A property of the derivative of an entire function

Walter Bergweiler* and Alexandre Eremenko†

July 22, 2011

Abstract

We prove that the derivative of a non-linear entire function is unbounded on the preimage of an unbounded set.

MSC 2010: 30D30. Keywords: entire function, normal family.

1 Introduction and results

The main result of this paper is the following theorem conjectured by Allen Weitsman (private communication):

Theorem 1. *Let f be a non-linear entire function and M an unbounded set in $\mathbf{C}$. Then $f'(f^{-1}(M))$ is unbounded.*

We note that there exist entire functions f such that $f'(f^{-1}(M))$ is bounded for every bounded set M, for example, $f(z) = e^z$ or $f(z) = \cos z$.

Theorem 1 is a consequence of the following stronger result:

Theorem 2. *Let f be a transcendental entire function and $\varepsilon > 0$. Then there exists $R > 0$ such that for every $w \in \mathbf{C}$ satisfying $|w| > R$ there exists $z \in \mathbf{C}$ with $f(z) = w$ and $|f'(z)| \geq |w|^{1-\varepsilon}$.*

---

*Supported by the Deutsche Forschungsgemeinschaft, Be 1508/7-1, and the ESF Networking Programme HCAA.

†Supported by NSF grant DMS-1067886.

The example $f(z)=\sqrt{z}\sin\sqrt{z}$ shows that that the exponent $1-\varepsilon$ in the last inequality cannot be replaced by 1. The function $f(z)=\cos\sqrt{z}$ has the property that for every $w\in\mathbf{C}$ we have $f'(z)\to 0$ as $z\to\infty$, $z\in f^{-1}(w)$.

We note that the Wiman–Valiron theory [20, 12, 4] says that there exists a set $F\subset[1,\infty)$ of finite logarithmic measure such that if

$$|z_r|=r\notin F \quad\text{and}\quad |f(z_r)|=\max_{|z|=r}|f(z)|,$$

then

$$f(z)\sim\left(\frac{z}{z_r}\right)^{\nu(r,f)}f(z_r)\quad\text{and}\quad f'(z)\sim\frac{\nu(r,f)}{r}f(z)$$

for $|z-z_r|\leq r\nu(r,f)^{-1/2-\delta}$ as $r\to\infty$. Here $\nu(r,f)$ denotes the central index and $\delta>0$. This implies that the conclusion of Theorem 2 holds for all w satisfying $|w|=M(r,f)$ for some sufficiently large $r\notin F$. However, in general the exceptional set in the Wiman–Valiron theory is non-empty (see, e.g., [3]) and thus it seems that our results cannot be proved using Wiman–Valiron theory.

Acknowledgment. We thank Allen Weitsman for helpful discussions.

2 Preliminary results

One important tool in the proof is the following result known as the Zalcman Lemma [21]. Let

$$g^{\#}=\frac{|g'|}{1+|g|^2}$$

denote the spherical derivative of a meromorphic function g.

Lemma 1. *Let F be a non-normal family of meromorphic functions in a region D. Then there exist a sequence (f_n) in F, a sequence (z_n) in D, a sequence (ρ_n) of positive real numbers and a non-constant function g meromorphic in $\mathbf{C}$ such that $\rho_n\to 0$ and $f_n(z_n+\rho_n z)\to g(z)$ locally uniformly in $\mathbf{C}$. Moreover, $g^{\#}(z)\leq g^{\#}(0)=1$ for $z\in\mathbf{C}$.*

We say that $a\in\overline{\mathbf{C}}$ is a *totally ramified* value of a meromorphic function f if all a-points of f are multiple. A classical result of Nevanlinna says that a non-constant function meromorphic in the plane can have at most 4 totally ramified values, and that a non-constant entire function can have at most 2 finite totally ramified values. Together with Zalcman's Lemma this yields the following result [5, 13, 14]; cf. [22, p. 219].

Lemma 2. *Let F be a family of functions meromorphic in a domain D and M a subset of $\overline{\mathbf{C}}$ with at least 5 elements. Suppose that there exists $K \geq 0$ such that for all $f \in F$ and $z \in D$ the condition $f(z) \in M$ implies $|f'(z)| \leq K$. Then F is a normal family.*

If all functions in F are holomorphic, then the conclusion holds if M has at least 3 elements.

Applying Lemma 2 to the family $\{f(z+c) : c \in \mathbf{C}\}$ where f is an entire function, we obtain the following result.

Lemma 3. *Let f be an entire function and M a subset of $\mathbf{C}$ with at least 3 elements. If f' is bounded on $f^{-1}(M)$, then $f^{\#}$ is bounded in $\mathbf{C}$.*

It follows from Lemma 3 that the conclusion of Theorems 1 and 2 holds for all entire functions for which $f^{\#}$ is unbounded.

We thus consider entire functions with bounded spherical derivative. The following result is due to Clunie and Hayman [6]. Let

$$M(r,f) = \max_{|z|\leq r} |f(z)| \quad \text{and} \quad \rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

denote the maximum modulus and the order of f.

Lemma 4. *Let f be an entire function for which $f^{\#}$ is bounded. Then $\log M(r,f) = O(r)$ as $r \to \infty$. In particular, $\rho(f) \leq 1$.*

We will include a proof of Lemma 4 after Lemma 6.

The following result is due to Valiron [20, III.10] and H. Selberg [17, Satz II].

Lemma 5. *Let f be a non-constant entire function of order at most 1 for which 1 and -1 are totally ramified. Then $f(z) = \cos(az+b)$, where $a, b \in \mathbf{C}$, $a \neq 0$.*

We sketch the proof of Lemma 5. Put $h(z) = f'(z)^2/(f(z)^2-1)$. Then h is entire and the lemma on the logarithmic derivative [9, p.94, (1.17)], together with the hypothesis that $\rho(f) \leq 1$, yields that $m(r,h) = o(\log r)$ and hence that h is constant. This implies that f has the form given. Another proof is given in [10]

The next lemma can be extracted from the work of Pommerenke [16, Sect. 5], see [8, Theorem 5.2].

Lemma 6. *Let f be an entire function and $C > 0$. If $|f'(z)| \leq C$ whenever $|f(z)| = 1$, then $|f'(z)| \leq C|f(z)|$ whenever $|f(z)| \geq 1$.*

Lemma 6 implies the theorem of Clunie and Hayman mentioned above (Lemma 4). For the convenience of the reader we include a proof of a slightly more general statement, which is also more elementary than the proofs of Clunie, Hayman and Pommerenke; see also [1, Lemma 1].

Let $G = \{z : |f(z)| > 1\}$ and $u = \log|f|$. Then $|f'/f| = |\nabla u|$ and our statement which implies Lemmas 4 and 6 is the following.

Proposition. *Let G be a region in the plane, u a harmonic function in $\overline{G}$, positive in G, and such that for $z \in \partial G$ we have $u(z) = 0$ and $|\nabla u(z)| \leq 1$. Then $|\nabla u(z)| \leq 1$ for $z \in G$, and $u(z) \leq |z| + O(1)$ as $z \to \infty$.*

Proof. It is enough to consider the case of unbounded G with non-empty boundary. For $a \in G$, consider the largest disc B centered at a and contained in G. The radius $d = d(a)$ of this disc is the distance from a to ∂G. There is a point $z_1 \in \partial B$ such that $u(z_1) = 0$. Put $z(r) = a + r(z_1 - a)$, where $r \in (0, 1)$. Harnack's inequality gives

$$\frac{u(a)}{d(1+r)} \leq \frac{u(z(r))}{d(1-r)} = \frac{u(z(r)) - u(z_1)}{d(1-r)}.$$

Passing to the limit as $r \to 1$ we obtain

$$u(a) \leq 2d(a)|\nabla u(z_1)| \leq 2d(a).$$

This holds for all $a \in G$. Now we take the gradient of both sides of the Poisson formula and, noting that $u(a + d(a)e^{it}) \leq 2d(a + d(a)e^{it}) \leq 4d(a)$, obtain the estimate

$$|\nabla u(a)| \leq \frac{1}{\pi d(a)} \int_{-\pi}^{\pi} |u(a + d(a)e^{it})| dt \leq 8.$$

So ∇u is bounded in G. As the complex conjugate of ∇u is holomorphic in G and $|\nabla u(z)| \leq 1$ at all boundary points z of G, except infinity, the Phragmén–Lindelöf theorem [15, III, 335] gives that $|\nabla u(z)| \leq 1$ for $z \in G$. This completes the proof of the Proposition.

We recall that for a non-constant entire function f the maximum modulus $M(r) = M(r, f)$ is a continuous strictly increasing function of r. Denote by

φ the inverse function of M. Clearly, for $|w| > |f(0)|$ the equation $f(z) = w$ has no solutions in the open disc of radius $\varphi(|w|)$ around 0. The following result of Valiron ([18, 19], see also [7]) says that for functions of finite order this equation has solutions in a somewhat larger disc.

Lemma 7. *Let f be a transcendental entire function of finite order and $\eta > 0$. Then there exists $R > |f(0)|$ such that for all $w \in \mathbf{C}, |w| \geq R$, the equation $f(z) = w$ has a solution z satisfying $|z| < \varphi(|w|)^{1+\eta}$.*

We note that Hayman ([11], see also [2, Theorem 3]) has constructed examples which show that the assumption about finite order is essential in this lemma.

3 Proof of Theorem 2

Suppose that the conclusion is false. Then there exists $\varepsilon > 0$, a transcendental entire function f and a sequence (w_n) tending to ∞ such that $|f'(z)| \leq |w_n|^{1-\varepsilon}$ whenever $f(z) = w_n$. By Lemma 3, the spherical derivative of f is bounded, and we may assume without loss of generality that

$$f^{\#}(z) \leq 1 \quad \text{for } z \in \mathbf{C}. \tag{1}$$

We may also assume that $f(0) = 0$. It follows from (1) that $|f'(z)| \leq 2$ if $|f(z)| = 1$, and thus Lemma 6 yields

$$\left|\frac{f'(z)}{f(z)}\right| \leq 2 \quad \text{if } |f(z)| \geq 1. \tag{2}$$

It also follows from (1), together with Lemma 4, that $\rho(f) \leq 1$. We may thus apply Lemma 7 and find that if $\eta > 0$ and if n is sufficiently large, then there exists ξ_n satisfying

$$|\xi_n| \leq \varphi(|w_n|)^{1+\eta} \quad \text{and} \quad f(\xi_n) = w_n.$$

We put

$$\tau_n = \varphi(|w_n|)^{1+2\eta}$$

and define

$$\Phi_n(z) = \frac{w_n - 2f(\tau_n z)}{w_n} = 1 - 2\frac{f(\tau_n z)}{w_n}.$$

Then $\Phi_n(0) = 1$, $\Phi_n(\xi_n/\tau_n) = -1$, and $\xi_n/\tau_n \to 0$ as $n \to \infty$. Thus the sequence (Φ_n) is not normal at 0, and we may apply Zalcman's Lemma (Lemma 1) to it. Replacing (Φ_n) by a subsequence if necessary, we thus find that

$$g_n(z) = \Phi_n(z_n + \rho_n z) = 1 - \frac{2}{w_n} f(\tau_n z_n + \tau_n \rho_n z) \to g(z)$$

locally uniformly in $\mathbf{C}$, where $|z_n| \leq 1$, $\rho_n > 0$, $\rho_n \to 0$, and g is a non-constant entire function with bounded spherical derivative. With $\zeta_n = \tau_n z_n$ and $\mu_n = \tau_n \rho_n$ we have

$$g_n(z) = 1 - \frac{2}{w_n} f(\zeta_n + \mu_n z), \tag{3}$$

and

$$g_n'(z) = -\frac{2\mu_n}{w_n} f'(\zeta_n + \mu_n z). \tag{4}$$

We may assume that $\rho_n \leq 1$ and hence $|\zeta_n| \leq \tau_n$ and $\mu_n \leq \tau_n$ for all n.

If $g_n(z) = 1$, then $f(\zeta_n + \mu_n z) = 0$, hence $|f'(\zeta_n + \mu_n z)| \leq 1$ by (1). Since $\mu_n \leq \tau_n$, we deduce that

$$|g_n'(z)| \leq \frac{2\tau_n}{w_n} \quad \text{if } g_n(z) = 1. \tag{5}$$

If $g_n(z) = -1$, then $f(\zeta_n + \mu_n z) = w_n$, and hence $|f'(\zeta_n + \mu_n z)| \leq |w_n|^{1-\varepsilon}$ by our assumption. Thus

$$|g_n'(z)| \leq \frac{2\mu_n}{|w_n|} |w_n|^{1-\varepsilon} \leq \frac{2\tau_n}{|w_n|^{\varepsilon}} \quad \text{if } g_n(z) = -1. \tag{6}$$

It follows from the definition of τ_n that

$$\tau_n = o(|w_n|)^{\delta}) \quad \text{as } n \to \infty, \tag{7}$$

for any given $\delta > 0$.

We deduce from (5), (6) and (7) that $g'(z) = 0$ whenever $g(z) = 1$ or $g(z) = -1$. Since g has bounded spherical derivative, we conclude from Lemmas 3 and 4 that $g(z) = \cos(az + b)$. Without loss of generality, we may assume that $g(z) = \cos z$ so that $g'(z) = -\sin z$. In particular, there exist sequences (a_n) and (b_n) both tending to 0, such that $g_n(a_n) = 1$ and $g_n'(b_n) = 0$. From (5) we deduce that

$$|g_n'(a_n)| \leq \frac{2\tau_n}{|w_n|}. \tag{8}$$

Noting that $g''(z) = -\cos z$ we find that

$$g_n'(a_n) = g_n'(a_n) - g_n'(b_n) = \int_{b_n}^{a_n} g_n''(z)dz \sim b_n - a_n \tag{9}$$

as $n \to \infty$, and thus

$$|b_n - a_n| \leq \frac{3\tau_n}{|w_n|} \tag{10}$$

for large n, by (8). This implies that

$$|g_n(b_n) - 1| = |g_n(b_n) - g_n(a_n)| = \left| \int_{a_n}^{b_n} g_n'(z)dz \right| \leq 2|b_n - a_n| \leq \frac{6\tau_n}{|w_n|} \tag{11}$$

for large n.

We put

$$h_n(z) = g_n(z + b_n) - g_n(b_n)$$

and note that $h_n(0) = 0$, $h_n'(0) = g_n'(b_n) = 0$ and

$$h_n(z) \to \cos z - 1 \quad \text{as } n \to \infty.$$

It follows that

$$\frac{h_n(z)}{z^2} \to \frac{\cos z - 1}{z^2} \quad \text{as } n \to \infty,$$

which implies that there exists $r > 0$ such that

$$\frac{1}{4} \leq \frac{|h_n(z)|}{|z^2|} \leq \frac{3}{4} \quad \text{for } |z| \leq r. \tag{12}$$

and large n.

Now we fix any $\gamma \in (0, 1/2)$ and put

$$c_n = b_n + \frac{1}{|w_n|^\gamma}.$$

Then

$$g_n(c_n) - 1 = h_n(|w_n|^{-\gamma}) + g(b_n) - 1$$

and thus, using (11) and (12) we obtain for large n:

$$|g_n(c_n) - 1| \leq \left| h_n(|w_n|^{-\gamma}) \right| + |g(b_n) - 1| \leq \frac{3}{4|w_n|^{2\gamma}} + \frac{6\tau_n}{|w_n|} \leq \frac{1}{|w_n|^{2\gamma}}. \tag{13}$$

Similarly

$$|g_n(c_n)-1| \geq \left|h_n(|w_n|^{-\gamma})\right| - |g(b_n)-1| \geq \frac{1}{5|w_n|^{2\gamma}}. \tag{14}$$

On the other hand, arguing as in (9), we have

$$g_n'(c_n) = g_n'(c_n) - g_n'(b_n) = \int_{b_n}^{c_n} g_n''(z)dz \sim b_n - c_n = -\frac{1}{|w_n|^\gamma},$$

and thus

$$|g_n'(c_n)| \geq \frac{1}{2|w_n|^\gamma} \tag{15}$$

for large n. Put $v_n = \zeta_n + \mu_n c_n$. Then

$$f(v_n) = \frac{w_n}{2}(1-g_n(c_n)) \quad \text{and} \quad f'(v_n) = \frac{w_n}{2\mu_n} g_n'(c_n),$$

by (3) and (4). Hence

$$\frac{1}{10}|w_n|^{1-2\gamma} \leq |f(v_n)| \leq \frac{1}{2}|w_n|^{1-2\gamma}, \tag{16}$$

by (13) and (14) while

$$|f'(v_n)| \geq \frac{|w_n|^\gamma}{2\mu_n}.$$

Since $|f(v_n)| \geq 1$ for large n, by (16), this contradicts (2) and (7).

References

[1] M. Barrett and A. Eremenko, A generalization of a theorem of Clunie and Hayman, to appear in Proc. Amer. Math. Soc., arXiv: 1011.3907.

[2] W. Bergweiler, Order and lower order of composite meromorphic functions, Michigan Math. J., 36 (1989) 135–146.

[3] W. Bergweiler, On meromorphic functions that share three values and on the exceptional set in Wiman–Valiron theory, Kodai Math. J. 13 (1990) 1–9.

[4] W. Bergweiler, P. J. Rippon and G. M. Stallard, Dynamics of meromorphic functions with direct or logarithmic singularities, Proc. London Math. Soc., 97 (2008) 368–400.

[5] H. H. Chen and X. H. Hua, Normality criterion and singular directions, in "Proceedings of the Conference on Complex Analysis (Tianjin, 1992)", Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994, pp. 34–40.

[6] J. Clunie and W. Hayman, On the spherical derivative of integral and meromorphic functions, Comm. Math. Helv., 40 (1966) 117–148.

[7] A. Edrei and W. Fuchs, On the zeros of $f(g(z))$ where f and g are entire functions, J. Anal. Math., 2 (1964) 243–255.

[8] A. Eremenko, Normal holomorphic curves from parabolic regions to projective spaces, arXiv: 0710.1281.

[9] A. A. Goldberg and I. V. Ostrovskii, Value distribution of meromorphic functions. Transl. Math. Monographs 236, American Math. Soc., Providence, R. I., 2008.

[10] А. Гольдберг, В. Таирова, О целых функциях с двумя конечными вполне разветвленными значениями, Записки мех-мат факултета Харьковского университета и Харьковского мат. общ., 29 (1963) 67–78 (Russian).

[11] W. Hayman, Some integral functions of infinite order, Math. Notae, 20 (1965) 1–5.

[12] W. Hayman, The local growth of power series: a survey of the Wiman–Valiron method, Canad. Math. Bull., 17 (1974) 317–358.

[13] A. Hinkkanen, Normal families and Ahlfors's five islands theorem, New Zealand J. Math., 22 (1993) 2, 39–41.

[14] P. Lappan, A uniform approach to normal families, Rev. Roumaine Math. Pures Appl., 39 (1994) 691–702.

[15] G. Pólya and G. Szegö, Problems and theorems in analysis. I, Springer, New York, 1998.

[16] Ch. Pommerenke, Estimates for normal meromorphic functions, Ann. Acad. Sci. Fenn., Ser. A I, 476 (1970) 10pp.

[17] H. Selberg, Über einige Eigenschaften bei der Werteverteilung der meromorphen Funktionen endlicher Ordnung, Avh. Det Norske Videnskaps-Akademi i Oslo I. Matem.-Naturvid. Kl., 7 (1928) 17pp.

[18] G. Valiron, Sur les fonctions entières et leurs fonctions inverses, Comptes Rendus, 173 (1921) 1059–1061.

[19] G. Valiron, Sur un théorème de M. Fatou, Bull. Sci. Math., 46 (1922) 200–208.

[20] G. Valiron, Lectures on the general theory of entire functions, Chelsea, New York, 1949.

[21] L. Zalcman, A heuristic principle in complex function theory, Amer. Math. Monthly, 82 (1975) 813–817.

[22] L. Zalcman, Normal families: new perspectives, Bull. Amer. Math. Soc. (N. S.), 35 (1998) 215–230.

*Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel,
Ludewig-Meyn-Str. 4, 24098 Kiel, Germany
bergweiler@math.uni-kiel.de*

*Department of Mathematics, Purdue University, West Lafayette, IN 47907
eremenko@math.purdue.edu*