

# Square complexes and simplicial nonpositive curvature

Tomasz Elsner<sup>a\*</sup> and Piotr Przytycki<sup>b†</sup>

<sup>a</sup> *Mathematical Institute, University of Wrocław,  
Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland  
e-mail: elsner@math.uni.wroc.pl*

<sup>b</sup> *Institute of Mathematics, Polish Academy of Sciences,  
Śniadeckich 8, 00-956 Warsaw, Poland  
e-mail: pprzytyc@mimuw.edu.pl*

## Abstract

We prove that each nonpositively curved square  $\mathcal{VH}$ -complex can be turned functorially into a locally 6-large simplicial complex of the same homotopy type. It follows that any group acting geometrically on a CAT(0) square  $\mathcal{VH}$ -complex is systolic. In particular the product of two finitely generated free groups is systolic, which answers a question of Daniel Wise. On the other hand, we exhibit an example of a compact non- $\mathcal{VH}$  nonpositively curved square complex, whose fundamental group is neither systolic, nor even virtually systolic.

## 1 Introduction

In this note we compare nonpositively curved square  $\mathcal{VH}$ -complexes (introduced in [Wis96]) and locally 6-large simplicial complexes (introduced in [JŚ06]). First we describe locally 6-large and systolic complexes. The definitions we use are taken from [JŚ07], with a slight modification allowing simplices in a locally 6-large simplicial complex not to be embedded. Nevertheless, the definition of a systolic complex coincides with the one in [JŚ07].

**Definition 1.1.** A *generalised simplicial complex* is a set  $\mathcal{S}$  of affine simplices together with a set  $\mathcal{E}$  (closed under compositions) of affine embeddings of simplices of  $\mathcal{S}$  onto the faces of simplices of  $\mathcal{S}$  (attaching maps), such that for any proper face  $\tau$  of any simplex  $\sigma \in \mathcal{S}$  there is *precisely one* attaching map onto  $\tau$ .

A *(generalised) simplicial map* between generalised simplicial complexes is a set of affine maps commuting with the attaching maps and mapping each source simplex onto one of the target simplices.

---

\*Partially supported by MNiSW grant N N201 541 738.

†Partially supported by MNiSW grant N N201 541 738 and the Foundation for Polish Science.

The *geometric realisation* of a generalised simplicial complex  $(\mathcal{S}, \mathcal{E})$  is the quotient space  $\mathcal{S}/\mathcal{E}$ . The quotient map of a generalised simplicial map is the *geometric realisation* of such a map. We will abuse the language and not distinguish between simplicial complexes or simplicial maps and their geometric realisations.

The *link* of a simplex  $\sigma$  in a complex  $X = (\mathcal{S}, \mathcal{E})$  is the (generalised) simplicial complex  $X_\sigma = (\mathcal{S}_\sigma, \mathcal{E}_\sigma)$  where the set  $\mathcal{S}_\sigma$  is obtained by taking for each attaching map  $\phi_{\sigma, \tau}: \sigma \rightarrow \tau$  the maximal subsimplex of  $\tau$  disjoint from the image of  $\sigma$  and  $\mathcal{E}_\sigma$  is the set of restrictions of the maps in  $\mathcal{E}$ .

Subsequently, we refer to a generalised simplicial complex simply as a *simplicial complex* and use the phrase *simple simplicial complex* when referring to a standard simplicial complex (in which simplices are embedded and the intersection of two simplices, if non-empty, is a single simplex).

**Definition 1.2.** A simplicial complex is *simple* if it does not contain an edge joining a vertex to itself, or a pair of simplices with the same boundary (e.g. a double edge). A simple simplicial complex is *flag* if any complete subgraph (a clique) of its 1-skeleton spans a simplex.

A *cycle without diagonals* in a simplicial complex  $X$  is an embedded simplicial loop such that there are no edges in  $X$  connecting a pair of its nonconsecutive vertices.

A simplicial complex is *locally 6-large* if all of its vertex links are flag and do not contain cycles of length 4 or 5 without diagonals. A connected and simply connected locally 6-large simplicial complex is called *systolic* (i.e. systolic complexes are the universal coverings of connected locally 6-large complexes).

A group admitting a geometric action on a systolic complex is called *systolic*.

The original definition of local 6-largeness in [JS07] requires that we check the flagness and the absence of short cycles without diagonals for the link at *any* simplex. However, for higher-dimensional simplices it is a direct consequence of those properties for the links at the vertices.

Similarly as for simplicial complexes, we allow cells in square complexes not to be embedded. The formal definition of a (generalised) square complex is the same as of a generalised simplicial complex, except for putting *vertices, edges and squares* in place of *simplices*. The only thing that needs to be rephrased is the definition of the link.

**Definition 1.3.** The *link* at a vertex  $v$  of a (generalised) square complex  $X = (\mathcal{S}, \mathcal{E})$  is a 1-dimensional (generalised) simplicial complex  $X_v = (\mathcal{S}_v, \mathcal{E}_v)$

(a graph), where  $\mathcal{S}_v$  is obtained by taking for each attaching map  $\phi_{v,\sigma}: v \rightarrow \sigma$  the vertex of  $\sigma$  opposite to  $v$  (if  $\sigma$  is an edge) or the diagonal of  $\sigma$  opposite to  $v$  (if  $\sigma$  is a square) and  $\mathcal{E}_v$  is the set of restrictions of the maps in  $\mathcal{E}$ .

A square complex is called a  $\mathcal{VH}$ -complex if its 1-cells can be partitioned into two classes  $V$  and  $H$  called *vertical* and *horizontal* edges, respectively, and the attaching map of each square alternates between the edges of  $V$  and  $H$ . In other words, the link at each vertex is a bipartite graph with independent sets of vertices coming from edges of  $V$  and  $H$ .

Note that the link of a  $\mathcal{VH}$ -complex at a vertex may have double edges.

**Definition 1.4.** A square complex is *nonpositively curved* (or *locally CAT(0)*) if the link at any vertex does not contain embedded combinatorial cycle of length less than 4. For a  $\mathcal{VH}$  complex this reduces to the property that there are no double edges in the links at vertices.

For a general definition of CAT(0) and nonpositively curved spaces (not needed in our article) see [BH99]. Note only, that a simply connected space which is nonpositively curved is CAT(0) ([BH99, Theorem 4.1]).

**Example 1.5.** The product of two trees is a CAT(0)  $\mathcal{VH}$ -complex. If a group acts freely by isometries on the product of two trees and preserves the coordinates, then the quotient square complex is a nonpositively curved  $\mathcal{VH}$ -complex.

The paper is divided into two parts. In Section 2 we provide a functorial construction turning a nonpositively curved square  $\mathcal{VH}$ -complex into a locally 6-large simplicial complex of the same homotopy type (in particular turning a CAT(0)  $\mathcal{VH}$ -complex into a systolic complex). The main application of the construction is:

**Theorem 1.6** (see Corollary 2.6). *The fundamental group of a compact non-positively curved  $\mathcal{VH}$ -complex is systolic.*

The first application of Theorem 1.6 is the answer to a question posed by Daniel Wise in [Wis05]:

**Corollary 1.7.** *The product of two finitely generated free groups is systolic.*

We also obtain a series of consequences of Theorem 1.6 by applying it to the examples of nonpositively curved  $\mathcal{VH}$ -complexes (some with exotic properties) given by Daniel Wise in [Wis96].

**Corollary 1.8** (compare [Wis96, Corollary 2.8]). *The fundamental group of an alternating knot complement is systolic.*

**Corollary 1.9** (compare [Wis96, Theorem 5.5]). *There exists a systolic group, which is not residually finite.*

One can arrange for even a stronger property:

**Corollary 1.10** (compare [Wis96, Theorem 5.13]). *There exists a systolic group, which has no finite-index subgroups.*

In Section 3 we show that the  $\mathcal{VH}$ -hypothesis in Theorem 1.6 is necessary:

**Theorem 1.11** (see Theorem 3.2). *There exists a compact non- $\mathcal{VH}$  nonpositively curved square complex, whose fundamental group is not systolic, nor even virtually systolic.*

**Acknowledgements.** We thank Daniel Wise for motivating us, for his suggestions and discussions.

## 2 Nonpositively curved $\mathcal{VH}$ -complexes are systolic

Our main construction yields a way of turning a nonpositively curved  $\mathcal{VH}$ -complex into a locally 6-large simplicial complex.

**Construction 2.1.** Let  $X$  be a  $\mathcal{VH}$  complex with the sets  $E_V$  and  $E_H$  of vertical and horizontal edges, respectively. Denote by  $V$  and  $S$  the sets of vertices and squares of  $X$ , respectively. We construct an associated simplicial complex  $X^*$  called the *simplicification* of  $X$ , which has the same homotopy type as  $X$ .

First we divide each vertical edge  $e \in E_V$  in two and subdivide each square  $s \in S$  into six triangles, as in the Figure 1(a), obtaining a (generalised) simplicial complex  $\hat{X}$  (a triangulation of  $X$ ). The vertices of  $\hat{X}$  (which will correspond to the vertices of  $X^*$ ) are in bijective correspondence with the elements of  $V \cup E_V \cup S$ . We denote those vertices by  $v^*$ ,  $e^*$ ,  $s^*$ , for  $v \in V$ ,  $e \in E_V$ ,  $s \in S$ , respectively.

The link of  $\hat{X}$  at a vertex  $e^*$  is isomorphic to the suspension of a set of  $n$  points, where  $n$  is the number of squares  $s \in S$  with a vertical edge  $e$  (counted with multiplicities, i.e. a square with both vertical edges equal to  $e$  is counted twice). The union  $\hat{Y}_e$  of all the simplices of  $\hat{X}$  containing the vertex  $e^*$  is isomorphic to the suspension of an  $n$ -pod, where some pairs of vertices may be identified.

The complex  $X^*$  is obtained from  $\hat{X}$  by attaching simplices  $\sigma_v^+ = v_+^* e^* s_1^* \dots s_n^*$  and  $\sigma_v^- = v_-^* e^* s_1^* \dots s_n^*$  for each vertex  $e^*$ , where  $v_+$  and  $v_-$  are the endpoints

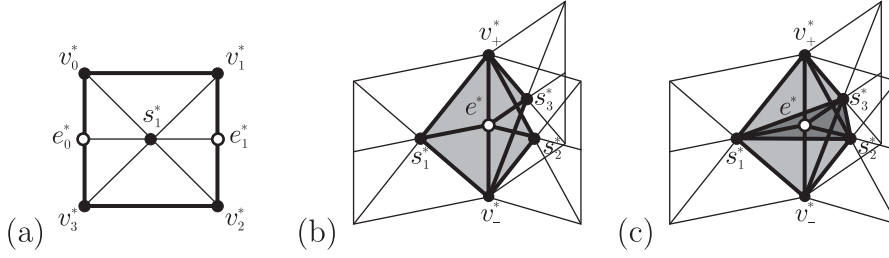


Figure 1: (a) the triangulation of  $X$  (b)  $\hat{Y}_e \subset \hat{X}$  (c)  $Y_e^* \subset X^*$

of the edge  $e \in E_V$  and  $s_1, \dots, s_n$  are the squares adjacent to the vertical edge  $e \in E_V$  (counted with multiplicities).

The link of  $e^*$  in  $X^*$  is the suspension of an  $(n-1)$ -simplex and the union  $Y_e^*$  of all the simplices of  $X^*$  containing the vertex  $e^*$  is isomorphic to the suspension of an  $n$ -simplex, where some pairs of vertices may be identified.

**Proposition 2.2.** *A  $\mathcal{VH}$  square complex  $X$  and its simplexification  $X^*$  have the same homotopy type.*

*Proof.* As the triangulation  $\hat{X}$  of  $X$  (defined in Construction 2.1) embeds into  $X^*$ , we only need to prove that  $X^*$  deformation retracts onto  $\hat{X}$ . Since for distinct  $e_0, e_1 \in E_V$  we have  $Y_{e_0}^* \cap Y_{e_1}^* \subset \hat{X}$  it is enough to show that for any  $e \in E_V$  the complex  $Y_e^*$  deformation retracts onto  $Y_e^* \cap \hat{X} = \hat{Y}_e$ .

If  $Y_e^*$  is a simple complex (i.e. the suspension of the simplex with vertices  $s_1^*, \dots, s_n^*, e^*$ ), then denoting by  $S$  the suspension and by  $C$  the cone operator, we have

$$\hat{Y}_e = S(C(\{s_1^*, \dots, s_n^*\})) \subset S(C(\sigma(s_1^*, \dots, s_n^*))) = Y_e^*,$$

where  $\sigma(s_1^*, \dots, s_n^*)$  is the simplex with vertices  $s_1^*, \dots, s_n^*$ .

Consider the retraction  $r : C(\sigma(s_1^*, \dots, s_n^*)) \rightarrow C(\{s_1^*, \dots, s_n^*\})$  defined to be the affine extension of the map from the first barycentric subdivision, which preserves the subcomplex  $C(\{s_1^*, \dots, s_n^*\})$  and maps the barycentres of the remaining simplices to the cone vertex. It is easy to see that  $r$  can be extended to a deformation retraction. By suspending the deformation retraction, we obtain a deformation retraction from  $Y_e^*$  onto  $\hat{Y}_e$ .

If  $Y_e^*$  is not simple, then it is a quotient space of the suspension of a simplex obtained by identifying some pairs of vertices. In that case the deformation retraction from  $Y_e^*$  onto  $\hat{Y}_e$  is the quotient of the map described above.  $\square$

**Remark 2.3.** Note that Construction 2.1 is functorial. Namely, let  $f : X \rightarrow Y$  be a *combinatorial map* between  $\mathcal{VH}$  complexes (i.e. mapping cells onto

cells of the same dimension, in our case mapping edges to edges and squares to squares). Assume also that  $f$  preserves the sets of vertical and horizontal edges. Then  $f$  induces a canonical combinatorial map  $f^*: X^* \rightarrow Y^*$ . Moreover, we have  $(f \circ g)^* = f^* \circ g^*$  and  $id^* = id$ . In particular, if  $f$  is invertible (in other words is a *combinatorial isomorphism*; it induces an isometry between the geometric realisations), then so is  $f^*$ . Finally, note that if a group  $G$  acts properly (cocompactly, geometrically) on  $X$ , then its induced action on  $X^*$  is also proper (cocompact, geometric).

We are now ready for our main result.

**Theorem 2.4.** *If  $X$  is a nonpositively curved  $\mathcal{VH}$  complex, then its simplexification  $X^*$  is locally 6-large.*

Before giving the proof, we list a few consequences, obtained by applying Proposition 2.2 and Remark 2.3.

**Corollary 2.5.** *If  $X$  is a  $CAT(0)$   $\mathcal{VH}$  complex, then its simplexification  $X^*$  is systolic. If  $G$  acts geometrically on  $X$ , then  $G$  is systolic.*

There are two notable applications of Corollary 2.5.

**Corollary 2.6** (Theorem 1.6). *The fundamental group of a compact non-positively curved  $\mathcal{VH}$  complex is systolic.*

The second application promotes Wise's aperiodic flat construction ([Wis96, Construction 7.1]) into the systolic setting.

**Definition 2.7.** A *flat* in a systolic complex  $X$  is a subcomplex  $\mathbb{E}_\Delta^2 \subset X$  which is isomorphic to the equilaterally triangulated plane (the triangulation with 6 triangles adjacent to each vertex) and whose 1-skeleton is isometrically embedded into  $X^{(1)}$  (with the combinatorial metric).

**Corollary 2.8.** *There exists compact a locally 6-large simplicial complex, whose universal cover (which is systolic) contains a flat, which is not the limit of a sequence of periodic flats.*

It remains to prove our main result.

*Proof of Theorem 2.4.* We need to check that the link of  $X^*$  at any vertex is flag and does not contain cycles of length 4 or 5 without diagonals. It is immediate for any vertex  $e^*$ , where  $e \in E_V$ , as the link of  $e^*$  is the suspension of a simplex.

The link of  $X$  at a vertex  $s^*$ ,  $s \in S$  has the form of two suspensions of simplices,  $S\sigma_{m-1}$  and  $S\sigma_{n-1}$  ( $m$  and  $n$  being the numbers of squares adjacent

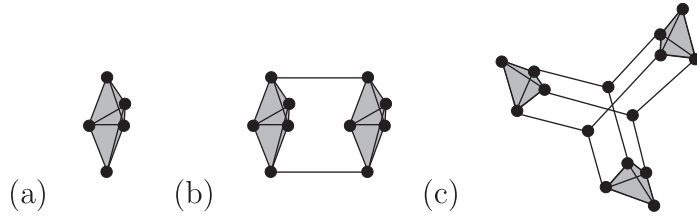


Figure 2: Sample link of  $X^*$  at a vertex (a)  $e^*$  (b)  $s^*$  (c)  $v^*$

to the vertical edges of  $s$ ), whose top and bottom vertices are connected by an edge (the case  $m = n = 3$  is depicted in Figure 2(b)). It is clear that it is flag and any cycle without diagonals in that link has length at least 6.

Now let  $L$  be the link of  $X^*$  at  $v^*$ ,  $v \in V$ . Then  $L$  is the union of:

- a set of simplices (one  $n_e$ -simplex for each vertical edge  $e \in E_V$  with an endpoint  $v$ , where  $n_e$  is the number of squares  $s \in S$  adjacent to  $e$ , counted with multiplicities) and
- a set of  $m_e$ -pods (one  $m_e$ -pod for each horizontal edge  $e \in E_H$  issuing from  $v$ , where  $m_e$  is the number of squares adjacent to  $e$ , counted with multiplicities),

where each endpoint of each  $m_e$ -pod is identified with a different vertex of one of the simplices (Figure 2(c) shows the link  $L$  in the case when the neighbourhood of  $v \in V$  is the product of two tripods). It is clear that  $L$  is flag, and any cycle without diagonals in  $L$  needs to pass through at least two  $m_e$ -pods and two simplices, which makes its length at least 6.  $\square$

### 3 Examples of nonpositively curved square complexes which are not systolic

In the next part of the paper we show that our theorem cannot be improved to include all nonpositively curved square complexes. Namely, we construct an example of a compact nonpositively curved square complex, whose fundamental group is not systolic. Later, we use that example to show a compact nonpositively curved square complex, whose fundamental group is neither systolic, nor even virtually systolic (Theorem 1.11).

Let  $K$  be the square complex presented in Figure 3, built up of two squares. It has only one vertex and the link at this vertex is shown in Figure 4. Thus we see that  $K$  is a nonpositively curved square complex, but not a  $\mathcal{VH}$ -complex. We will show that  $\pi_1(K)$  is not a systolic group.

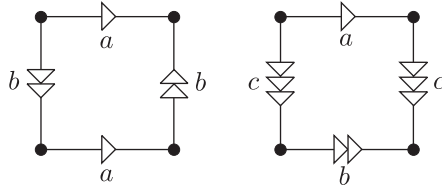


Figure 3: The nonpositively curved square complex  $K$

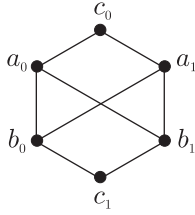


Figure 4: The link of  $K$  at the only vertex

**Theorem 3.1.** *The group  $\pi_1(K)$  is not systolic.*

*Proof.* The group

$$\pi_1(K) = \langle a, b, c \mid ba = ab^{-1}, a = cbc^{-1} \rangle$$

is an HNN-extension of the fundamental group of a Klein bottle, so it has a subgroup  $H = \langle a, b \rangle$ , which is isomorphic to the fundamental group of a Klein bottle, in particular is virtually  $\mathbb{Z}^2$ .

Suppose  $\pi_1(K)$  is systolic, i.e. acts geometrically on some systolic simplicial complex  $X$ . As a corollary from the systolic flat torus theorem (precisely by Corollary 6.2(1) together with Theorem 5.4 in [Els09]) we have that  $H$ , as a virtually  $\mathbb{Z}^2$  group, acts properly on a systolic flat in  $X$  (see Definition 2.7). If the fundamental group of a Klein bottle  $\langle a, b \mid ba = ab^{-1} \rangle$  acts properly (by combinatorial isomorphisms) on an equilaterally triangulated plane  $\mathbb{E}_\Delta^2$  (shown in Figure 5), then the axis of the glide reflection  $a$  is  $l$  and the direction of the translation  $b$  is  $k$  or vice versa.

The elements  $a^2$  and  $b^2$  act by translations on  $\mathbb{E}_\Delta^2$  (with axes  $k$  and  $l$ ). The 1-skeleton of  $\mathbb{E}_\Delta^2$  with the combinatorial metric is isometrically embedded into the 1-skeleton of  $X$  (by Definition 2.7), so the lines  $\hat{k}$  and  $l$  (marked in Figure 5) are invariant geodesics (in the 1-skeleton) for  $a^2$  and  $b^2$ . By [Els10, Proposition 3.10] the geodesic  $l$  is quasi-convex in the 1-skeleton of  $X$  equipped with the combinatorial metric (i.e. any geodesic in  $X^{(1)}$  with both endpoints on  $l$  is contained in the  $\delta$ -neighbourhood of  $l$ , for some universal  $\delta$ ). The geodesic  $\hat{k}$  is clearly not quasi-convex (every point of  $\mathbb{E}_\Delta^2$  lies on some geodesic with both endpoints on  $\hat{k}$ ).



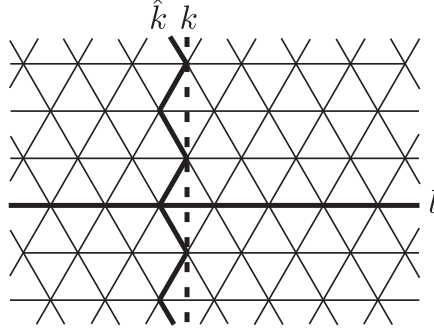


Figure 5:  $\mathbb{E}_\Delta^2$

Since  $a^2 = cb^2c^{-1}$ , the translation  $a^2$  has two invariant geodesics:  $\hat{k}$  and  $c(l)$  (or  $l$  and  $c(\hat{k})$ ). Two invariant geodesics of an isometry acting by a translation on both of them are at finite Hausdorff distance, so either both  $\hat{k}$  and  $c(l)$  are quasi-convex, or none of them is ([Els10, Proposition 3.11]). That contradicts the fact that  $l$  is quasi-convex, while  $\hat{k}$  is not.  $\square$

As we have just shown, the fundamental group of  $K$  is not systolic, however it is virtually systolic (there is a 2-leaf covering  $\tilde{K}$ , which is a  $\mathcal{VH}$ -complex, so  $\pi_1(\tilde{K})$  is systolic by Theorem 2.4). Now we use the complex  $K$  to construct a square complex  $S$ , whose fundamental group is not even virtually systolic.

Let  $E$  be the compact nonpositively curved  $\mathcal{VH}$ -complex which has no connected finite coverings, constructed by Wise in [Wis96, Theorem 5.13]. Let  $\sigma$  be any loop in  $E$  consisting entirely of horizontal edges. We can subdivide the complex  $K$  such that all loops  $a$ ,  $b$  and  $c$  have the same combinatorial length as the loop  $\sigma$ . Now we define  $\bar{E}$  and  $\bar{\bar{E}}$  to be two copies of  $E$  and let

$$S = (E \cup \bar{E} \cup \bar{\bar{E}}) \cup K / \sim,$$

where  $\sim$  is the identification of  $\sigma$ ,  $\bar{\sigma}$  and  $\bar{\bar{\sigma}}$  with  $a$ ,  $b$  and  $c$ , respectively. Then  $S$  is a nonpositively curved (non- $\mathcal{VH}$ ) square complex.

**Theorem 3.2** (Theorem 1.11). *The group  $\pi_1(S)$  is not virtually systolic, where  $S$  is the nonpositively curved square complex defined above.*

*Proof.* We first argue that  $\pi_1(S)$  is not systolic. Since

$$\pi_1(S) = \pi_1(K) *_{a=\sigma} \pi_1(E) *_{b=\bar{\sigma}} \pi_1(\bar{E}) *_{c=\bar{\bar{\sigma}}} \pi_1(\bar{\bar{E}})$$

is an amalgam product, the inclusion  $K \subset S$  induces injection  $\pi_1(K) \rightarrow \pi_1(S)$ . To conclude that  $\pi_1(S)$  is not systolic, we can recall the fact that a

finitely presented subgroup of a torsion-free systolic group is systolic itself ([Wis05]), while  $\pi_1(K)$  is not systolic (Theorem 3.1). An equivalent way of arriving to that conclusion is to repeat for  $S$  the argument used for  $K$  in the proof of Theorem 3.1.

To prove that  $\pi_1(S)$  is not virtually systolic, we show that it has no finite-index subgroups (i.e.  $S$  has no connected non-trivial finite coverings). Let  $p : \tilde{S} \rightarrow S$  be a connected finite covering. Since  $E \subset S$  has no connected non-trivial finite coverings,  $p^{-1}(E)$  is a disjoint union of copies of  $E$ . In particular, any lift  $\tilde{a}$  of the loop  $a$  has the same length as  $a$ . The same holds for the loops  $b$  and  $c$ . As  $a$ ,  $b$  and  $c$  together with the three copies of  $\pi_1(E)$  generate  $\pi_1(S)$ , that implies that  $p$  is a trivial covering.  $\square$

## References

- [BH99] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.
- [Els09] T. Elsner, *Flats and the flat torus theorem in systolic spaces*, *Geometry&Topology* **13** (2009), 661-698.
- [Els10] ———, *Flats and the flat torus theorem in systolic spaces* (2010), submitted.
- [JŚ06] T. Januszkiewicz and J. Świątkowski, *Simplicial nonpositive curvature*, *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 1–85.
- [JŚ07] ———, *Filling invariants of systolic complexes and groups*, *Geometry&Topology* **11** (2007), 727–758.
- [Wis05] D. T. Wise, *Sixtolic complexes and their fundamental groups* (2005), in preparation.
- [Wis96] ———, *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups* (1996), Ph.D. thesis.