^*$ -algebras,  $\theta_1 : C \rightarrow A_1$  and  $\theta_2 : C \rightarrow A_2$  are  $*$ -homomorphisms and consider the associated amalgamated product  $A_1 *_C A_2$  and canonical  $*$ -homomorphisms  $\iota_j : A_j \rightarrow A_1 *_C A_2$ . If  $\varphi_j : A_j \rightarrow D$  and  $\varphi_2 : A_2 \rightarrow D$  are  $*$ -anti-homomorphisms such that  $\varphi_1 \circ \theta_1 = \varphi_2 \circ \theta_2$ , then there is a unique  $*$ -anti-homomorphism  $\Phi : A_1 *_C A_2 \rightarrow D$  such that  $\Phi \circ \iota_j = \varphi_j$ .*

*Proof.* Both  $\Upsilon_D \circ \varphi_1$  and  $\Upsilon_D \circ \varphi_2$  are  $*$ -homomorphisms, and  $\Upsilon_D \circ \varphi_1 \circ \theta_1 = \Upsilon_D \circ \varphi_2 \circ \theta_2$ . There is a unique  $*$ -homomorphism  $\Psi : A_1 *_C A_2 \rightarrow D^{\text{op}}$  such that  $\Psi \circ \iota_j = \Upsilon_D \circ \varphi_j$ . Let  $\Phi = \Upsilon_{D^{\text{op}}} \circ \Psi$ . This is a  $*$ -anti-homomorphism and

$$\Phi \circ \iota_j = \Upsilon_{D^{\text{op}}} \circ \Psi \circ \iota_j = \Upsilon_{D^{\text{op}}} \circ \Upsilon_D \circ \varphi_j = \varphi_j.$$

If  $\Phi' : A_1 *_C A_2 \rightarrow D$  is also a  $*$ -anti-homomorphism with  $\Phi' \circ \iota_j = \varphi_j$  then  $\Upsilon_D \circ \Phi'$  is a  $*$ -homomorphism and

$$\Upsilon_D \circ \Phi' \circ \iota_j = \Upsilon_D \circ \varphi_j$$

so  $\Upsilon_D \circ \Phi' = \Psi$  and so

$$\Phi' = \text{id}_D \circ \Phi' = \Upsilon_{D^{\text{op}}} \circ \Upsilon_D \circ \Phi' = \Upsilon_{D^{\text{op}}} \circ \Psi = \Phi. \quad \square$$

What the lemma shows is that not only is  $A_1 *_C A_2$  universal for  $*$ -homomorphisms out of  $A_1$  and  $A_2$ , it is also universal for anti- $*$ -homomorphisms. We will use this fact to put a  $\tau$ -structure on the amalgamated product.

**Theorem 5.2.** *Suppose  $(C, \tau_0)$ ,  $(A_1, \tau_1)$  and  $(A_2, \tau_2)$  are  $C^{*,\tau}$ -algebras and  $\theta_1 : C \rightarrow A_1$  and  $\theta_2 : C \rightarrow A_2$  are  $*$ - $\tau$ -homomorphisms. The  $C^{*,\tau}$ -algebra  $A_1 *_C A_2$  becomes a  $C^{*,\tau}$ -algebra with the unique operation  $\tau$  making both  $\iota_j : A_j \rightarrow A_1 *_C A_2$  into  $*$ - $\tau$ -homomorphisms. Moreover,  $(A_1 *_C A_2, \tau)$  is the amalgamated product of  $(A_1, \tau_1)$  and  $(A_2, \tau_2)$  over  $(C, \tau_0)$ .*

*Proof.* We have anti-homomorphisms  $T_0 : C \rightarrow C$  and  $T_j : A_j \rightarrow A_j$  defined by  $T_0(c) = c^{\tau_0}$  and  $T_j(a) = a^{\tau_j}$ . The statement that  $\theta_j$  is a  $*$ - $\tau$ -homomorphism translates to  $\theta_j \circ T_0 = T_j \circ \theta_j$ . So we have

$$\iota_1 \circ T_1 \circ \theta_1 = \iota_1 \circ \theta_1 \circ T_0 = \iota_2 \circ \theta_2 \circ T_0 = \iota_2 \circ T_2 \circ \theta_2.$$

Therefore there is a unique  $*$ -anti-homomorphism  $T : A_1 *_C A_2 \rightarrow A_1 *_C A_2$  such that  $T(\iota_j(a)) = \iota_j(a^{\tau_j})$ . Certainly  $T \circ T$  is a  $*$ -homomorphism,

and since it fixes  $\iota_j(a)$  it is the identity map. We thus can make  $A_1 *_C A_2$  into a  $C^{*,\tau}$ -algebra by  $x^\tau = T(x)$ . For example,

$$(\iota_1(a)\iota_2(b)\iota_1(c))^\tau = \iota_1(c^{\tau_1})\iota_2(b^{\tau_2})\iota_1(a^{\tau_1}).$$

If we are given  $\varphi_j : A_j \rightarrow B$ , two  $*\text{-}\tau$ -homomorphisms such that  $\varphi_1 \circ \theta_1 = \varphi_2 \circ \theta_2$ , then there is a unique  $*$ -homomorphism  $\Phi : A_1 *_C A_2 \rightarrow B$  such that  $\Phi(\iota_j(a)) = \varphi_j(a)$  for all  $a$  in  $A_j$ . To finish, we must show that  $\Phi$  is actually a  $*\text{-}\tau$ -homomorphism. Given a product

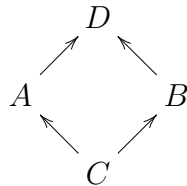
$$w = \iota_1(a_1)\iota_2(b_1)\iota_1(a_2)\iota_2(b_2) \cdots \iota_1(a_n)\iota_2(b_n)$$

we find

$$\begin{aligned} \Phi(w^\tau) &= \Phi(\iota_2(b_n^{\tau_2})\iota_1(a_n^{\tau_1}) \cdots \iota_2(b_1^{\tau_2})\iota_1(a_1^{\tau_1})) \\ &= \varphi_2(b_n^{\tau_2})\varphi_1(a_n^{\tau_1}) \cdots \varphi_2(b_1^{\tau_2})\varphi_1(a_1^{\tau_1}) \\ &= \varphi_2(b_n)^\tau \varphi_1(a_n)^\tau \cdots \varphi_2(b_1)^\tau \varphi_1(a_1)^\tau \\ &= (\varphi_1(a_1)\varphi_2(b_1) \cdots \varphi_1(a_n)\varphi_2(b_n))^\tau \\ &= (\Phi(\iota_1(a_1)\iota_2(b_1) \cdots \iota_1(a_n)\iota_2(b_n)))^\tau \\ &= (\Phi(w))^\tau. \end{aligned}$$

For the other three types of words (ending in  $\iota_1(A_1)$  or beginning in  $\iota_2(A_2)$ ) we also find  $\Phi(w^\tau) = \Phi(w)^\tau$ . Since  $\Phi$  is linear and continuous, we conclude  $\Phi(x^\tau) = \Phi(x)^\tau$  for all  $x$  in  $A_1 *_C A_2$ .  $\square$

We can talk about pushout diagrams instead of amalgamated products. If  $C$  is large relative to  $A_1$  or  $A_2$  this tends to be a more fitting description. What Theorem 5.2 shows is that when a diagram



is a diagram of  $*\text{-}\tau$ -homomorphisms and  $C^{*,\tau}$ -algebras, if it is a pushout in the category of  $C^*$ -algebras then it is a pushout in the category of  $C^{*,\tau}$ -algebras.

**Definition 5.3.** We say that a  $*\text{-}\tau$ -homomorphism is *proper* if it is proper as a  $*$ -homomorphism.

Recall (or see [4]) that a  $*$ -homomorphism  $\varphi : A \rightarrow B$  is said to be proper if for some (equivalently every) approximate unit  $e_\lambda$  of  $A$  the image  $\varphi(e_\lambda)$  is an approximate unit for  $B$ . Using Cohen factorization we find [21] that this is equivalent to the condition  $B = \varphi(A)B$ .

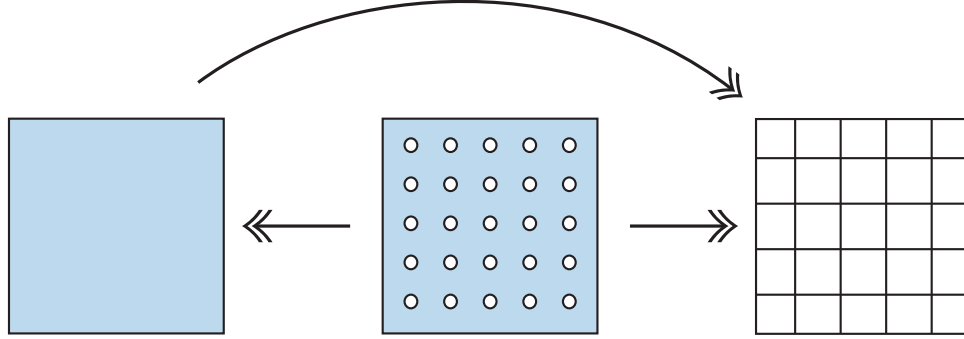


FIGURE 6.1. How we treat each 2-cell in a 2-dimensional CW complex  $X$ .

**Theorem 5.4.** *If a diagram of extensions*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_1 & \longrightarrow & C_1 & \longrightarrow & B \longrightarrow 0 \\
 & & \uparrow \alpha & & \uparrow & & \parallel \\
 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B \longrightarrow 0
 \end{array}$$

of  $C^{*,\tau}$ -algebras is given, with  $\alpha$  proper, then the left square is a pushout.

*Proof.* This is an immediate corollary of this statement in the category of  $C^*$ -algebras [4, Corollary 4.3] and Theorem 5.2.  $\square$

## 6. THE SPHERE, TORUS AND OTHER 2D SITUATIONS

Suppose  $X$  is a two-dimensional, finite CW complex. Assume the 2-cells are given as unit squares, and on that unit square a lattice of size  $\frac{1}{m}$ , at the center point of every square created by the lattice grid we insert a circle so create a space  $X^{[m]}$ . Let  $\Gamma_m$  denote the closed subset of  $X$  corresponding to replacing each 2-cell with the 1-D mesh. Consult figure 6.1.

**Theorem 6.1.** *Suppose*

$$\varphi : C(X, \text{id}) \rightarrow \prod (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n)$$

is a unital  $*$ - $\tau$ -homomorphism with  $\tau_n = \sharp$  for all  $n$  or  $\tau_n = \mathbb{T}$  for all  $n$ . If  $K_{-2}(\varphi) = 0$  then there exists a  $*$ - $\tau$ -homomorphism  $\psi$  so that

$$\begin{array}{ccc}
 C(X^{[n]}, \text{id}) & & \\
 \uparrow \iota & \searrow \psi & \\
 C(X, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n)
 \end{array}$$

commutes.

*Proof.* We can prove this by induction if we show the following. Suppose  $Y$  is an open set in a compact metrizable space  $X$  and that  $Y$  is homeomorphic to the open unit disk  $\Omega$ . Let  $Z$  denote the result of removing the closed set in  $Y$  that corresponds to the open disk of radius  $\frac{1}{2}$  at the center of  $\Omega$ , but then map  $Z$  onto  $X$  by continuous function that set fixes points in  $X \setminus Y$  and corresponds to the map

$$\{(x, y) \mid \frac{1}{2} \leq x^2 + y^2 < 1\} \rightarrow \{(x, y) \mid x^2 + y^2 < 1\} = \Omega$$

that operates as

$$(x, y) \mapsto \left( \frac{\sqrt{x^2 + y^2} - \frac{1}{2}}{\sqrt{x^2 + y^2}} x, \frac{\sqrt{x^2 + y^2} - \frac{1}{2}}{\sqrt{x^2 + y^2}} y \right).$$

We wish to show the given extension property

$$\begin{array}{ccc} C(Y, \text{id}) & & \\ \uparrow \iota & \dashrightarrow \psi & \\ C(X, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) \end{array}$$

where  $\varphi$  is assumed unital and  $\psi$  is required to be unital. We have

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_0(\Omega^{[1]}, \text{id}) & \longrightarrow & C(Y, \text{id}) & \longrightarrow & C(X \setminus Y, \text{id}) \longrightarrow 0 \\ & & \alpha \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & C_0(\Omega, \text{id}) & \longrightarrow & C(X, \text{id}) & \longrightarrow & C(X \setminus Y, \text{id}) \longrightarrow 0 \end{array}$$

a commuting diagram of  $C^{*,\tau}$ -algebras and morphisms, with exact rows. The map  $\Omega^{[1]} \rightarrow \Omega$  inducing  $\alpha$  is proper, so  $\alpha$  is proper. By Theorem 5.2 the left square is a pushout, so we get a map  $\varphi$  that is a  $*$ - $\tau$ -homomorphism by using Corollary 4.3. (We gain a map from  $C_0(\Omega^{[1]}, \text{id})$  and use that and the given map from  $C(X)$  to get a map from  $C(Y)$ .) As the unit in  $C(Y)$  comes from the unit in  $C(X)$  it is automatic that  $\psi$  is unital.  $\square$

**Theorem 6.2.** *Suppose  $X$  is a finite, two-dimensional CW complex. Then for any unital  $*$ - $\tau$ -homomorphism*

$$\varphi : C(X, \text{id}) \rightarrow \prod (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{d_n}(\mathbb{C}), \tau_n)$$



with  $\tau_n = \sharp$  for all  $n$  or  $\tau_n = \mathbb{T}$  for all  $n$ , there exists a unital  $*$ - $\tau$ -homomorphism  $\psi$  making the following diagram commute:

$$\begin{array}{ccc}
 & & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) \\
 & \nearrow \psi & \downarrow \pi \\
 C(X, \text{id}) & \xrightarrow{\varphi} & \prod (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n) / \bigoplus (\mathbf{M}_{2d_n}(\mathbb{C}), \tau_n)
 \end{array}$$

*Proof.* Let  $\mathcal{F}$  be a finite generating set for  $C(X)$ . By an intertwining argument as in the proof of Theorem 3.1 in [3] we can show it suffices to find for each  $\epsilon > 0$  a  $\psi$  so that  $\|\psi(f) - \pi \circ \varphi(f)\| < \epsilon$  for all  $f$  in  $\mathcal{F}$ . By extending  $\varphi$  to  $C(X^{[m]}, \text{id})$  for large  $m$  and then by retracting  $X^{[m]}$  to  $\Gamma_n$  we get that  $\varphi$  approximately factors through  $C(\Gamma_n, \text{id})$ . A map that factors through  $C(\Gamma_n, \text{id})$  will lift exactly, by the semiprojectivity of  $C(\Gamma_n, \text{id})$  which was established in our previous paper, [19].  $\square$

If we can describe  $C(X)$  by generators and relations, we get corollaries about approximation of tuples of real or self-dual matrices. Two spaces for which this works well are the familiar sphere and torus.

We have thus proven the two remaining main theorems, Theorems 1.5 and 1.6.

## 7. CONTROLLING MAXIMUM WANNIER SPREAD

Commuting sets of Hermitian matrices can be simultaneously diagonalized by a unitary matrix. Thus our main results have equivalent formulations in terms of simultaneous approximate diagonalization. Simultaneous approximate diagonalization arises in signal process [2] and the resulting algorithms have been utilized in first principals molecular dynamics [8].

There are many ways to measure the failure of a set of matrices to be diagonal. Essentially every unitarily invariant norm of the matrices gives a means to measure this. If the matrices can be interpreted as position operators, one can instead ask for “exponential localization” which really only makes sense if considering a class of finite models of increasing lattice size. We bring this up because it is probably the most physically relevant measure. Naturally, it is hard to even define, much less compute numerically. Within the choices of norms, the operator norm would be the most physically relevant, while something like the Frobenius norm the easiest to incorporate into an efficient algorithm.

In condensed matter physics, one often looks for a basis for an energy band that is well localized. When computed numerically, generally one attempt to minimize the *total* Wannier spread, which corresponds to

minimizing off-diagonal parts in the the Frobenius norm. We consider theoretical bounds on minimizing the *maximum* Wannier spread.

Given  $X_1, \dots, X_d$  Hermitian matrices in  $\mathbf{M}_n(\mathbb{C})$  and a unit vector  $\mathbf{b}$ , the *Wannier spread* of  $\mathbf{b}$  with respect to the set  $\mathcal{X}$  of these of  $X_r$  we define as

$$\sigma_{\mathcal{X}}^2(\mathbf{b}) = \sum_{r=1}^d \langle X_r^2 \mathbf{b}, \mathbf{b} \rangle - \langle X_r \mathbf{b}, \mathbf{b} \rangle^2.$$

Given an orthonormal subset  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$  of  $\mathbb{C}^n$  we define its *total Wannier spread* with respect to  $\mathcal{X}$  as

$$\sum_j \sigma_{\mathcal{X}}^2(\mathbf{b}_j)$$

and its *maximum Wannier spread* with respect to  $\mathcal{X}$  as

$$\mu_{\mathcal{X}}(\mathcal{B}) = \max_j \sigma_{\mathcal{X}}^2(\mathbf{b}_j).$$

For  $\mathcal{X} = \{X_1, \dots, X_d\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_d\}$ , both sets of Hermitian matrices on  $\mathbb{C}^n$ , define

$$\|\mathcal{X}\| = \max_r \|X_r\|$$

and

$$\text{dist}(\mathcal{X}, \mathcal{Y}) = \max_r \|X_r - Y_r\|.$$

**Lemma 7.1.** *Suppose  $\mathcal{X} = \{X_1, \dots, X_d\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_d\}$  are sets of Hermitian matrices on  $\mathbb{C}^n$  with  $\|\mathcal{X}\| \leq 1$  and  $\|\mathcal{Y}\| \leq 1$ . For any orthonormal subset  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$  of  $\mathbb{C}^n$ ,*

$$\mu_{\mathcal{X}}(\mathcal{B}) \leq \mu_{\mathcal{Y}}(\mathcal{B}) + 4d \cdot \text{dist}(\mathcal{X}, \mathcal{Y}).$$

*Proof.* For any unit vector  $\mathbf{b}$  we find

$$|\langle X_r^2 \mathbf{b}, \mathbf{b} \rangle - \langle Y_r^2 \mathbf{b}, \mathbf{b} \rangle| \leq \|X_r^2 - Y_r^2\| \leq 2 \|X_r - Y_r\|$$

and

$$|\langle X_r \mathbf{b}, \mathbf{b} \rangle^2 - \langle Y_r \mathbf{b}, \mathbf{b} \rangle^2| \leq 2 |\langle X_r \mathbf{b}, \mathbf{b} \rangle - \langle Y_r \mathbf{b}, \mathbf{b} \rangle| \leq 2 \|X_r - Y_r\|$$

and so

$$|\sigma_{X_r}^2(\mathbf{b}) - \sigma_{Y_r}^2(\mathbf{b})| \leq 4 \text{dist}(\mathcal{X}, \mathcal{Y}).$$

Therefore

$$|\sigma_{\mathcal{X}}^2(\mathbf{b}_j) - \sigma_{\mathcal{Y}}^2(\mathbf{b}_j)| = \left| \sum_r \sigma_{X_r}^2(\mathbf{b}_j) - \sum_r \sigma_{Y_r}^2(\mathbf{b}_j) \right| \leq 4d \cdot \text{dist}(\mathcal{X}, \mathcal{Y})$$

for all  $j$ . □

**Lemma 7.2.** *Suppose  $\mathcal{Y} = \{Y_1, \dots, Y_d\}$  is a set of commuting Hermitian matrices on  $\mathbb{C}^n$ . There exists an orthonormal basis  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  of  $\mathbb{C}^n$  for which  $\mu_{\mathcal{Y}}(\mathcal{B}) = 0$ .*

*Proof.* We take  $\mathcal{B}$  to be an orthonormal basis of common eigenvectors of the  $Y_r$  and then notice

$$\langle Y_r^2 \mathbf{b}_j, \mathbf{b}_j \rangle - \langle Y_r \mathbf{b}_j, \mathbf{b}_j \rangle^2 = 0. \quad \square$$

**Lemma 7.3.** *Suppose  $W : \mathbb{C}^{n_1} \rightarrow \mathbb{C}^{n_2}$  is an isometry. Let  $P = WW^*$ . Suppose  $\mathcal{X} = \{X_1, \dots, X_d\}$  is a set of Hermitian matrices on  $\mathbb{C}^{n_2}$  with  $\|\mathcal{X}\| \leq 1$  and  $\|[X_r, P]\| \leq \delta$  for all  $r$ . Suppose  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$  is an orthonormal set in  $\mathbb{C}^{n_1}$ . The orthonormal set  $W\mathcal{B} = \{W\mathbf{b}_1, \dots, W\mathbf{b}_k\}$  and the set  $W^*\mathcal{X}W = \{W^*X_1W, \dots, W^*X_dW\}$  satisfies*

$$\mu_{\mathcal{X}}(W\mathcal{B}) \leq \mu_{W^*\mathcal{X}W}(\mathcal{B}) + 8d\delta.$$

*Proof.* Let  $Y_r = PX_rP + (I - P)X_r(I - P)$ . Then

$$\|X_r - Y_r\| \leq 2\|PX_r(I - P)\| \leq 2\|[P, X_r]\|$$

so  $\text{dist}(\mathcal{X}, \mathcal{Y}) \leq 2\delta$  and so

$$\mu_{\mathcal{X}}(W\mathcal{B}) \leq \mu_{\mathcal{Y}}(W\mathcal{B}) + 8d\delta,$$

where  $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_d\}$ . For any  $\mathbf{b}$  in  $\mathbb{C}^{n_1}$  we find

$$\langle Y_r W\mathbf{b}, W\mathbf{b} \rangle = \langle W^*Y_r W\mathbf{b}, \mathbf{b} \rangle$$

and

$$\begin{aligned} \langle Y_r^2 W\mathbf{b}, W\mathbf{b} \rangle &= \langle W^*Y_r^2 P W\mathbf{b}, \mathbf{b} \rangle \\ &= \langle W^*Y_r W W^*Y_r W\mathbf{b}, \mathbf{b} \rangle \\ &= \langle (W^*Y_r W)^2 \mathbf{b}, \mathbf{b} \rangle. \end{aligned}$$

This implies

$$\mu_{W^*\mathcal{Y}W}(\mathcal{B}) = \mu_{\mathcal{Y}}(W\mathcal{B}).$$

However,  $W^*Y_r W = W^*X_r W$  so

$$\mu_{W^*\mathcal{X}W}(\mathcal{B}) = \mu_{W^*\mathcal{Y}W}(\mathcal{B}) = \mu_{\mathcal{Y}}(W\mathcal{B}). \quad \square$$

**Proposition 7.4.** *Suppose  $P$  is a projection in  $\mathbf{M}_n(\mathbb{C})$  and  $\hat{\mathcal{X}} = \{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4\}$  is a representation of  $\mathcal{T}'_0$  and  $\|[P, \hat{X}_r]\| \leq \delta$  for all  $r$ . Let  $n_1$  be the rank of  $P$  and suppose  $W : \mathbb{C}^{n_1} \rightarrow \mathbb{C}^n$  is any isometry with  $WW^* = P$ . Let  $X_r = W^*\hat{X}_r W$ .*

- (1)  $\|\mathcal{X}\| \leq 1$ .
- (2)  $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$  is a representation of  $\mathcal{T}'_{2\delta}$ .

- (3) if there is a representation  $\mathcal{Y}$  of  $\mathcal{T}'_0$  with  $\text{dist}(\mathcal{X}, \mathcal{Y}) \leq \epsilon$  then there is an orthogonal basis  $\mathcal{B}$  of  $P\mathbb{C}^n$  with

$$\mu_{\hat{\chi}}(\mathcal{B}) \leq 8d\delta + 4d\epsilon.$$

*Proof.* (1) The norm condition is easy, as it certainly is for for the  $\hat{X}_r$ .

(2) For any  $r$  and  $s$ ,

$$\begin{aligned} \|X_r X_s - X_s X_r\| &= \left\| W^* \hat{X}_r P \hat{X}_s W - W^* \hat{X}_s P \hat{X}_r W \right\| \\ &\leq \left\| \hat{X}_r P \hat{X}_s - \hat{X}_s P \hat{X}_r \right\| \\ &\leq \left\| \hat{X}_r P \hat{X}_s - \hat{X}_r \hat{X}_s P \right\| + \left\| \hat{X}_s \hat{X}_r P - \hat{X}_s P \hat{X}_r \right\| \\ &\leq 2\delta \end{aligned}$$

and, for  $(r, s)$  equal  $(1, 2)$  or  $(3, 4)$ , we find

$$\begin{aligned} &\|X_r^2 + X_s^2 - I\| \\ &= \left\| W^* \hat{X}_r P \hat{X}_r W + W^* \hat{X}_s P \hat{X}_s W - W^* P W \right\| \\ &\leq \left\| \hat{X}_r P \hat{X}_r + \hat{X}_s P \hat{X}_s - P \right\| \\ &\leq \left\| \hat{X}_r P \hat{X}_r - \hat{X}_r^2 P \right\| + \left\| \hat{X}_s P \hat{X}_s - \hat{X}_s^2 P \right\| + \left\| \hat{X}_r^2 P + \hat{X}_s^2 P - P \right\| \\ &\leq 2\delta. \end{aligned}$$

(3) Follows from (1) and (2) and Lemmas 7, 7.1 and 7.3.  $\square$

**Definition 7.5.** If we had made a different choice of partial isometry,  $W_1$ , then the resulting 4-tuples would be unitarily equivalent to the first, since  $W^* W_1$  is a unitary. Therefore, the Bott index,

$$\text{Bott}(X_1 + iX_2, X_3 + iX_4)$$

does not depend on the choice of  $W$ . We therefore define

$$\text{Bott}(P; \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) = \text{Bott}(X_1 + iX_2, X_3 + iX_4)$$

so long as  $\delta$  is small enough to ensure a spectral gap in the Bott matrix  $B(H_1, H_2, H_3)$ .

In this form, depending on a projection almost commuting with commuting matrices, our Bott index is essentially the same as the Chern number as calculated on finite samples, in [22].

If the projection  $P$  and the  $\hat{X}_r$  are real, then we can select the isometry  $W$  to be real which means that the  $X_r$  will be real.

If the projection  $P$  and the  $\hat{X}_r$  are self-dual in  $\mathbf{M}_{2N}(\mathbb{C})$ , we can use the spectral theorem for normal quaternionic matrices to find  $W$  an isometry with  $WW^* = P$  and  $W^* = -Z_{N_1}W^T Z_N$ . This means

$$X_r^\sharp = -Z_{N_1}W^T \hat{X}_r \overline{W} Z_{N_1} = -W^* Z_N \hat{X}_r Z_N W = W^* \hat{X}_r W$$

so the  $X_r$  are self-dual. We can unambiguously define

$$\text{Pf-Bott}(P; \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) = \text{Pf-Bott}(X_1 + iX_2, X_3 + iX_4).$$

We finish then with corollaries to the theorems in §1.6 about a lattice on a torus. Similar results are true for lattices on sphere, but these are less popular in physics.

**Theorem 7.6.** *For every  $\epsilon > 0$ , there is a  $\delta$  in  $(0, \frac{1}{8})$  so that, for every 4-tuple of commuting Hermitian matrices  $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4$  with in  $\mathbf{M}_n(\mathbb{C})$  with*

$$\hat{X}_1^2 + \hat{X}_2^2 = \hat{X}_3^2 + \hat{X}_4^2 = I,$$

*if  $P$  is a projection with  $\|[P, \hat{X}_r]\| \leq \delta$  for all  $r$ , and if*

$$\text{Bott}(P; \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) = 0,$$

*then there there is an orthonormal basis  $\mathbf{b}_1, \dots, \mathbf{b}_{n_1}$  of the subspace  $PC^n$  so that*

$$\left\langle \hat{X}_r^2 \mathbf{b}_j, \mathbf{b}_j \right\rangle - \left\langle \hat{X}_r \mathbf{b}_j, \mathbf{b}_j \right\rangle^2 \leq \epsilon$$

*for all  $r$  and all  $j$ .*

**Theorem 7.7.** *For every  $\epsilon > 0$ , there is a  $\delta > 0$  so that, for every 4-tuple of commuting Hermitian matrices  $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4$  with  $\mathbf{M}_n(\mathbb{R})$  with*

$$\hat{X}_1^2 + \hat{X}_2^2 = \hat{X}_3^2 + \hat{X}_4^2 = I,$$

*if  $P$  is a real projection with  $\|[P, \hat{X}_r]\| \leq \delta$  for all  $r$ , then there there is an orthonormal basis  $\mathbf{b}_1, \dots, \mathbf{b}_{n_1}$  of the subspace  $P\mathbb{R}^n$  so that*

$$\left\langle \hat{X}_r^2 \mathbf{b}_j, \mathbf{b}_j \right\rangle - \left\langle \hat{X}_r \mathbf{b}_j, \mathbf{b}_j \right\rangle^2 \leq \epsilon$$

*for all  $r$  and all  $j$ .*

**Theorem 7.8.** *For every  $\epsilon > 0$ , there is a  $\delta$  in  $(0, \frac{1}{8})$  so that, for every 4-tuple of commuting Hermitian, self-dual matrices  $\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4$  with  $\mathbf{M}_{2N}(\mathbb{C})$  with*

$$\hat{X}_1^2 + \hat{X}_2^2 = \hat{X}_3^2 + \hat{X}_4^2 = I,$$

*if  $P$  is a self-dual projection with  $\|[P, \hat{X}_r]\| \leq \delta$  for all  $r$ , and if*

$$\text{Pf-Bott}(P; \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4) = 1,$$

then there there are vectors  $\mathbf{b}_1, \dots, \mathbf{b}_{N_1}$  so that

$$\mathbf{b}_1, \dots, \mathbf{b}_{N_1}, \mathcal{T}\mathbf{b}_1, \dots, \mathcal{T}\mathbf{b}_{N_1}$$

is an orthonormal basis of the subspace  $PC^{2N}$  and

$$\left\langle \hat{X}_r^2 \mathbf{b}_j, \mathbf{b}_j \right\rangle - \left\langle \hat{X}_r \mathbf{b}_j, \mathbf{b}_j \right\rangle^2 \leq \epsilon$$

for all  $r$  and all  $j$ .

## 8. DISCUSSION

Our results on localization have very minimal assumptions. We do not require that the projection  $P$  arises from a gap in the spectrum of the a Hamiltonian. We also allow that the distribution of sites within the torus can be very irregular. We pay for this with very weak localization in the conclusion. In contrast, one can study specific models and find exponential decay in a basis made from time-reversal pairs [24].

Our results might have application in signal processing in areas such as hypercomplex processes [25] and blind source separation [2, 26].

## ACKNOWLEDGMENTS

We thank Matthew Hastings for his suggestions regarding the present article and for his role in laying out a program relating the theory of almost commuting matrices to condensed matter physics.

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